Representations of Heisenberg Groups

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Abstract

The theorem of Marshall Stone and John von Neumann states that for a Heisenberg group $G$, there exists a unique irreducible representation of $G$ such that the centre of $G$ acts normally. David Mumford extended this result to affine group schemes, but the proof was only published later by Tsutomu Sekiguchi. This thesis is a survey of the representation theory of Heisenberg groups, focussed mainly on the Stone-von Neumann theorem.

We begin with the construction of the irreducible representations of Heisenberg groups over a finite field of prime characteristic $p \neq 2$. After that we prove a special case of the classical Stone-von Neumann theorem, and finally we elucidate the proof in Sekiguchi’s paper.

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Chapter 1

Introduction

The final and most interesting result in this thesis is the Stone-von Neumann theorem for algebraic Heisenberg groups. Algebraic Heisenberg groups are affine group schemes $\mathcal{G}$ over an algebraically closed field $k$ lying in a central extension

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathcal{G} \rightarrow K \rightarrow 0,$$

where $K$ is a finite abelian affine group scheme and $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$ is the centre of $\mathcal{G}$.

The Stone-von Neumann theorem states that for an algebraic Heisenberg group $\mathcal{G}$, there is exactly one irreducible representation for which the centre of $\mathcal{G}$ acts normally; we call this representation the Heisenberg representation. This statement can be found in Proposition 3.7 of [Mumford et al., 1991] and additional details of the proof are provided in the Appendix of [Sekiguchi, 1977].

Heisenberg groups were first studied for their role in quantum mechanics, and the Stone-von Neumann theorem established the equivalence of the matrix mechanics formulation of QM and Schrödinger’s wave mechanics. A history of the theorem can be found in [Rosenberg, 2003], and a general introduction to Heisenberg groups is given in [Semmes, 2002].

A brief overview of representations and character theory is given in this chapter; good references are [Fulton and Harris, 1991], [Simon, 1996], and [Serre, 1977]. Finite Heisenberg groups are introduced in Chapter 2, and their finite-dimensional irreducible representations are explicitly constructed.

A proof of a special case of the Stone-von Neumann theorem for continuous Heisenberg groups is given in Chapter 3, following Section 1 of [Mumford et al., 1991]. The canonical example of a continuous Heisenberg group is $H_3(\mathbb{R})$, the group of upper-triangular 3-by-3 matrices with entries in $\mathbb{R}$ having 1’s on the diagonal. One particular realisation of the Heisenberg representation, the theta representation, is notable for the fact that the Jacobi theta function is invariant under the action of a discrete subgroup.

As algebraic Heisenberg groups are defined using the language of affine group schemes, an introduction to these objects is given in Chapter 4. There is a contravariant equivalence between the category of affine group schemes and the category of Hopf algebras,
also introduced in Chapter 4, and the proof of the Stone-von Neumann theorem for algebraic Heisenberg groups in Chapter 5 relies heavily on this equivalence.

1.1 Representations

In this chapter $G$ will be an abstract group.

**Definition 1.** A *representation of $G$, or $G$-module*, is a vector space $V$ together with a group homomorphism $\rho : G \rightarrow \text{Aut}(V)$. When the homomorphism is clear from context, we often abuse notation and simply write $g$ for the image of $g \in G$ under $\rho$. The dimension of $V$ is called the *degree* of the representation.

**Definition 2.** Let $V$ and $W$ be representations of $G$. A *(G-linear) map* between representations is a vector space map $\varphi$ such that the square

$$
\begin{array}{ccc}
V & \xrightarrow{\varphi} & W \\
g \downarrow & & \downarrow g \\
V & \xrightarrow{\varphi} & W
\end{array}
$$

commutes for all $g \in G$. Such maps are also called *$G$-module homomorphisms*, or *intertwining operators*. If there exists a $G$-module homomorphism between two representations $V$ and $W$ that is also a vector-space isomorphism, we say that $V$ and $W$ are *isomorphic representations*.

**Definition 3.** Let $V$ be a representation of $G$. A *subrepresentation* or *submodule* of $V$ is a vector subspace $W$ of $V$ that is invariant under the action of $G$; that is, $GW \subseteq W$. A representation that has no proper non-zero subrepresentations is said to be *irreducible*. If a representation $V$ can be written as the direct sum of irreducible representations, it is said to be *completely reducible*.

**Lemma 1** *(Schur’s Lemma)*. Let $\varphi$ be a $G$-module homomorphism between two irreducible representations $V$ and $W$ of $G$, where $V$ and $W$ are finite-dimensional vector spaces over an algebraically closed field $k$. Then either $\varphi = 0$ or $\varphi$ is an isomorphism. If $V \cong W$ then $\varphi = \lambda I$ for some $\lambda \in k$, where $I$ is the identity map.

**Proof.** Suppose that $\varphi \neq 0$. Then $\text{Im} \varphi$ and $\text{Ker} \varphi$ are subrepresentations of $W$ and $V$, respectively. Since $V$ is irreducible and $\text{Ker} \varphi \neq V$, we must have $\text{Ker} \varphi = 0$. Similarly, since $W$ is irreducible and $\text{Im} \varphi \neq 0$, we must have $\text{Im} \varphi = W$.

Now suppose that $V = W$. Then $\varphi$ has an eigenvalue $\lambda$, since $k$ is algebraically closed. Then

$$\text{Ker} (\varphi - \lambda I) \neq \{0\}$$

and so is not an isomorphism. Thus $\varphi - \lambda I = 0$. 

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If $V$ is a representation of $G$, then $V \oplus V$ and $V \otimes V$ are also representations of $G$ via the actions

$$g(v \oplus w) = gv \oplus gw, \quad \text{and} \quad g(v \otimes w) = gv \otimes gw.$$ 

If $V$ is finite-dimensional and $\rho : G \rightarrow \text{Aut}(V)$ is a representation of $G$, we can define a representation $\hat{\rho}$ of $G$ on the dual space $V^*$ via

$$\hat{\rho} : V^* \rightarrow V^* : \varphi \mapsto \varphi \circ \rho(g^{-1}).$$

This definition preserves the pairing $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow k$, since

$$\langle \hat{\rho}(g)(\varphi), \rho(g)(v) \rangle = \varphi \rho(g^{-1}) \rho(g) v = \langle \varphi, v \rangle.$$ 

Given two representations $(\rho, V)$ and $(\sigma, W)$ of $G$, we can use the above to make $\text{Hom}(V, W) \cong V^* \otimes W$ into a representation of $G$ by defining

$$g\varphi = \sigma(g) \circ \varphi \circ \rho(g^{-1}),$$

for $\varphi \in \text{Hom}(V, W)$. That is, $g\varphi$ is defined such that the square

$$\begin{array}{ccc}
V & \xrightarrow{\varphi} & W \\
\rho(g) \downarrow & & \downarrow \sigma(g) \\
V & \xrightarrow{g\varphi} & W
\end{array}$$

commutes.

Schur’s lemma and the above constructions will show up in the proof of the Stone-von Neumann theorem in section 5.1.

We need some additional results about representations of finite groups for constructing the irreducible representations of finite Heisenberg groups in the next section. We state them here without proof.

**Definition 4.** Let $V$ be a finite-dimensional representation of $G$, where $V$ is a complex vector space. Then the *character* of $V$ is the function $\chi_V : G \rightarrow \mathbb{C}$ given by $g \mapsto \text{Tr}(g|_V)$. Since $\text{Tr}(gh^{-1}) = \text{Tr}(g)$, the character of the representation is a function defined on conjugacy classes of $G$. Such functions are called *class functions*.

**Proposition 2.** Let $(\rho, V)$ and $(\sigma, W)$ be two representations of a finite group $G$. We define a scalar product on the representations by

$$\langle \rho, \sigma \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g),$$

where $|G|$ denotes the order of $G$. Then

$$\langle \rho, \sigma \rangle = \begin{cases} 
1 & \text{if } V \cong W, \\
0 & \text{if } V \not\cong W.
\end{cases}$$

[Fulton and Harris, 1991, Equation (2.10)]
Proposition 3. If \( \varphi \) is the character of a representation \( V \), then \( \langle \varphi, \varphi \rangle = 1 \) if and only if \( V \) is irreducible. [Serre, 1977, Theorem 5]

Proposition 4. If \( \{ V_i \} \) is the set of distinct irreducible finite representations of a finite group \( G \), then
\[
|G| = \sum_i (\dim V_i)^2.
\]

[Fulton and Harris, 1991, Corollary 2.18]
Chapter 2

Finite Heisenberg Groups

We begin our discussion by constructing the irreducible representations of finite Heisenberg groups. We see some techniques that will reappear in the proof of the Stone-von Neumann theorem, both for continuous and algebraic Heisenberg groups. In the later proofs, we construct something akin to the symplectic form that exists in the finite case. For the continuous Heisenberg groups, we use this to single out a maximal isotropic subgroup, which plays a large role in constructing the Heisenberg representation.

Definition 5. Let $\mathbb{F} = \mathbb{F}_q$ be the field with $q = p^m$ elements, where $p$ is an odd prime, and let $V$ be a finite-dimensional vector space over $\mathbb{F}$. Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ be a bilinear map that is

(a) *Non-degenerate:* if $v \in V$, then there exists $w \in V$ such that $\langle v, w \rangle \neq 0$, and

(b) *Skew-symmetric:* if $v, w \in V$, then $\langle v, w \rangle = -\langle w, v \rangle$.

Such a map is called a *symplectic form* on $V$. A vector space over $\mathbb{F}_q$ with a symplectic form on it is necessarily of even dimension, since the symplectic form defines an invertible skew-symmetric matrix $A$, and

$$
\det A = \det A^T = \det(-A) = (-1)^n \det A,
$$

where $n = \dim V$.

The *Heisenberg group* associated to this data is the group with underlying set $\mathbb{F} \times V$, and group operation

$$(t, v)(t', w) = (t + t' + \langle v, w \rangle, v + w). \quad (2.1)$$

The identity of $G$ is $(0, 0)$ and the inverse of an element $(t, x) \in G$ is $(-t, -x)$. The obvious inclusion $\mathbb{F} \hookrightarrow \mathbb{F} \times V = G$ is a group homomorphism identifying $\mathbb{F}^+$ with the centre of $G$. We can picture the situation as a central extension of $V$ by $\mathbb{F}^+$:

$$
0 \to \mathbb{F}^+ \to G \to V \to 0. \quad (2.2)
$$

Compare this to the central extensions (3.1) and (5.1).
Example 1. The polarised Heisenberg group is the set $H_n(\mathbb{F}) = \mathbb{F} \times \mathbb{F}^n \times \mathbb{F}^n$ with product
\[(t, x, y)(t', x', y') = \left(t + t' + \frac{1}{2}(x \cdot y' - y \cdot x'), x + x', y + y'\right),\]
where $\cdot$ is the usual inner product.

Example 2. The classical Heisenberg group is the set of matrices of the form
\[
\begin{pmatrix}
1 & x & t \\
0 & I_n & y \\
0 & 0 & 1
\end{pmatrix}
\]
with entries in $\mathbb{F}$, and the usual matrix product:
\[
\begin{pmatrix}
1 & x & t \\
0 & I_n & y \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & x' & t' \\
0 & I_n & y' \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & x + x' & t + t' + x \cdot y' \\
0 & I_n & y + y' \\
0 & 0 & 1
\end{pmatrix}.
\]

The polarised Heisenberg group $H_n(\mathbb{F})$ is related to the classical Heisenberg group via the group isomorphism
\[
\varphi : \begin{pmatrix}
1 & x & t \\
0 & I_n & y \\
0 & 0 & 1
\end{pmatrix} \mapsto \left(t - \frac{1}{2}x \cdot y, x, y^T\right).
\]

This map is an isomorphism of sets, so we just need to check that it is a group homomorphism:
\[
\varphi \left(\begin{pmatrix}
1 & x & t \\
0 & I_n & y \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & x' & t' \\
0 & I_n & y' \\
0 & 0 & 1
\end{pmatrix}\right)
= \varphi \begin{pmatrix}
1 & x + x' & t + t' + x \cdot y' \\
0 & I_n & y + y' \\
0 & 0 & 1
\end{pmatrix}
= \left(t + t' + x \cdot y' - \frac{1}{2}(x + x') \cdot (y + y'), x + x', y^T + y'^T\right)
= \left(t + t' - \frac{1}{2}(x \cdot y + y' \cdot x') + \frac{1}{2}(x \cdot y' - x' \cdot y), x + x', y^T + y'^T\right)
= \left(t - \frac{1}{2}x \cdot y, x, y^T\right)\left(t' - \frac{1}{2}x' \cdot y', x', y'^T\right)
= \varphi \begin{pmatrix}
1 & x & t \\
0 & I_n & y \\
0 & 0 & 1
\end{pmatrix}\varphi \begin{pmatrix}
1 & x' & t' \\
0 & I_n & y' \\
0 & 0 & 1
\end{pmatrix}.
\]

2.1 Representations of Finite Heisenberg Groups

In this section we prove the following:
Theorem 5. Let $F = \mathbb{F}_q$, where $q = p^m$ for an odd prime $p$, and let $V$ be a vector space over $F$ of dimension $2n$ together with a symplectic form $\langle \cdot, \cdot \rangle : V \times V \to F$. Let $G$ be the finite Heisenberg group associated to this data, and suppose that $W$ is an irreducible representation of $G$. Then $W$ is either one of the $q^{2n}$ one-dimensional representations (characters) of $G$, or one of the $(q - 1)$ representations of degree $q^n$.

We begin by classifying the characters of $G$.

Lemma 6. Let $F = \mathbb{F}_q$ be as above. Then there is a group isomorphism between $F^+$ and the group of characters of $\mathbb{F}_q$, given by $a \mapsto \psi_a$, where

$$
\psi_a(t) = \exp\left(\frac{2\pi i}{p} \text{Tr}(at)\right),
$$

and

$$
\text{Tr} : F \to \mathbb{F}_p : t \mapsto t + t^p + t^{p^2} + \cdots + t^{p^{m-1}}.
$$

Proof. Note that if $t \in F$ then $\text{Tr}(t) \in \mathbb{F}_p$, since we are working in characteristic $p$ and

$$
\text{Tr}(t)^p = \left(t + t^p + t^{p^2} + \cdots + t^{p^{m-1}}\right)^p = t^p + t^{p^2} + \cdots + t^{p^m} = \text{Tr}(t).
$$

The map $\text{Tr}$ is also additive, since

$$
\text{Tr}(x + y) = (x + y)^p + (x + y)^{p^2} + \cdots + (x + y)^{p^{m-1}}
= (x^p + y^p) + (x^{p^2} + y^{p^2}) + \cdots + (x^{p^{m-1}} + y^{p^{m-1}}) = \text{Tr}(x) + \text{Tr}(y).
$$

Thus

$$
\psi_{a+b}(t) = \exp\left(\frac{2\pi i}{p} \text{Tr}((a+b)t)\right)
= \exp\left(\frac{2\pi i}{p} (\text{Tr}(at) + \text{Tr}(bt))\right)
= \exp\left(\frac{2\pi i}{p} \text{Tr}(at)\right) \exp\left(\frac{2\pi i}{p} \text{Tr}(bt)\right) = \psi_a(t)\psi_b(t),
$$

showing that $\psi$ is a group homomorphism.

We can see that this map is one-to-one by using the identity

$$
\sum_{t \in F} \psi_a(t) = q \delta_{a,0}
$$
and Proposition 2:

\[
\frac{1}{|F|} \sum_{t \in F} \psi_a(t) \psi_b(t) = \frac{1}{|F|} \sum_{t \in F} \psi_1(-at) \psi_1(bt) = \frac{1}{|F|} \sum_{t \in F} \psi_1((b-a)t) = \frac{1}{|F|} \sum_{t \in F} \psi_{b-a}(t) = \begin{cases} 
1 & \text{if } a = b, \\
0 & \text{if } a \neq b.
\end{cases}
\]

\[\square\]

**Lemma 7.** Let \( \psi_1 \) be as in the previous lemma, and let \( v \in V \). Define

\[ \Psi_v : G \rightarrow \mathbb{C}^* : (t, w) \mapsto \psi_1(\langle v, w \rangle). \]

Then \( \Psi_v \) is a character of \( G \).

**Proof.** We need only check that \( \Psi_v \) is multiplicative:

\[
\Psi_v((t, w)(t', w')) = \Psi_v((t + t' + \langle w, w' \rangle, w + w')) = \psi_1(\langle v, w + w' \rangle) = \psi_1(\langle v, w \rangle + \langle v, w' \rangle) = \psi_1(\langle v, w \rangle) \psi_1(\langle v, w' \rangle) = \Psi_v((t, w)) \Psi_v((t', w')).
\]

\[\square\]

**Lemma 8.** Let \( v, w \in V \), and let \( \Psi_v, \Psi_w \) be as above. If \( \Psi_v = \Psi_w \) then \( v = w \).

**Proof.** If \( \Psi_v = \Psi_w \) then for all \( x \in V \),

\[ \psi_1(\langle v, x \rangle) = \psi_1(\langle w, x \rangle). \]

But then

\[ \psi_1(\langle v - w, x \rangle) = 1, \]

or \( \langle v - w, x \rangle = 0 \) for all \( x \in V \). Since \( \langle \cdot, \cdot \rangle \) is non-degenerate, \( v = w \).

\[\square\]

We now have \( |V| = q^{2n} \) non-isomorphic characters of \( G \), and we claim that these are all that exist. We will argue this later using Proposition 4; see Equation (2.3). For now we look at the higher-dimensional irreducible representations of \( G \).

**Proposition 9.** Let \( V \) be a finite-dimensional vector space with a symplectic form \( \langle \cdot, \cdot \rangle \). Then there exists a basis

\[ \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \]

for \( V \) such that

\[ \langle \alpha_i, \beta_j \rangle = \delta_{ij}, \quad \text{and} \quad \langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0. \]

Such a basis is called a **Darboux basis.**
Let \(\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}\) be a Darboux basis for \(V\) as above, and let
\[
H = \text{Span} \{\alpha_1, \ldots, \alpha_n\} \quad \text{and} \quad K = F \times H \subset G.
\]
(The subgroup \(K\) plays the role of the maximal isotropic subgroup in the next section.)

**Lemma 10.** Let \(a \in F^\times\) and let \(\psi_a(k) = \psi_a(t)\) for \(k = (t, x) \in K\). Then
\[
\mathcal{H}_a := \{f : G \to \mathbb{C} \mid f(kg) = \psi_a(k)f(g), \ \text{all } k \in K, g \in G\}
\]
is a complex vector space of dimension \(q^n\).

**Proof.** We can give a basis for \(\mathcal{H}_a\) as follows. \(K\) is a normal abelian subgroup of \(G\), and since \(|K| = |F| \times |H| = q^{n+1}\), we have a set \(S = \{s_0, \ldots, s_{q^n-1}\}\) of \(q^n\) coset representatives of \(K\) in \(G\). In fact, \(S\) is isomorphic as a group to \(F^n\) under addition, and if \(g \in G\), there exists a unique pair \((k, s_j) \in K \times S\) such that \(g = ks_j\).

We define elements \(f_i\) of \(\mathcal{H}_a\) by
\[
f_i : G \to \mathbb{C} : g = ks_j \mapsto \psi_a(k)\delta_{ij}, \quad 0 \leq i \leq q^n - 1.
\]
Then if \(f \in \mathcal{H}_a\),
\[
f(ks_j) = \psi_a(k)f(s_j) = \sum_i f(s_i)\psi_a(k)\delta_{ij} = \sum_i f(s_i)f_i(k),
\]
so \(\{f_i\}\) is a spanning set for \(\mathcal{H}_a\).

Now suppose that for \(\{\lambda_i\} \subset \mathbb{C}\) we have
\[
\sum_i \lambda_i f_i = 0.
\]
Then
\[
0 = \sum_i \lambda_i f_i(s_j) = \lambda_j \psi_a(k),
\]
so \(\lambda_j = 0\), showing that the \(f_i\) are linearly independent and thus form a basis for \(\mathcal{H}_a\).

**Lemma 11.** Let \(a \in F^\times\) and let \(\mathcal{H}_a\) be as in the previous lemma. Then the map \(\rho : G \to \text{Aut}(\mathcal{H}_a)\) defined by
\[
(\rho(g)f)(h) = f(hg)
\]
is an irreducible representation of degree \(q^n\).

**Proof.** If \(g_1, g_2, h \in G\) and \(f \in \mathcal{H}_a\) then
\[
(\rho(g_1g_2)f)(h) = f(hg_1g_2) = (\rho(g_2)f)(hg_1) = \rho(g_1)(\rho(g_2)f)(h),
\]
so \(\rho\) defines a representation.
Now suppose there exists a non-trivial subrepresentation $W$ of $H_a$, and let $f \in W$, $f \neq 0$. Then if $x' \in H$, the action of $(0, x')$ on $f$ is

\[
(0, x')f ((t, x)(0, y)) = f ((t, x)(0, y)(0, x')) \\
= f ((t + \langle x, y \rangle, x + y)(0, x')) \\
= f ((t + \langle x, y \rangle + \langle y, x' \rangle, x + x' + y)) \\
= f ((t + 2 \langle y, x' \rangle, x + x')(0, y)) \\
= \psi_a(t + 2 \langle y, x' \rangle)f(y) \\
= \psi_a(t)\psi_a(2 \langle y, x' \rangle)f(y).
\]

Since $\langle y, \cdot \rangle$ is surjective if $y \neq 0$, we have

\[
\left( \sum_{x' \in H} (0, x')f \right) ((t, x)(0, y)) = \psi_a(t)f(y) \sum_{x' \in H} \psi_a(2 \langle y, x' \rangle) \\
= \begin{cases} 
\psi_a(t)|H| f(0) & \text{if } y = 0, \\
0 & \text{if } y \neq 0.
\end{cases}
\]

But this is just a scalar multiple of $f_0$, and since $W$ is a subrepresentation, $f_0 \in W$.

For $y \in S$, we have $(0, y)f_0 = f_y$, so in fact $W$ must be $H_a$. \qed

**Lemma 12.** If $a, b \in \mathbb{F}^\times$ and $H_a \cong H_b$ then $a = b$.

**Proof.** By Schur’s lemma, an intertwining operator between $H_a$ and $H_b$ must be a scalar $\lambda \in \mathbb{C}$. But then we would require

\[
\psi_a(t) = \psi_b(t),
\]

or $\psi_{a-b}(t) = 1$ for all $t \in \mathbb{F}$. Thus $a = b$. \qed

**Proof (Theorem 5).** We have $q^{2n}$ characters of $G$ and $(q - 1)$ irreducible representations of degree $q^n$. Summing the squares of the dimensions of these representations, we get

\[
q^{2n} + (q - 1)q^{2n} = q^{2n+1} = |G|.
\]

Thus by Proposition 4 we have constructed all irreducible representations of $G$. \qed

For further discussion of the representations of finite Heisenberg groups, see [Benson and Ratcliff, 2008] and [Misaghian, 2009].

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Chapter 3

Continuous Heisenberg Groups

In this chapter we move from a finite field $\mathbb{F}$ to the complex numbers. Our aim is to construct the unique irreducible representation of a given Heisenberg group $G$ such that the centre acts naturally. That is, if $\lambda \in Z(G)$ then $\rho(\lambda) = \lambda \cdot \text{id}$. This representation is called the Heisenberg representation. We follow the definitions and arguments of Section 1 of [Mumford et al., 1991].

Throughout this section, $G$ and $K$ are locally compact groups such that $K$ is abelian and $G$ is a central extension of $K$ by $C^*_1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. That is to say, we have a short exact sequence

$$1 \rightarrow C^*_1 \rightarrow G \rightarrow K \rightarrow 0,$$

where $C^*_1$ is a normal subgroup of $G$ lying in the centre of $G$. We also assume that $G$ admits a continuous section over $K$, so that $G = C^*_1 \times K$, as a set. When $K$ is finite these are commonly known as finite Heisenberg groups, but we use the name continuous here to distinguish them from Heisenberg groups over $\mathbb{F}$.

The group law on $G = C^*_1 \times K$ is given by

$$\left( \lambda, x \right) \left( \mu, y \right) = \left( \lambda \mu \psi(x, y), x + y \right),$$

(cf. Equation (2.1)) where

$$\psi : K \times K \rightarrow C^*_1$$

is a 2-cocycle; that is

$$\psi(x, y)\psi(x + y, z) = \psi(x, y + z)\psi(y, z).$$

The 2-cocycle condition is equivalent to the associativity of the multiplication:

$$((\lambda, x) (\mu, y)) (\eta, z) = (\lambda \mu \psi(x, y), x + y) (\eta, z) = (\lambda \mu \eta \psi(x, y), x + y + z),$$

$$(\lambda, x) ((\mu, y), (\eta, z)) = (\lambda, x) (\mu \eta \psi(y, z), y + z) = (\lambda \mu \eta \psi(x, y + z), x + y + z).$$
Given such a $\psi$, the identity $(\mu, y)$ of the group must satisfy

$$(\lambda, 0)(\mu, y) = (\lambda \mu \psi(0, y), y) = (\lambda, 0),$$

so it is clear that $y = 0$ and $\mu \psi(0, 0) = 1$. Thus we see that the identity is $(\psi(0, 0)^{-1}, 0)$. (We will soon see that this is in fact just $(1, 0)$.)

So if $(\lambda, x)^{-1} = (\mu, y)$, we have

$$(\lambda \mu \psi(x, y), x + y) = \left(\psi(0, 0)^{-1}, 0\right).$$

Thus $y = -x$ and $\mu^{-1} = \lambda \psi(x, -x) \psi(0, 0)$.

Now fix two points $x$ and $y$ in $K$, and choose liftings of them $\tilde{x}$ and $\tilde{y}$ in $G$. Then $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ lies in $C_1^*$, since

$$j(\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}) = x + y - x - y = 0.$$  

This product is independent of the choice of liftings $\tilde{x} = (\lambda, x)$ and $\tilde{y} = (\mu, y)$, since in the product the $\lambda$’s and $\mu$’s all cancel. So we have a map

$$e : K \times K \longrightarrow C_1^* : (x, y) \mapsto \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}.$$  

**Proposition 13.** We have the following equalities:

$$e(x + y, z) = e(x, z)e(y, z),$$  

$$e(x, y + z) = e(x, y)e(x, z),$$  

$$e(x, x) = 1,$$  

$$e(x, y) = e(y, x)^{-1}, \quad \text{and}$$  

$$e(x, y) = \frac{\psi(x, y)}{\psi(y, x)}.$$  

Note that the first four equations mirror those of the symplectic form we had for finite Heisenberg groups in Definition 5. We call a map satisfying these conditions an \textit{alternating bicharacter} or a \textit{skew-symmetric bihomomorphism}.  

**Proof.** Equations (3.5) and (3.6) are obvious from the definition of $e$. We begin with (3.7).

Let $\tilde{x} = (1, x)$ and $\tilde{y} = (1, y)$, so $\tilde{x}\tilde{y} = (\psi(x, y), x + y)$ and

$$\tilde{x}^{-1}\tilde{y}^{-1} = (\tilde{y}\tilde{x})^{-1} = (\psi(y, x), x + y)^{-1} = \left(\psi(y, x)^{-1}\psi(x + y, -x - y)^{-1}\psi(0, 0)^{-1}, -x - y\right).$$

Then

$$\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = \left(\psi(x, y)\psi(y, x)^{-1}\psi(0, 0)^{-1}, 0\right).$$
However, we have
\[ e(x, x) = 1 = \frac{\psi(x, x)}{\psi(x, x)} \psi(0, 0)^{-1}, \]
so the identity of the group is actually \((1, 0)\) and
\[ e(x, y) = \frac{\psi(x, y)}{\psi(y, x)}. \]

Equation (3.3) follows from (3.7) and the identities
\begin{align*}
\psi(x, y)\psi(x + y, z) &= \psi(x, y + z)\psi(y, z), & (3.8) \\
\psi(z, x)\psi(x + z, y) &= \psi(z, x + y)\psi(x, y), & \text{and} & (3.9) \\
\psi(x, z)\psi(x + z, y) &= \psi(x, y + z)\psi(z, y), & (3.10)
\end{align*}
as follows:
\begin{align*}
e(x + y, z) &= \frac{\psi(x + y, z)}{\psi(z, x + y)} \\
&= \frac{\psi(x, y + z)\psi(y, z)}{\psi(z, x)\psi(x + z, y)} \\
&= \frac{\psi(x, z)\psi(y, z)}{\psi(z, x)\psi(z, y)} = e(x, z)e(y, z)
\end{align*}
The proof of (3.4) is similar to the proof of (3.3). 

Let \(\widehat{K}\) be the character group of \(K\), also known as the Pontryagin dual, and define \(\varphi : K \rightarrow \widehat{K}\) by
\[ \varphi(x)(y) = e(x, y). \] (3.11)
Thus we have
\[ ghg^{-1} = \varphi(\bar{g})(\bar{h})h, \] (3.12)
for all \(g, h \in G\), where \(\bar{g} = \pi(g)\) and \(\bar{h} = \pi(h)\).

**Definition 6.** \(G\) is a Heisenberg group if \(\varphi\) is an isomorphism.

We now use the map \(e\) to identify a certain class of subgroups of \(K\) called the isotropic subgroups. Partially ordered by inclusion, the maximal elements of this class, the maximal isotropic subgroups, play an important role in the proof of Stone-von Neumann. We also note that if \(G\) is a Heisenberg group then \(C^*_1\) is the centre of \(G\).

**Lemma 14.** Let \(G\) be a Heisenberg group, and let \(H\) be a closed subgroup of \(K\). Then the following are equivalent:

1. \(e|_{H \times H} \equiv 1\) and \(H\) is maximal with this property.
2. \(\varphi\) restricts to an isomorphism between \(H\) and \(\widehat{K}/\bar{H} \subset \widehat{K}\).
(3) $H = H^\perp$, where $H^\perp = \{ x \in K \mid e(x, y) = 1, \text{all } y \in H \}$.

**Proof.** Let $x \in \varphi^{-1}(\hat{K}/H) \subset K$. Then if $y \in H$, $e(x, y) = \varphi(x)(y) = 1$, so $\varphi^{-1}(\hat{K}/H) \subset H^\perp$. Also if $x \in H^\perp$ then $\varphi(x) \in \hat{K}/H$, so we have

$$H^\perp = \varphi^{-1}(\hat{K}/H).$$

Thus (2) $\iff$ (3).

Now suppose that $e|_{H \times H} \equiv 1$ and $H$ is maximal with this property. Take any $x \in H^\perp$, and let $H'$ be the closure of $H + \mathbb{Z}x \subset K$. Then $e|_{H' \times H'} \equiv 1$, and since $H$ is a maximal subgroup with this property, we must have $x \in H$, or $H^\perp \subset H$.

**Definition 7.** A subgroup $H$ satisfying $e|_{H \times H} \equiv 1$ is called *isotropic*. A subgroup $H$ satisfying any of the conditions in Lemma 14 is called *maximal isotropic*.

**Proposition 15.** Let $H$ be a closed subgroup of $K$. Then the following are equivalent:

1. $H$ is isotropic.
2. $\pi : G \longrightarrow K$ splits over $H$.

**Proof.** Suppose that $\sigma$ is a splitting of $\pi : G \longrightarrow H$, and for $x, y \in H$, let $\tilde{x} = \sigma(x)$, $\tilde{y} = \sigma(y)$. Then

$$e(x, y) = \tilde{x} \tilde{y} \tilde{x}^{-1} \tilde{y}^{-1} = \sigma(x) \sigma(y) \sigma(-x) \sigma(-y) = \sigma(0) = 1.$$

Now let $H$ be isotropic. Then $\pi^{-1}(H)$ is commutative, and by taking duals of (3.1), we have the short exact sequence

$$0 \longleftarrow \mathbb{Z} \longleftarrow \hat{\pi^{-1}(H)} \longleftarrow \hat{H} \longleftarrow 0.$$

Let $\zeta \in \hat{\pi^{-1}(H)}$ be a lifting of $1 \in \mathbb{Z}$. Then $\zeta : H \longrightarrow \mathbb{C}$ is a character of $H$ such that if $\lambda \in \mathbb{C}^*_1$ then $\zeta(\lambda) = \lambda$. We also have

$$\pi|_{\text{Ker } \zeta} : \text{Ker } \zeta \longrightarrow H$$

is an isomorphism, and we take the section of $\pi$ to be $\pi|_{\text{Ker } \zeta}^{-1}$. □

### 3.1 A Special Case of the Stone-von Neumann Theorem

We are now ready to prove the Stone-von Neumann Theorem for the case where $K$ is finite.
Theorem 16 (Stone-von Neumann). Let $G$ be a Heisenberg group such that $K$ is finite. Then there exists a unique irreducible unitary representation $U : G \rightarrow \text{Aut}(\mathcal{H}_0)$ of $G$ such that if $\lambda \in \mathbb{C}^*$, then

$$U_\lambda = \lambda \cdot \text{id}.$$ 

If $H$ is a maximal isotropic subgroup of $K$, and $\sigma(x) = (\alpha(x), x)$ is a splitting of $\pi$ over $H$, then the representation $(U, \mathcal{H}_0)$ may be realised by

$$\mathcal{H}_0 = \left\{ \text{measurable functions } f : K \rightarrow \mathbb{C} \text{ such that} \right. $$

- $a) \ f(x + h) = \alpha(h)^{-1} \psi(h, x)^{-1} f(x), \text{ all } h \in H, \text{ and}$
- $b) \ \int_{K/H} |f(x)|^2 \, dx < \infty$

where $G$ acts on $\mathcal{H}_0$ by

$$\left( U_{(\lambda, y)} f \right)(x) = \lambda \psi(x, y) f(x + y). \quad (3.13)$$

This representation is called the Heisenberg representation of $G$ and is denoted $L^2(K//H)$.

The proof follows [Mumford et al., 1991], and proceeds in three parts. We first decompose an arbitrary representation $\mathcal{H}$ of $G$ into a direct sum of subspaces. We next show that all but one of the summands are empty, and we obtain a $G$-module isomorphism from $\mathcal{H}$ to $L^2(\hat{H}, \mathcal{K}_n)$, defined in Equation (3.14), where $n$ depends upon $\mathcal{H}$.

The final step introduces another isomorphism of representations, $W$, taking $L^2(\hat{H}, \mathcal{K}_n)$ to $L^2(K//H) \otimes \mathcal{K}_n$, where $\mathcal{K}_n$ is the standard $n$-dimensional Hilbert space. From this we can conclude that if $\mathcal{H}$ is irreducible then $n = 1$ and any irreducible representation $\mathcal{H}$ of $G$ is isomorphic to $L^2(K//H)$.

Since we are considering only the case where $K$ is finite, we can ignore the integrability condition, but the same result holds for infinite groups $K$. An outline of a proof for the general case can be found in [Mumford et al., 1991].

Proof.

Step I. Suppose that $H$ is a maximal isotropic subgroup of $K$, and that $\sigma(x) = (\alpha(x), x)$ is a splitting of $\pi$ over $H$.

Let $(U, \mathcal{H})$ be a unitary representation of $G$. We can decompose $\mathcal{H}$ under the action of the abelian subgroup $\sigma(H)$, writing

$$\mathcal{H} = \bigoplus_{\zeta \in \hat{H}} \mathcal{H}_\zeta,$$

where

$$\mathcal{H}_\zeta = \{ a \in \mathcal{H} \mid U_{\sigma(x)} a = \zeta(x) a, \text{ for all } x \in H \}.$$
We can group the $\zeta$’s by the dimension of $H_\zeta$, allowing us to rewrite this direct sum. To do so we define
\[ H_n = \{ \zeta \in \hat{H} \mid \dim H_\zeta = n \}, \]
and
\[ L^2(\hat{H}_n, \mathcal{K}_n) = \left\{ f : \hat{H}_n \longrightarrow \mathcal{K}_n \right\}, \tag{3.14} \]
where $\mathcal{K}_n$ is the standard $n$-dimensional Hilbert space. Then for $\zeta \in \hat{H}_n$, $H_\zeta$ is isomorphic to $\mathcal{K}_n$. A function $f \in L^2(\hat{H}_n, \mathcal{K}_n)$ is determined by its value on each $\zeta \in \hat{H}_n$, so using this fact and the previous statement, we have
\[ L^2(\hat{H}_n, \mathcal{K}_n) = \bigoplus_{\zeta \in \hat{H}_n} L^2(\{\zeta\}, \mathcal{K}_n) = \bigoplus_{\zeta \in \hat{H}_n} \mathcal{K}_n. \]
It follows that
\[ \mathcal{H} = \bigoplus_{n=0}^{\infty} \bigoplus_{\zeta \in \hat{H}_n} \mathcal{H}_\zeta \cong \bigoplus_{n=0}^{\infty} \bigoplus_{\zeta \in \hat{H}_n} \mathcal{K}_n = \bigoplus_{n=0}^{\infty} L^2(\hat{H}_n, \mathcal{K}_n). \tag{3.15} \]
We can make this an isomorphism of representations by letting
\[ (U_{\sigma(x)}f)(\zeta) = \zeta(x)f(\zeta), \tag{3.16} \]
or equivalently,
\[ (U_{(1,x)}f)(\zeta) = \alpha(x)^{-1}\zeta(x)f(\zeta). \]

Step II. Let $\pi(g) = y$ and $h = (\lambda, x)$. Then by (3.12), we have
\[ ghg^{-1} = \varphi(y)(x)h. \tag{3.17} \]
So if $U_h$ is the character $\zeta \in \hat{H}$, then $U_{ghg^{-1}}$ is the character $\varphi(y) + \zeta$, since $\pi^{-1}(H) = \mathbb{C}_1^* \times H$ is a normal subgroup and
\[
U_{ghg^{-1}}a = \varphi(y)(x) U_h a \\
= \varphi(y)(x) \zeta(x) a \\
= (\varphi(y) + \zeta)(x) a.
\]
It follows that
\[ \dim \mathcal{H}_\zeta = \dim \mathcal{H}_{\varphi(y) + \zeta}. \]
But $\varphi$ is surjective, so $\dim \mathcal{H}_\zeta$ does not depend on $\zeta$, and by (3.15) we have
\[ \mathcal{H} \cong L^2(\hat{H}, \mathcal{K}_n). \]
for some $n$.

Step III. We quote the following result from [Mumford et al., 1991]:
Lemma 17. Automorphisms $V$ of $L^2(\hat{H}, \mathcal{K}_n)$ that commute with the action of $H$ are given by

$$ (Vf)(\zeta) = A(\zeta)(f(\zeta)), \quad (3.18) $$

where $A(\zeta) : \mathcal{K}_n \to \mathcal{K}_n$ is a unitary map.

Note that $e$ measures the lack of commutativity of the operators $U_g$. We have

$$ U(\lambda, x)U(\mu, y) = \psi(x, y)U(\lambda \mu, x + y) $$

so by (3.16), we have

$$ (Vf)(\zeta) = (U(1, x)f)(\zeta - \phi(x)). \quad (3.20) $$

Then if $y \in H$,

$$ (VU_{\sigma(y)}f)(\zeta) = (U(1, x)U_{\sigma(y)}f)(\zeta - \phi(x)) $$

$$ = e(x, y)(U_{\sigma(y)}U(1, x)f)(\zeta - \phi(x)) $$

$$ = e(x, y)(\zeta - \phi(x))U(1, x)f)(\zeta - \phi(x)) $$

$$ = e(x, y)(\zeta - \phi(x))(Vf)(\zeta) $$

$$ = \zeta(y)(Vf)(\zeta) $$

$$ = (U_{\sigma(y)}Vf)(\zeta), $$

so the operator $V$ commutes with the action of $H$.

Thus by Equation (3.20) and Lemma 17,

$$ U(1, x)f(\zeta) = A_x(\zeta)(f(\zeta + \phi(x))), $$

for some unitary isomorphisms $A_x(\zeta)$ of $\mathcal{K}_n$.

Step IV. Define

$$ L^2(K//H, \mathcal{K}_n) = \left\{ f : K \to \mathcal{K}_n \text{ such that } \begin{array}{l} f(x + h) = \alpha(h)^{-1}\psi(h, x)^{-1}f(x), \quad \forall h \in H \end{array} \right\} $$

$$ = L^2(K//H) \otimes \mathcal{K}_n, $$

and let $W : L^2(\hat{H}, \mathcal{K}_n) \to L^2(K//H, \mathcal{K}_n)$ be the map given by

$$ (Wf)(x) = U(1, x)f(e) = A_x(e)f(\phi(x)). $$
We will show that this $W$ is a unitary isomorphism of representations of $G$, and so $H \cong L^2(K//H, \mathcal{K}_n)$.

We can see that $W$ is unitary, since $A$ is unitary and by the previous equation,

$$
\int_{K/H} ||Wf(x)||^2 \, dx = \int_{K/H} ||f(\phi(x))||^2 \, dx
$$

$$
= \int_{\tilde{H}} ||f(\zeta)||^2 \, d\zeta.
$$

The last equality holds since $\phi : K/H \to \tilde{H}$ is an isomorphism by Lemma 14.

To see that $Wf \in L^2(K//H, \mathcal{K}_n)$, we compute:

$$(Wf)(x + h) = (U_{(1,x+h)}f)(e)$$

$$= (U_{(\psi(h,x),-1)}U_{(1,x)}f)(e)$$

$$= \psi(h, x)^{-1} \alpha(h)^{-1} e(h)(U_{(1,x)}f)(e)$$

$$= \psi(h, x)^{-1} \alpha(h)^{-1} (Wf)(x),$$

We also need to check that $W$ satisfies Equation (3.13), generalised to $L^2(K//H) \otimes \mathcal{K}_n$. We have

$$(WU_{(1,y)}f)(x) = (U_{(1,x)}U_{(1,y)}f)(e)$$

$$= U_{(\psi(x,y),x+y)}f(e)$$

$$= \psi(x, y)U_{(1,x+y)}f(e) = \psi(x, y)(Wf)(x + y).$$

Finally, $W$ is surjective, as $f(\zeta) \in \mathcal{K}_n$ can take arbitrary values and thus so can $Wf$, since

$$(Wf)(x) = A_e(f(\phi(x))).$$

We can now conclude that $\mathcal{H}$ is unitarily isomorphic to $L^2(K//H) \otimes \mathcal{K}_n$ as a $G$-module. In particular, if $\mathcal{H}$ is irreducible we must have $n = 1$ and

$$\mathcal{H} \cong L^2(K//H).$$

\[ \square \]

### 3.2 Representations of Real Heisenberg Groups

In this section we consider Heisenberg groups

$$1 \to \mathbb{C}_1^* \to H(V) \to V \to 0$$

where $V$ is a real vector space. The construction of the Heisenberg representation in Theorem 16 relied upon a choice of maximal isotropic subgroup $H$; this choice can give rise to different models, or realisations, of the Heisenberg representation, though of course by the theorem they are all isomorphic to one another.
Example 3. Let $V = \mathbb{R}^{2g}$ with symplectic form $\langle \cdot, \cdot \rangle: V \times V \to \mathbb{R}$ given by

$$\langle x, y \rangle = x_1 \cdot y_2 - x_2 \cdot y_1,$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then as a set

$$H(V) \cong H(2g, \mathbb{R}) := \mathbb{C}^*_1 \times \mathbb{R}^{2g}.$$

We have a 2-cocycle $\psi(x, y) = \exp(\pi i \langle x, y \rangle)$ giving the group operation

$$(\lambda, x)(\mu, y) = (\lambda \mu \exp(\pi i \langle x, y \rangle), x + y).$$

We define two subspaces

$$W_1 = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right\} \subset \mathbb{R}^{2g},$$

and we take $H$, the maximal isotropic subgroup in the proof of Theorem 16, to be $W_2$.

Then by Stone-von Neumann, we have the Heisenberg representation

$$\mathcal{H}_0 = \left\{ \text{measurable functions } f: \mathbb{R}^{2g} \to \mathbb{C} \text{ such that} \right\}

\begin{array}{l}
\text{a) } f \left( x + \begin{pmatrix} 0 \\ y_2 \end{pmatrix} \right) = \exp(\pi i y_2 \cdot x_1) f(x), \text{ all } h \in H, \text{ and} \\
\text{b) } \int_{W_1} \left| f \left( \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \right) \right|^2 dx_1 < \infty
\end{array}

The first condition shows that $f$ is determined by its values on $W_1$, so we define $\varphi = f|_{W_1}$.

Then the Heisenberg representation is just

$$\mathcal{H}_1 = \left\{ \varphi: \mathbb{R}^g \to \mathbb{C} \mid \int_{W_1} |\varphi(x_1)|^2 dx_1 < \infty \right\},$$

where

$$U_{(1, y_1)} \varphi(x_1) = \psi(x_1, y_1) \varphi(x_1 + y_1),$$

$$= \varphi(x_1 + y_1) \quad \text{and}$$

$$U_{(1, y_2)} \varphi(x_1) = \psi(x_1, y_2) f \left( \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \right)$$

$$= \exp(2\pi i x_1 \cdot y_2) \varphi(x_1),$$

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or in general,
\[
U(\lambda, y_1 + y_2) \varphi(x_1) = \lambda \exp(-\pi i y_1 \cdot y_2) U(1, y_1) U(1, y_2) \varphi(x_1)
\]
\[
= \lambda \exp(-\pi i y_1 \cdot y_2) \exp(2\pi i x_1 \cdot y_2) U(1, y_1) \varphi(x_1)
\]
\[
= \lambda \exp(\pi i (2x_1 \cdot y_2 - y_1 \cdot y_2)) \varphi(x_1 + y_1).
\]

This realisation may be characterised as the group of unitary maps of \(L^2(\mathbb{R}^g)\) consisting of translations and multiplication by characters.

**Example 4.** There is another maximal isotropic subgroup of \(\mathbb{R}^{2g}\), the lattice \(H = \mathbb{Z}^{2g}\). We give a splitting of \(\pi\) for \(H\) by
\[
\sigma(n) = (\exp(\pi i n_1 \cdot n_2), n), \quad \text{where } n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}.
\]

Then the realisation of the Heisenberg representation corresponding to this data is
\[
\mathcal{H}_2 = \left\{ \text{measurable functions } f: \mathbb{R}^{2g} \rightarrow \mathbb{C} \right\}
\]
\[
\text{such that a) } f(x + n) = \exp(\pi i n_1 \cdot n_2) \exp(-\pi i A(n, x)) f(x), \text{ all } n \in \mathbb{Z}^{2g}, \text{ and b) } \int_{\mathbb{R}^{2g}/\mathbb{Z}^{2g}} |f(x)|^2 < \infty
\]
with group action
\[
((\lambda, y)f)(x) = \lambda \exp(\pi i A(x, y)) f(x + y).
\]

### 3.3 The Jacobi Theta Function

We briefly introduce the Jacobi theta function and show that it is invariant under a certain discrete subgroup of a Heisenberg group \(G\). For a much deeper analysis and discussion, see [Mumford, 1994, Ch. 1].

We choose a complex number \(\tau\) such that \(\text{Im } \tau > 0\). For all \(x\) and \(y\) in \(\mathbb{R}\) we define operators \(S_y\) and \(T_x\) on the space of holomorphic functions:
\[
(S_y f)(z) = f(z + y), \quad (3.21)
\]
\[
(T_x f)(z) = \exp(\pi i x^2 \tau + 2\pi i x z) f(z + x \tau). \quad (3.22)
\]

Then \(S\) and \(T\) form 1-parameter groups, since
\[
S_y (S_y' f) = S_{y+y'} f \quad \text{and} \quad T_x (T_x' f) = T_{x+x'} f.
\]

These operators fail to commute in general, since
\[
S_y (T_x f)(z) = (T_x f)(z + y)
\]
\[
= \exp \left( \pi i x^2 \tau + 2\pi i x (z + y) \right) f(z + y + x \tau), \quad \text{and}
\]
\[
T_x (S_y f)(z) = \exp \left( \pi i x^2 \tau + 2\pi i x z \right) (S_y f)(z + x \tau)
\]
\[
= \exp \left( \pi i x^2 \tau + 2\pi i x z \right) f(z + y + x \tau),
\]
so
\[ S_y T_x = \exp (2\pi i y) T_x S_y. \]
The group generated by the \( T_x \)'s and \( S_y \)'s is, as a set, \( G = \mathbb{C}_1^* \times \mathbb{R} \times \mathbb{R} \).

An element \((\lambda, a, b) \in G\) gives us the transformation
\[
((\lambda, x, y)f)(z) = \lambda \exp \left( \pi i x^2 \tau + 2\pi i x z \right) f(z + y + x\tau).
\]

Writing down the group operation, it is clear that \( G \) is a Heisenberg group:
\[
(\lambda, x, y) (\mu, x', y') = (\lambda \mu \exp (2\pi i y'), x + x', y + y').
\]

We define the Jacobi theta function \( \vartheta : \mathbb{C} \times \mathcal{H} \rightarrow \mathbb{C} \), where \( \mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \} \), by
\[
\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp \left( \pi i n^2 \tau + 2\pi i n z \right).
\] (3.23)

It is clear from the definition that
\[
\vartheta(z + 1, \tau) = \vartheta(z, \tau).
\] (3.24)

Translating \( z \) by \( \tau \), we see that
\[
\vartheta(z + \tau, \tau) = \sum_{n \in \mathbb{Z}} \exp \left( \pi i n^2 \tau + 2\pi i n (z + \tau) \right)
\]
\[
= \sum_{n \in \mathbb{Z}} \exp \left( \pi i (n + 1)^2 \tau - \pi i \tau + 2\pi i n z \right).
\]

If we let \( m = n + 1 \), we have
\[
\vartheta(z + \tau, \tau) = \sum_{n \in \mathbb{Z}} \exp \left( \pi i m^2 \tau - \pi i \tau + 2\pi i m z - 2\pi i z \right)
\]
\[
= \exp (-\pi i \tau - 2\pi i z) \vartheta(z, \tau).
\] (3.25)

Putting together Equations (3.24) and (3.25), we have
\[
\vartheta(z + a\tau + b, \tau) = \exp (-\pi i a^2 \tau - 2\pi i a z) \vartheta(z, \tau), \quad \text{all } (a, b) \in \mathbb{Z} \times \mathbb{Z},
\] (3.26)
by induction on \( a \).

If we define \( \vartheta_r(z) = \vartheta(z, \tau) \), then \( \vartheta_r \) is invariant under the action of the subgroup
\[
\{1\} \times \mathbb{Z} \times \mathbb{Z} \subset G,
\]
since
\[
((1, a, b)\vartheta_r)(z) = \exp \left( \pi i a^2 \tau + 2\pi i a z \right) \vartheta_r(z + \tau)
\]
\[
= \vartheta_r(z).
\]

In fact, up to scalar multiplication, \( \vartheta \) is the unique entire function invariant under this subgroup. For a proof of this statement, see [Mumford, 1994].
Chapter 4

An Introduction to Affine Group Schemes

In the next chapter we prove the Stone-von Neumann theorem in the context of affine group schemes. This chapter sets the scene for the proof, defining the objects we are working with and giving some of their properties.

Group objects are objects $G$ in a category $\mathcal{C}$ that allow us to put a group structure on the set of morphisms $\text{Hom}(X, G)$ for all objects $X \in \mathcal{C}$. We take as known the definitions of categories and functors, but no further knowledge of category theory is assumed. We are only concerned here with group objects in the category of affine schemes over a base scheme.

Affine schemes are algebro-geometric objects, and are the building blocks of schemes, in a way that is analogous to open subsets of $\mathbb{R}^n$ being the building blocks of manifolds. However, we do not need the generality of schemes in what follows, and we focus solely on affine schemes.

Affine group schemes are then group objects in the category of affine schemes over a base scheme. They are dual to Hopf algebras, and the representation theory of Hopf algebras generalises the representation theory of groups and modules. We also give an alternative definition of affine group schemes over $\text{Spec } k$ as representable functors from the category of $k$-algebras to the category of groups.

A gentle introduction to schemes can be found in [Nelson, 2009]. For finite group schemes, see [Pink, 2005], and for affine group schemes, see [Waterhouse, 1979], [Jantzen, 1987], and [Farnsteiner].

4.1 Group Objects

In order to define group objects, we need some additional structure in our category. Given that we need group-like structure, it is not surprising that we require a concept of products for the objects in our category.
Definition 8. A category \( \mathcal{C} \) is said to have \textit{arbitrary finite products} if the following conditions hold:

a) For any objects \( X \) and \( Y \) in \( \mathcal{C} \), there exists an object \( X \times Y \in \mathcal{C} \) and morphisms \( \pi_X : X \times Y \rightarrow X \) and \( \pi_Y : X \times Y \rightarrow Y \) such that the map

\[
(X \times Y)(Z) \rightarrow X(Z) \times Y(Z) : \phi \mapsto (\pi_X \circ \phi, \pi_Y \circ \phi)
\]

is bijective. Here we denote by \( X(Y) \) the set of morphisms from \( Y \) to \( X \) in \( \mathcal{C} \).

b) There exists an object \( * \in \mathcal{C} \) such that for all objects \( X \) in \( \mathcal{C} \), there is exactly one morphism from \( X \) to \( * \). Such an object is unique up to unique isomorphism and is called the \textit{terminal object}.

The terminal object in a category with arbitrary finite products may be thought of as the ‘empty product’.

Definition 9. Let \( G \) be an object in \( \mathcal{C} \), where \( \mathcal{C} \) is a category with arbitrary finite products, and let \( \Delta \) be the diagonal map taking \( g \in G \) to \( (g, g) \in G \times G \). Suppose we have morphisms

\[
\mu : G \times G \rightarrow G, \quad \iota : G \rightarrow G, \quad \text{and} \quad e : * \rightarrow G,
\]

such that the following diagrams commute:

\[
\begin{align*}
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mu \times \text{id}} & G \times G \\
\downarrow{\text{id} \times \mu} & & \downarrow{\mu} \\
G \times G & \xrightarrow{\mu} & G \\
\end{array}
\end{align*}
\tag{4.1}
\]

\[
\begin{align*}
\begin{array}{ccc}
G \times G & \xrightarrow{\mu \circ (\text{id} \times \iota)} & G \\
\downarrow{\Delta} & & \downarrow{\mu \circ (\text{id} \times \iota)} \\
G \times G & \xrightarrow{\mu \circ (\text{id} \times \iota)} & G \\
\end{array}
\end{align*}
\tag{4.2}
\]

\[
\begin{align*}
\begin{array}{ccc}
G \times G & \xrightarrow{\mu \circ (e \times \text{id})} & G \\
\downarrow{\Delta} & & \downarrow{\mu \circ (e \times \text{id})} \\
G \times G & \xrightarrow{\mu \circ (e \times \text{id})} & G \\
\end{array}
\end{align*}
\tag{4.3}
\]

Then \( G \) is called a \textit{group object} in \( \mathcal{C} \). (In the last two diagrams we abuse notation by writing \( e \) for the map \( G \rightarrow * \rightarrow G \).)
A group object is *commutative* or *abelian* if the triangle

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\sigma} & G \times G \\
\downarrow{\mu} & & \downarrow{\mu} \\
G & \xrightarrow{f} & G \\
\end{array}
\]

commutes, where \( \sigma \) swaps the order of the factors.

**Proposition 18.** Let \( G \) be a group object in a category \( C \), and let \( X \) be an object in \( C \). Then the set of morphisms \( G(X) \) forms a group, with multiplication, inverse, and identity given by composition with \( \mu \), \( \iota \), and \( e \), respectively.

The converse of this proposition is also true, so we may take the definition of a group object to be: an object \( G \) and a morphism \( \mu : G \times G \to G \) such that for all objects \( X \) in \( C \), \( G(X) \) forms a group with multiplication \( (\psi, \varphi) \mapsto \mu \circ (\psi, \varphi) \).

This way of viewing group objects will lead to a nice interpretation of affine group schemes over \( \text{Spec} \ k \) as representable functors from the category of \( k \)-algebras to the category of groups.

**Definition 10.** Let \( G \) and \( H \) be two group objects in a category \( C \). A *group object homomorphism* is a morphism \( f \) in \( H(G) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
G \times G & \xrightarrow{f \times f} & H \times H \\
\downarrow{\mu_G} & & \downarrow{\mu_H} \\
G & \xrightarrow{f} & H \\
\end{array}
\]

### 4.2 Affine Schemes

To define affine schemes, we begin by taking the set of prime ideals of a ring, and we call this set its spectrum. We make this set a topological space by endowing it with the Zariski topology, and to each open set of this topological space, we associate a ring in a certain manner, giving us a sheaf of rings on the space. This final entity is an affine scheme.

In this section all rings are assumed to be commutative with identity.

**Definition 11.** If \( R \) is a ring, we define the *spectrum of \( R \)* to be the set of all prime ideals of \( R \) and denote this by \( \text{Spec} \ R \). On this set we define the closed sets to be exactly those of the form

\[ V(I) = \{ P \in \text{Spec} \ R \mid I \subseteq P \}, \]

where \( I \) is any ideal of \( R \). This gives us a topology on \( \text{Spec} \ R \), called the *Zariski topology*. It is clear from this definition that the closed points correspond to the maximal ideals of \( R \).
We can give a basis for the Zariski topology by defining for each \( f \in R \) an open set \( \text{Spec } R_f \):

\[
\text{Spec } R_f = \{ P \in \text{Spec } R \mid f \notin P \}.
\]

To each open set \( \text{Spec } R_f \) we can associate a ring \( R_f \), the localisation of \( R \) at \( f \). In this way we have formed a sheaf of rings on \( \text{Spec } R \).

**Definition 12.** If \( X \) is a topological space, let \( \mathcal{C} \) be the category comprised of objects that are the open sets of \( X \) and for each inclusion of open sets \( U \subseteq V \) we have a corresponding morphism \( U \hookrightarrow V \). A sheaf of rings on \( X \) is a contravariant functor \( \mathcal{O}_X \) from \( \mathcal{C} \) to the category of rings that satisfies the following compatibility criterion: if \( \{ U_\alpha \} \) is an open cover of a set \( U \subseteq X \), and \( \{ f_\alpha \in \mathcal{O}_X(U_\alpha) \} \) is a collection such that

\[
\mathcal{O}_X(U_\alpha \cap U_\beta \hookrightarrow U_\alpha)(f_\alpha) = \mathcal{O}_X(U_\alpha \cap U_\beta \hookrightarrow U_\beta)(f_\beta)
\]

then there exists a unique \( f \in \mathcal{O}_X(U) \) such that

\[
f_\alpha = \mathcal{O}_X(U_\alpha \hookrightarrow U)(f), \quad \text{all } \alpha.
\]

We also denote the image of an inclusion \( U \hookrightarrow V \) by \( \text{res}_{V,U} : \mathcal{O}_X(V) \to \mathcal{O}_X(U) \), since we often think of this as function restriction. Keeping this in mind, the contravariance of the functor and the compatibility criterion seem natural.

**Definition 13.** An affine scheme is a topological space \( X \) with a sheaf of rings \( \mathcal{O}_X \) such that \( X \) is homeomorphic to \( \text{Spec } R \) for some ring \( R \), and \( \mathcal{O}_{\text{Spec } R} \cong \mathcal{O}_X \).

Morphisms between affine schemes \( \text{Spec } R \) and \( \text{Spec } S \) correspond to ring maps from \( S \) to \( R \).

**Definition 14.** Let \( \text{Spec } R \) be an affine scheme and consider the collection of all \( R \)-modules. For any \( R \)-module \( A \) we have a ring map from \( R \) to \( A \), so by taking the dual maps from \( \text{Spec } A \) to \( \text{Spec } R \) we get the category of affine schemes over \( \text{Spec } R \), and we then call \( \text{Spec } R \) the base scheme.

A morphism from \( \text{Spec } A \) to \( \text{Spec } B \) in this category is a map \( f \) of affine schemes such that the triangle

\[
\begin{array}{ccc}
\text{Spec } A & \xrightarrow{f} & \text{Spec } B \\
\downarrow & & \downarrow \\
\text{Spec } R & \rightarrow & \text{Spec } R
\end{array}
\]

commutes.

For a category of affine schemes over a base scheme \( \text{Spec } R \), it is clear from the construction that \( \text{Spec } R \) is the terminal object. In the category of \( R \)-modules, we have
a pushout of rings $A$ and $B$ given by the tensor product $A \otimes_R B$. Since these categories are dual, we now have a pullback, or fibre product, corresponding to the tensor product:

$$
\begin{array}{ccc}
X \times_S Y & \longrightarrow & X \\
\downarrow & & \downarrow f \\
Y & \longrightarrow & \text{Spec } R \\
B & \longleftarrow & B \\
\uparrow g^* & & \uparrow g \\
A & \longleftarrow & \text{Spec } R
\end{array}
$$

Thus we can define the fibre product to be

$$
\text{Spec } A \times_{\text{Spec } R} \text{Spec } B = \text{Spec } (A \otimes_R B).
$$

This product gives us enough structure to define group objects in a category of affine schemes over a base scheme.

**Definition 15.** A morphism of schemes $f : X \longrightarrow Y = \text{Spec } A$ is of **finite type** if $X$ has a finite open affine cover $\{U_i = \text{Spec } A_i\}$ such that each $A_i$ is a finitely generated $A$-algebra. If $f : X \longrightarrow Y$ is of finite type then we say that $X$ is of finite type over $Y$.

### 4.3 Affine Group Schemes

**Definition 16.** An **affine group scheme** is a group object in the category of affine schemes over a base scheme.

**Definition 17.** Let $G = \text{Spec } A$ be an affine commutative group scheme over $R$. If $A$ is a finite $R$-algebra and free of rank $m$ as an $R$-module, we say that $G$ is a **finite group scheme of order** $m$.

We now introduce an alternative, more categorical, point of view of affine group schemes, following [Waterhouse, 1979]. For a fixed field $k$, we wish to associate to each $k$-algebra $A$ a group $G(A)$ such that if $\varphi : A \longrightarrow B$ is a $k$-algebra homomorphism, we have an induced group homomorphism $G(A) \longrightarrow G(B)$. We also want the identity map on $A$ to be sent to the identity on $G(A)$. So we are looking for a functor from $k$-algebras to groups. Affine group schemes over the one-point space $\text{Spec } k$ turn out be such functors.

To make this precise, we require some definitions from category theory.

**Definition 18.** Let $F$ and $G$ be covariants functors from $\mathcal{C}$ to $\mathcal{D}$. A **natural transformation** $\eta : F \longrightarrow G$ associates to each object $X \in \mathcal{C}$ a morphism $\eta_X : F(X) \longrightarrow G(X)$ in $\mathcal{D}$ such that for all morphisms $f : X \longrightarrow Y$ in $\mathcal{C}$, the following square commutes:

$$
\begin{array}{ccc}
F(X) & \longrightarrow & F(Y) \\
\downarrow \eta_X & & \downarrow \eta_Y \\
G(X) & \longrightarrow & G(Y)
\end{array}
$$
If for all objects $X \in \mathcal{C}$, the morphism $\eta_X$ is an isomorphism in $\mathcal{D}$, we say that $\eta$ is a natural isomorphism.

**Definition 19.** Let $\mathcal{C}$ be a category such that for all objects $A$ and $B$ in $\mathcal{C}$, $\text{Hom}(A, B)$ is a set; such a category is called locally small. We can then define for an object $A \in \mathcal{C}$ a covariant functor $\text{Hom}(A, \cdot)$ from $\mathcal{C}$ to the category of sets by sending $B \in \mathcal{C}$ to $\text{Hom}(A, B)$, and a morphism $m : B \to C$ to the morphism $\varphi \mapsto m \circ \varphi \in \text{Hom}(A, C)$.

Let $F$ be a functor from $\mathcal{C}$ to the category of sets. If there exists an object $A \in \mathcal{C}$ and a natural isomorphism between $F$ and $\text{Hom}(A, \cdot)$, we say that $F$ is representable.

**Example 5.** Consider the ‘forgetful functor’ $U$ from the category of groups to sets, taking a group $G$ to its underlying set, and a group homomorphism $f : G \to H$ to the corresponding function between sets, essentially ‘forgetting’ the group structure.

For each group $G$ we have a bijection $\eta_G$ between the underlying set $U(G)$ of $G$ and the set $\text{Hom}(\mathbb{Z}, G)$. In particular, $g \in G$ corresponds to the group homomorphism taking $1 \in \mathbb{Z}$ to $g$. In addition, for each group map $f : G \to H$ we can see that the square

$$
\begin{array}{ccc}
\text{U}(G) & \xrightarrow{U(f)} & \text{U}(H) \\
\eta_G \downarrow & & \eta_H \downarrow \\
\text{Hom}(\mathbb{Z}, G) & \xrightarrow{\text{Hom}(\mathbb{Z}, f)} & \text{Hom}(\mathbb{Z}, H)
\end{array}
$$

commutes. Thus $U$ is representable by the group $\mathbb{Z}$.

**Definition 20.** An affine group scheme over a field $k$ is a representable functor from $k$-algebras to groups.

**Example 6.** We have:

- $G_m$ is represented by $k[t, t^{-1}] := k[x, y] / (xy - 1)$.
- $G_a = \mathbb{A}^1$ is represented by $k[t]$.
- $\text{GL}_2(k)$ is represented by $k[a, b, c, d, z] / (z(ad - bc) - 1)$.

**Definition 21.** Let $G$ and $H$ be affine group schemes (in the functorial sense) over $k$. A map $\psi : G \to H$ is called natural if for any $k$-algebra map $\varphi : R \to S$ we have a commutative square

$$
\begin{array}{ccc}
G(R) & \xrightarrow{\psi} & H(R) \\
\varphi \downarrow & & \varphi \downarrow \\
G(S) & \xrightarrow{\psi} & H(S)
\end{array}
$$

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Example 7. The map \( \det : \GL_n \rightarrow \mathbb{G}_m \) is natural.

Lemma 19 (Yoneda’s Lemma). Let \( E \) and \( F \) be representable functors from a category \( C \) to the category of sets such that \( E \) and \( F \) are represented by \( k \)-algebras \( A \) and \( B \), respectively. Then the natural maps from \( E \) to \( F \) correspond to \( k \)-algebra homomorphisms from \( B \) to \( A \).

4.4 Hopf Algebras

Another way of viewing affine group schemes is to treat the corresponding functor not as a functor from \( k \)-algebras to groups, but from \( k \)-algebras to sets with the conditions given in Definition 9. That is, if \( \text{Spec} A \) is an affine group scheme, we have \( k \)-algebra maps \( \mu^* : A \rightarrow A \otimes A \), \( \iota^* : A \rightarrow A \), and \( e^* : A \rightarrow k \) such that the diagrams

\[
\begin{align*}
A \otimes A \otimes A & \xrightarrow{\mu^* \otimes \text{id}} A \otimes A \\
& \xleftarrow{\text{id} \otimes \mu^*} \\
A \otimes A & \xrightarrow{\mu^*} A
\end{align*}
\]

(4.5)

\[
\begin{align*}
A & \xrightarrow{\text{mult}} A \otimes A \\
& \xleftarrow{\text{mult}} \\
A \otimes A & \xrightarrow{(\iota^* \otimes \text{id}) \circ \mu^*} A
\end{align*}
\]

(4.6)

\[
\begin{align*}
A & \xrightarrow{\text{mult}} A \otimes A \\
& \xleftarrow{\text{mult}} \\
A \otimes A & \xrightarrow{(\iota^* \otimes \text{id}) \circ \mu^*} A
\end{align*}
\]

(4.7)

all commute.

These conditions say that \( A \) is both an algebra and a coalgebra, and we call \( \iota^* \) an antipode. Such a structure is called a Hopf algebra, and we will use its properties throughout the proof of Stone-von Neumann in the next chapter.
Chapter 5

Algebraic Heisenberg Groups

In this chapter, we follow the proof of the Stone von-Neumann theorem for algebraic Heisenberg groups given in the Appendix of [Sekiguchi, 1977]. Unlike the finite and continuous cases, the proof relies heavily on the relationship between the category of affine schemes to the category of Hopf algebras. We fix $k$ to be an algebraically closed field of characteristic $p > 0$, and all schemes are affine schemes over $k$.

Our first definition comes from [Mumford, 1970, p. 221]:

**Definition 22.** Let $G_m = \text{Spec } k[t, t^{-1}]$, and let $K$ be an abelian affine group scheme. A *theta group* is an affine group scheme $\mathcal{G}$ lying in a central extension

$$1 \longrightarrow G_m \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} K \longrightarrow 0,$$

(5.1)

(where the maps are all group scheme homomorphisms) such that

a) There exists an open cover $\{U_i\}$ of $K$ and sections $\sigma_i$ of $\pi$ on these $U_i$,

b) $\iota$ is a closed immersion such that $\text{Ker } \pi = G_m$, and

c) $G_m$ is in the centre of $\mathcal{G}$.

If $K$ is finite, then there exists a global section $\sigma : K \to \mathcal{G}$ of $\pi$ such that $\mathcal{G} \cong G_m \times K$; this is actually an isomorphism of schemes. There will also exist a morphism $f : K \times K \to G_m$ such that the group law on $\mathcal{G}$ is given by

$$(\lambda, k)(\mu, k') = (\lambda \mu f(k, k'), k + k').$$

We can define a map

$$e : K \times K \longrightarrow G_m : (\pi(x), \pi(y)) \mapsto xyx^{-1}y^{-1}.$$ 

As in Proposition 13, we have

$$e(x + y, z) = e(x, z)e(y, z),$$

(5.2)

$$e(x, y + z) = e(x, y)e(x, z),$$

(5.3)

$$e(x, x) = 1, \text{ and}$$

(5.4)

$$e(x, y) = \frac{f(x, y)}{f(y, x)}.$$  

(5.5)
Proposition 20. If $K$ is finite then the following conditions are equivalent:

a) $\iota(G_m)$ is equal to the centre of $G$.

b) The map $\gamma : K \rightarrow \tilde{K}$ defined by $\gamma(x)(y) = e(x, y)$ is an isomorphism.

Definition 23. A theta group $G = \text{Spec } A$ satisfying one of the conditions in Proposition 20, and where $K$ is finite, is called an algebraic Heisenberg group.

5.1 Representations of Algebraic Heisenberg Groups

Definition 24. Let $V$ be a finite dimensional vector space over $k$, and let $G$ be an algebraic Heisenberg group. A representation of $G$ on $V$ is a homomorphism $\rho$ from $G$ to $GL(V)$. If $\rho(\lambda) = \lambda^n . \text{id}$ for some fixed $n$ and all $\lambda \in G_m$, we say that $\rho$ is a representation of weight $n$.

Our main result is the Stone-von Neumann theorem, which states:

Theorem 21. Let $G$ be an algebraic Heisenberg group. Then there exists an irreducible representation of $G$ of weight 1, the Heisenberg representation, which is unique up to isomorphism. We call this representation the Heisenberg representation and it is of degree $d$, where the order of $K$ is $d^2$.

Our proof begins by looking at the structure of the $k$-algebra $A$, where $G = \text{Spec } A$. We will see that $A$ is isomorphic as a graded ring to $R \otimes k[t, t^{-1}]$, where $K = \text{Spec } R$, and it is within $A$ that we find the Heisenberg representation.

Step I. Let $\mu$, $e$, and $i$ be the multiplication, identity, and inverse of the affine group scheme $G$. Then restricted multiplication $\mu_{|G_m \times G} : G_m \times G \rightarrow G$ defines a $k$-algebra homomorphism

$$s : A \rightarrow k[t, t^{-1}] \otimes_k A.$$  

For any $a \in A$, let

$$s(a) = \sum_{i=-\infty}^{\infty} t^i \otimes \pi_i(a).$$

Then $\pi_i : A \rightarrow A$ is a $k$-linear map for each $i$, and we define $A_i = \text{Im}(\pi_i)$.

Lemma 22. $A$ is the direct sum of the $A_i$’s.

Proof. Let $e : \text{Spec } k \rightarrow G_m$ be the identity morphism of the group scheme $G_m$. Since $e$ is the identity of $\mu$, we have the following commutative diagram:
This gives the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{s} & k[t, t^{-1}] \otimes A \\
& \searrow^{1_A} & \downarrow^{e^* \otimes 1_A} \\
& & A
\end{array}
\]

where \(e^*\) is the morphism from \(k[t, t^{-1}]\) to \(k\) taking \(t\) to 1. Thus we have the equality

\[
a = e^* \otimes 1_A \circ s(a)
\]

\[
= e^* \otimes 1_A \left( \sum_i t^i \otimes \pi_i(a) \right)
\]

\[
= \sum_i \pi_i(a).
\]

It remains to show that the \(A_i\) are mutually disjoint. By the associativity of \(\mu\), we have the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\mu|_{G \times G}} & G_m \times G \\
\downarrow^{\mu|_{G_m \times G}} & & \downarrow^{\mu|_{G_m \times G \times G}} \\
G_m \times G & \xleftarrow{id \times \mu|_{G_m \times G}} & G_m \times G_m \times G
\end{array}
\]

which implies the commutativity of the square

\[
\begin{array}{ccc}
A & \xrightarrow{s} & k[t, t^{-1}] \otimes A \\
& \searrow^{s \otimes 1_A} & \downarrow^{(\mu|_{G_m \times G_m}) \otimes 1_A} \\
k[t, t^{-1}] \otimes A & \xrightarrow{1 \otimes s} & k[t, t^{-1}] \otimes k[t, t^{-1}] \otimes A
\end{array}
\]

Following the action of the arrows on \(a \in A\) right then down, we have

\[
1 \otimes s \circ s(a) = 1 \otimes s \left( \sum_i t^i \otimes \pi_i(a) \right)
\]

\[
= \sum_i t^i \otimes \left( \sum_j t^j \otimes \pi_j \pi_i(a) \right)
\]

\[
= \sum_{i,j} t^i \otimes t^j \otimes \pi_j \pi_i(a).
\]
Following down then right, we have

\[
\left( \mu|_{G_m \times G_m} \right)^* \otimes 1_A \circ s(a) = \left( \mu|_{G_m \times G_m} \right)^* \otimes 1_A \left( \sum_i t^i \otimes \pi_i(a) \right)
\]

\[
= \sum_i \left( \sum_j t^j \otimes \pi_j \left( t^i \right) \right) \otimes \pi_i(a)
\]

\[
= \sum_i t^i \otimes t^i \otimes \pi_i(a).
\]

This last equality follows from the fact that \( t^i = \sum_j \pi_j \left( t^i \right) \), since

\[
t^i \otimes t^i = \sum_j t^i \otimes \pi_j \left( t^i \right) \implies \pi_j \left( t^i \right) = \delta_{i,j} t^i.
\]

Thus we have

\[
\sum_{i,j} t^i \otimes t^j \otimes \pi_j \pi_i(a) = \sum_i t^i \otimes t^i \otimes \pi_i(a),
\]

which implies that

\[
\pi_j \pi_i(a) = \delta_{i,j} \pi_i(a),
\]

and so \( A = \bigoplus_i A_i \).

Each element \( f \in A \) corresponds to a \( k \)-linear map from \( k[t] \) to \( A \), and thus to a map from \( G = \text{Spec} A \) to \( \text{Spec} k[t] = \mathbb{A}^1 \). Viewing elements of \( A \) in this way, the \( A_i \)'s have the additional property that they can be characterised by their restriction to \( G_m \). The following lemma strongly hints that \( A_1 \) will contain the Heisenberg representation.

**Lemma 23.** Let \( f \in A \). Then \( f \in A_i \) if and only if for all \( k \)-algebras \( B \) we have

\[
f(\lambda x) = \lambda^i f(x),
\]

for all \( \lambda \in G_m(B) \) and \( x \in G(B) \).

**Proof.** Since \( G_m = \text{Spec} k[t, t^{-1}] \) and \( \mathbb{A}^1 = \text{Spec} k[t] \), we have a group action of \( G_m \) on \( \mathbb{A}^1 \) via the ring homomorphism

\[
\Delta : k[t] \rightarrow k[t, t^{-1}] \otimes k[t].
\]

Thus we can make sense of the equality \( f(\lambda x) = \lambda^i f(x) \) via the diagrams

\[
\begin{array}{ccc}
G_m & \xrightarrow{f} & \mathbb{A}^1 \\
\downarrow \lambda & & \downarrow \lambda^i \\
G & \xrightarrow{f} & \mathbb{A}^1
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
G_m & \xrightarrow{f} & \mathbb{A}^1 \\
\downarrow \lambda & & \downarrow \lambda^i \\
G & \xrightarrow{f} & \mathbb{A}^1
\end{array}
\]
We can see that equality holds by looking at the associated \( k \)-algebra homomorphisms. We have
\[
f(\lambda x) = f \circ \mu|_{G_m \times G} \circ (\lambda \times x) \circ \Delta : \text{Spec } B \to \mathbb{A}^1,
\]
so if \( f \in A_i \), we have
\[
f^*(t) = f \quad \text{and} \quad s(f) = t^i \otimes f,
\]
giving us
\[
\left( f \circ \mu|_{G_m \times G} \circ (\lambda \times x) \circ \Delta \right)^*(t) = (\text{mult} \circ (\lambda \times x)^* \circ s \circ f^*) (t) = (\text{mult} \circ (\lambda^* \otimes x^*) \circ (t^i \otimes f) = \lambda^*(t^i) x^*(f) \in B.
\]
On the other hand, we have
\[
\lambda^i f(x) = \nabla \circ \left( \lambda^i \times (f \circ x) \right) \circ \Delta : \text{Spec } B \to \mathbb{A}^1.
\]
This gives us the map
\[
\left( \nabla \circ \left( \lambda^i \times (f \circ x) \right) \circ \Delta \right)^*(t) = \left( \text{mult} \circ \left( \lambda^i \right)^* \otimes (x^* \circ f^*) \circ \nabla \right) (t) = \left( \lambda^i \right)^*(t) x^*(f) = \lambda^*(t^i) x^*(f).
\]
Now take \( f \in A \) and suppose that \( f(\lambda x) = \lambda f(x) \) for all \( \lambda \in G_m(B) \) and \( x \in G(B) \). Since \( A = \bigoplus_j A_j \), we can write
\[
f = \sum_j a_j, \quad a_j \in A_j.
\]
Then
\[
\left( \sum_j a_j \right) (\lambda x) = \sum_j \lambda^j a_j(x) = \lambda^i \sum_j a_j(x).
\]
Thus \( a_j = \delta_{i,j} a_i \) and \( f \in A_i \).

\[\Box\]

**Lemma 24.** The coproduct \( \mu^* : A \to A \otimes A \) satisfies
\[
\mu^*(A_i) \subset A_i \otimes A_i \quad \text{for all } i.
\]
**Proof.** We have the commutative diagram

\[
\begin{array}{ccc}
 k[t^{\pm 1}] \otimes A & \overset{i^* \otimes id}{\longrightarrow} & A \otimes A \\
 \downarrow s & & \downarrow \mu^* \\
 A & & A
\end{array}
\]

which along with the equality

\[s(f) = t^i \otimes f, \quad \text{if } f \in A_i\]

tells us that

\[f \in A_i \implies (i^* \otimes id \circ \mu^*) (f) = t^i \otimes f.\]

We define a map giving right translation by \(\lambda\),

\[R_\lambda = (p_1, \mu \circ (\lambda \times id)) : \text{Spec } B \times G \longrightarrow \text{Spec } B \times G,\]

where \(p_1\) is projection of the first factor. Suppose that \(f \in A_i\). Then by Lemma 23, for any \(k\)-algebra \(B\) and \(\lambda \in G_m(B)\), we have the commutative diagram

\[
\begin{array}{ccc}
 B[t] & \overset{1 \otimes f^*}{\longrightarrow} & B \otimes A \\
 \downarrow \text{mult by } \lambda^*(t)^i & & \downarrow (R_\lambda)^* \\
 B[t] & \overset{1 \otimes f^*}{\longrightarrow} & B \otimes A
\end{array}
\]

Let \(\mu^*(f) = \sum_{j,k} a_j \otimes b_k\), where \(a_j \in A_j\) and \(b_k \in A_k\). Then the map \((R_\lambda)^*\) takes \(1 \otimes f\) to

\[\lambda^* \circ i^* \otimes id \circ \mu^*(f)\]

which, from the above, is

\[\sum_{j,k} \lambda^*(t^j) \otimes b_k = \sum_{j,k} \lambda^*(t^k) \otimes a_j,\]

using the fact that \(G_m\) is the centre of \(G\).

Letting \(\lambda^*\) be the map taking \(t\) to \(1\), we have

\[\sum_j a_j = \sum_k b_k,\]

and in particular, \(a_j = b_j\), since \(A\) is the direct sum of the \(A_i\)'s.

So \(\mu^*(f) = \sum_j a_j \otimes a_j\). By the associativity of the square above, we have

\[\lambda^*(t^i) \otimes f = \sum_j \lambda^*(t^j) \otimes a_j.\]
Again letting $\lambda^*$ be the map taking $t$ to 1, we have
\[ f = \sum_j a_j, \]
which implies that
\[ a_j = \begin{cases} f & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \]

Lemma 25. $A_0 \cong R$, and there exists $t \in A$ such that $A_i = A_0 t^i$ for all $i$.

Proof. The map $\phi = \mu \circ (\iota \times \tau) : G_m \times K \to G$ is an isomorphism of spaces with $G_m$-actions, which gives us an isomorphism $\phi^* : A \to R \otimes k[t, t^{-1}]$ of graded rings.

Step II. The next lemma is a technical one that we require for Theorem 27.

Lemma 26. Let $\mathcal{G} = \text{Spec } A$ be an affine group scheme over a field $k$, with multiplication morphism $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and identity morphism $e : \text{Spec } k \to \mathcal{G}$.

Let $W$ be a non-trivial $k$-subspace of $A$ such that
\[ \mu^*(W) \subset W \otimes A. \]

Then

(a) $W$ is closed under any left-invariant $k$-linear map $D : A \to A$; that is, any $k$-linear map $D$ such that the square
\[
\begin{array}{ccc}
A & \xrightarrow{D} & A \\
\mu^* \downarrow & & \mu^* \\
A \otimes A & \xrightarrow{1_A \otimes D} & A \otimes A
\end{array}
\]
commutes, and

(b) For any $k$-rational point $x \in G$, there exists a function $f \in W$ such that $f(x) \neq 0$.

Proof (a). By the commutativity of the square, we have
\[ \mu^*(D(f)) = 1 \otimes D(\mu^*(f)) \in W \otimes A. \]

Since $e$ is the unit morphism, we have
\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\mu} & \mathcal{G} \times \mathcal{G} \\
\downarrow & & \downarrow \text{ex} \downarrow 1 \\
\mathcal{G}
\end{array}
\]
This gives the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\mu^*} & A \otimes A \\
\downarrow^1 & & \downarrow^{1 \otimes e^*} \\
A & & A
\end{array}
\]

so

\[
D(f) = (1 \otimes e^*)\mu^*(D(f)) \in (1 \otimes e^*)(W \otimes A) = W.
\]

**Proof** (b). Since \(\mu^*(W) \subset W \otimes A\), it suffices to prove this for the case \(x = e\). To see this, suppose \(f \in W\) and \(f(e) \neq 0\). Then

\[
f(e) = f(\mu(x, x^{-1})).
\]

If \(\mu(f) = \sum_i w_i \otimes a_i\), where \(w_i \in W\), we have

\[
e^*(f) = \left(x^* \otimes (x^{-1})^*\right) \mu^*(f)
= \left(x^* \otimes (x^{-1})^*\right) \sum_i w_i \otimes a_i
= \sum_i x^*(w_i) \otimes (x^{-1})^*(a_i) \neq 0.
\]

Since the summands cannot all be zero, choose any \(j\) such that \(x^*(w_j)(x^{-1})^*(a_j) \neq 0\). Then

\[
x^*(w_j) = e^*(f) \left((x^{-1})^*(a_j)\right)^{-1} \neq 0,
\]

so \(w_j(x) \neq 0\).

So let \(x = e\). Since \(W \neq 0\), there exists \(f\) in \(W\) such that the image of \(f\) in \(O_{e,G} \neq 0\). Then there exists a \(k\)-linear map \(D_0 : O_{e,G} \to k\) such that \(D_0(f) \neq 0\). Now let \(D_1\) be the map \(A \to O_{e,G} \xrightarrow{D_0} k\), and

\[
D = (1_A \otimes D_1) \circ \mu^* : A \to A.
\]

Then \(D\) is a left-invariant \(k\)-linear map, since

\[
D^* : G \to G = \mu \circ (\text{id} \times D_1^*) : x \mapsto xD_1^*,
\]

and

\[
\mu \circ (\text{id} \times \mu \circ (\text{id} \times D_1^*)) : (x, y) \mapsto xyD_1^*, \quad \text{and}
\]

\[
\mu \circ (\text{id} \times D_1) \circ \mu : (x, y) \mapsto xyD_1^*.
\]
showing that the square
\[ \begin{array}{ccc}
G \times G & \xrightarrow{\text{id} \times D^*} & G \times G \\
\mu & \downarrow & \mu \\
G & \xrightarrow{D^*} & G
\end{array} \]
commutes. Since \( f \in W \) we have \( D(f) \in W \) by part (a) and \( D(f)(e) \neq 0 \).

**Step III.** We define a group scheme action of \( G \times G \) on \( A_1 \) by
\[(g, h)f(x) = f(g^{-1}xh).\]
Then this is a representation, since
\[(g_1g_2, h_1h_2)f(x) = f(g_2^{-1}g_1^{-1}xh_1h_2) = (g_2, h_2)f(g_1^{-1}xh_1) = (g_1, h_1)((g_2, h_2)f)(x).\]

**Theorem 27.** \( A_1 \) is an irreducible \( G \times G \)-module.

**Proof.** Let \( W \) be a non-trivial subrepresentation of \( A_1 \). We assume here Sekiguchi’s result that \( W \) satisfies \( \mu^*(W) \subset W \otimes A \), allowing us to apply the result of Lemma 26, part (b).

By Lemma 25, we have \( A_0t = A_1 \), so \( W \) corresponds to an ideal \( I \) of \( A_0 \). If \( W \neq A_1 \) then \( I \) is a proper ideal of \( A_0 \) and is contained in a maximal ideal \( M \subset A_0 \). As \( M \) is maximal, it corresponds to a closed point \( x \in K = \text{Spec} A_0 \).

If \( y = \tau(x) \) then by Lemma 26 (b), there exists \( f \in W \) such that \( f(y) \neq 0 \). But then \( f = f't \) where \( f' \in I \), and \( f'(y) \neq 0 \). However, \( f' \in I \subset M \), so the image of \( y \) under \( f' \) must be 0, which gives us a contradiction. Thus \( A_1 \) is irreducible as a \( G \times G \)-module.

We are finally in the position to prove Theorem 21.

**Proof (Uniqueness).** Let \( V \) be a finite-dimensional \( k \)-vector space, and let \( \sigma : G \rightarrow \text{GL}(V) \) be a representation of weight 1. If \( V \) is an irreducible \( G \)-module, then \( V^* \otimes V \) is an irreducible \( G \times G \)-module with action
\[(g, h)(v^* \otimes w) = v^* \circ \sigma(g^{-1}) \otimes \sigma(h)w.\]
From Theorem 27, we know that \( A_1 \) is an irreducible \( G \times G \)-module with action
\[(g, h)f(x) = f(g^{-1}xh).\]
We also have a \( G \times G \)-module homomorphism \( \varphi \) between the two representations, given by
\[\varphi(v^* \otimes w)(x) = v^*(\sigma(x)w).\]
The map $\varphi$ is well-defined since it is bi-linear, and it is a $G \times G$-module homomorphism since

$$
\varphi((g, h)(v^* \otimes w))(x) = \varphi(v^* \circ (g^{-1}) \otimes \sigma(h)w)(x)
$$

$$
= v^* \circ (g^{-1})(\sigma(x)\sigma(h)w)
$$

$$
= v^*(\sigma(g^{-1}xh)w)
$$

$$
= ((g, h)\varphi(v^* \otimes w))(x),
$$

and so the square

\[
\begin{array}{ccc}
V^* \otimes V & \xrightarrow{\varphi} & A_1 \\
\downarrow{(g,h)} & & \downarrow{(g,h)} \\
V^* \otimes V & \xrightarrow{\varphi} & A_1
\end{array}
\]

commutes for all $(g, h) \in G \times G$.

We also need to check that $\varphi(v^* \otimes w) \in A_1$. Let $v^* \otimes w \in V^* \otimes V$ and $f = \varphi(v^* \otimes w)$. Then

$$
f(\lambda x) = v^*(\lambda \sigma(x)w)
$$

$$
= \lambda v^*(\sigma(x)w) = \lambda f(x).
$$

Since $\varphi$ is non-zero, by Schur’s lemma $\varphi$ is an isomorphism and $V^* \otimes V$ and $A_1$ are isomorphic as $G \times G$-modules.

Consider $G$ as the subgroup $\{1\} \times G \subset G \times G$, and let $G$ act on both $V^* \otimes V$ and $A_1$. Then choosing a basis for $V$, the action of $g$ on $V^* \otimes V$ can be written as the matrix $id \otimes \sigma(g)$, which is the same as the action of $g$ on $V^* \otimes \sigma(g)$, where $d = \dim V$.

So as a $G$-module, $A_1$ is isomorphic to

$$
\underbrace{V \oplus \cdots \oplus V}_{\text{dim } V \text{ copies}}
$$

via the same intertwining operator $\varphi$.

We now have $\dim A_1 = (\dim V)^2 = \dim A_0$, and $\dim V = d$.

If $W$ is any other representation of $G$ of weight 1, we then have

$$
W \oplus \cdots \oplus W \cong A_1 \cong V \oplus \cdots \oplus V
$$

as $G$-modules, so $W \cong V$.

**Proof (Existence).** It does not seem surprising in light of Lemma 23 that $A_1$ should be a weight-1 representation of $G$. Define an action of $G$ on $A_1$ by

$$
(\rho(g)f)(x) = f(xg).
$$

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This is known as the right-regular representation, and it is routine to check that

\[(\rho(gh)f)(x) = f(xgh) = (\rho(h)f)(xg) = \rho(g)\rho(h)f(x).\]

That the representation \(\rho\) is weight-1 follows from the characterisation of \(A_1\). If \(\lambda \in G_m\) then

\[(\rho(\lambda)f)(x) = f(x\lambda) = \lambda f(x),\]

so \(\rho(\lambda) = \lambda\,\text{id} \). Any subrepresentation of \(\rho\) will also be of weight 1, and since \(A_1\) is finite-dimensional, there exists an irreducible representation of \(\mathcal{G}\) of weight 1. \(\square\)


