Honours Thesis

Partial Hedges of Options: Estimation of Price and Default Risk under Stochastic Volatility

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Abstract

An investor may choose to partially hedge his or her obligations on a claim they have written by constructing a portfolio replicating the claim on only some outcomes. This partial hedge costs less than the complete hedge, however it may lead to default. Analytical results for price and default risk for partial hedges on vanilla options are developed via a Geometric Brownian Motion asset price model. Monte Carlo simulation is then used to estimate prices and risks for a stochastic volatility model, for which closed form solutions are not known, via a pathwise Euler approximation of the stochastic differential equations. The efficiency of these estimates is increased via the variance reduction techniques Antithetic Variates and Control Variates. Various partial hedging strategies are analysed, based on the asset price at maturity of the claim, the maximum price over the life of the claim and the time to this maximum.

I would like to thank my supervisor Associate Professor Kostya Borovkov for aiding me with some technical details of this project, but also for broadening my conceptual understanding of its underlying themes.

Contents

Chapter 1

Introduction

When an investor—institutional or otherwise—holds a portfolio of claims, a method must be in place to ensure the risk in the portfolio is minimised. This is generally done by the construction of hedges. The complete hedging of a contingent claim generally requires a dynamic approach. This involves the construction of a replicating portfolio made up of riskless bonds and the underlying asset, and continuously rebalancing the portfolio until the claim matures. While this approach is theoretically riskless, the practical realities of transaction costs and discrete rebalancing lead to losses on rebalancing trades and a non-zero portfolio risk.

This thesis presents a method for increasing profitability when hedging contingent claims. This method is to *partially* hedge the claim. The holder of a claim generally recieves a non-negative payoff on the whole sample space Ω and a positive payoff on some non-empty set of outcomes, say A. The set $\overline{A}$ is then the set where the payoff of the claim is 0. The idea of a partial hedge is to hedge the claim on some set $B \subset A$, so that if the outcome lies in $\overline{A} \cup B$ the claim is perfectly hedged, yet if the outcome lies in $A \backslash B$ the claim is not hedged, and the writer of the claim will suffer a loss. In what follows, the risk of suffering a loss on the partial hedge of a written claim is called the *default* risk.

The important point to be made is that the partial hedge is a *riskier* portfolio than the complete hedge, and indeed pays a premium for that risk. While construction of a partial hedge still suffers losses from rebalancing transaction costs, these losses are offset to a pre-specified degree by the initial gain from the positive difference between the claim price recieved and partial hedge price paid. However, on top of the risk incurred through discrete rebalancing there is the additional risk of default on the partial hedge. The investor is left with the typical question of optimal investment: how much risk should be assumed in the expectation of how much—in this case, initial—return.

In this thesis the market is assumed to be frictionless and trading takes place continuously, so that hedges may be continuously rebalanced without losses due to transaction costs. In this context we calculate the price and risk of default associated with a range of partial hedging strategies based on final asset prices, the asset price maximum and the time to maximum, over the trading period of the claim, and gauge how changing the hedge parameters changes the price and risk. An obvious suggestion for further work is to include practical adjustments for market realities—transaction costs and discrete rebalancing.

We make our calculations assuming firstly a Geometric Brownian Motion asset price process (constant volatility), then secondly a stochastic volatility model, with volatility following a mean-reverting square root process, as in the fifth model proposed in Musiela & Rutkowski (1997, ch. 6.3.4). This stochastic volatility model does not in general admit known closed form solutions for the evolution of the asset price. To overcome this, we estimate partial hedge prices and default risks via pathwise approximation of asset price and volatility and Monte Carlo simulation, using Microsoft Visual Basic for Applications 6.0. Variance reduction techniques, namely antithetic variates and control variates, are then used to increse the efficiency of our estimators.

Most significantly amongst others, this thesis owes itself to the work of Ben Ameur, Breton and L'Écuyer (1999).

Chapter 2

Preliminaries

At the heart of financial engineering is the use of probability theory to quantify market uncertainty, or *risk*. For the purposes of this thesis, we will model uncertainty over the time-period $t \in [0, T]$ via the filtered probability space: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, with $\mathcal{F} = \mathcal{F}_T$. We take $\mathbb{P}$ to be the *real-world* probability measure, which may potentially be estimated from real-world data. In what follows, we will assume every process is defined on our filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

2.1 Stochastic Calculus

We begin by stating a useful definition. Most of the processes we will define possess this property:

Definition 2.1.1 (Adapted Process) *A stochastic process U is* adapted *to the filtration $\mathbb{F}$ if U_t is an $\mathcal{F}_t$-measurable random variable $\forall t$. We call such a process* $\mathbb{F}$-adapted.

Central to mathematical finance is the process known as the Brownian Motion:

Definition 2.1.2 (Brownian Motion) *An $\mathbb{F}$-adapted process $\{W(t)\}_{t \geq 0}$, $W(0) = 0$ is called the* Brownian Motion *if the following hold:*
(i) $W(t) - W(s) \sim N(0, t-s)$, $0 \leq s < t$, where $N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2.
(ii) $W(t) - W(s)$ is independent of $\mathcal{F}_s$, $0 \leq s < t$.
(iii) For every $\omega \in \Omega$ the path $t \in [0, \infty) \mapsto W(t, \omega) \in \mathbb{R}$ is continuous.

Note that from the above we can infer that $\mathbb{E}[W(t)] = 0$, $\mathrm{Var}[W(t)] = t$ and $\mathrm{Cov}[W(s)W(t)] = \min\{s, t\}$, where $\mathbb{E}$ denotes the expectation operator.

Definition 2.1.3 (Martingale) *An $\mathbb{F}$-adapted process $\{X(t)\}_{t\geq 0}$, with finite mean $\mathbb{E}[X(t)]$, $\forall t \geq 0$, is called a* Martingale *if*

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s) \text{ a.s.}, \qquad 0 \leq s \leq t.$$

We note here that the Brownian Motion is a Martingale under $\mathbb{P}$, and moreover that it is a Markov process. This property of the Brownian Motion, combined with its continuity and normally distributed increments, make it a good candidate for modelling uncertainty in a continuous stochastic process. Importantly, the variation of the Brownian Motion process is unbounded, and we cannot define an integral with respect to the Brownian Motion in the conventional Lebesgue-Stieltjes sense. When we model uncertainty in a process using the Brownian Motion via stochastic differential equations, we need to develop a definition of the integral of the Brownian Motion:

Definition 2.1.4 (Itô Integral) *Suppose we have a sequence of partitions of $[0,t]$ $\tau_n(t) = \{t_1^{(n)} = t_0, t_1^{(n)} = t_1, \ldots, t^{(n)} = t_n\}$, with $t_0 = 0$, $t_n = t$, $t_i < t_j$ $\forall i < j$, and $\max_{0\leq j<n} |t_{j+1} - t_j| \to 0$ as $n \to \infty$. Let $\{f^{(n)}(t)\}$ be a sequence of $\mathbb{F}$-adapted simple random processes,*

$$f_i^{(n)}(t) = \begin{cases} X_i & \text{if } t_{i-1} \leq t < t_i, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $f^{(n)}(t)$ converges in probability to some $f(t)$ as $n \to \infty$, where $f(t)$ is an $\mathbb{F}$-adapted process with $\int_0^T \mathbb{E}[f^2(s)]ds < \infty$. Then

$$I_t^{(n)}(f) = \sum_{i=1}^{n} X_i(W(t_i) - W(t_{i-1})) \longrightarrow \int_0^t f(s)dW(s) = I_t(f) \text{ as } n \longrightarrow \infty$$

in probability. The random variable $I_t(f)$ is called the Itô Integral *of f.*

The Itô Integral has the following properties:
(i) $I_t(af+bg) = aI_t(f)+bI_t(g)$ for a,b constants and f,g satisfying the above conditions,
(ii) $I_t(f)$ is continuous and $\mathbb{F}$-adapted,
(iii) $\{I_t(f)\}_{0\leq t\leq T}$ is a Martingale, and $\mathbb{E}I_t(f) = \mathbb{E}I_0(f) = 0$,
(iv) $\text{Var}(I_t(f)) = \mathbb{E}I_t^2(f) = \int_0^t \mathbb{E}f^2(s)ds$.
(v) For any deterministic function $f(t)$, $0 \leq t \leq T$,

$$I_t(f) = \int_0^t f(s)dW(s) \sim N\Big(0, \int_0^t f_s^2(s)ds\Big).$$

The Itô Integral is usefully generalised to the following:

Definition 2.1.5 (Itô Process) *An* $\mathbb{F}$*-adapted process* $\{X(t)\}_{0\leq t\leq T}$ *is called an* Itô Process *if*

$$X(t) = X(0) + \int_0^t \mu(s,\omega)ds + \int_0^t \sigma(s,\omega)dW(s),$$

where $\mu(t,\omega)$ *is* $\mathbb{F}$*-adapted with* $\int_0^T |\mu(s,\omega)|ds < \infty$ *a.s., and* $\sigma(t,\omega)$ *is* $\mathbb{F}$*-adapted with* $\int_0^T \sigma^2(s,\omega)ds < \infty$ *a.s.*

The above Itô Process is often written in differential form:

$$dX(t) = \mu(t,\omega)dt + \sigma(t,\omega)dW(t).$$

Definition 2.1.6 (Stochastic Differential Equation) *Now assume that* $\mu(t,x)$ *and* $\sigma(t,x)$ *are given bounded and continuous functions, taking values in* $[0,\infty)\times\mathbb{R}$*. If* $\int_0^t \mu(s,X(s))ds$ *and* $\int_0^t \sigma(s,X(s))ds$ *exist, and there also exists an Itô Process*

$$X(t) = X(0) + \int_0^t \mu(s,X(s))ds + \int_0^t \sigma(s,X(s))dW(s),$$

then we say $\{X(t)\}$ *is a (strong) solution to the* stochastic differential equation

$$dX(t) = \mu(t,X(t))dt + \sigma(t,X(t))dW(t).$$

We will be using the following conventions for the multiplication of differentials:

$$\begin{aligned} dW(t)\cdot dW(t) &= dt, \\ dW(t)\cdot dt &= 0, \\ dt\cdot dt &= 0. \end{aligned} \tag{2.1}$$

The first of these refers to the key property of the quadratic variation of the Brownian Motion process.

We have the following formula for functions of continuous stochastic processes:

Definition 2.1.7 (Itô Formula) *Let* $f(x)$ *be twice continuously differentiable, and* $X(t)$ *be an Itô Process as defined above. Then*

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2.$$

From (2.1) we obtain the following expression for the Itô Formula:

$$df(X(t)) = f'(X(t)) \cdot dX(t) + \frac{1}{2}\sigma^2(t, X(t))dt. \tag{2.2}$$

Of crucial importance for mathematical finance is the notion of *change of measure*. We state the main measure-theoretic results used in pricing claims and assets in mathematical finance. The following is adapted from Bingham & Kiesel (1998, chapter 2.4).

Definition 2.1.8 (Absolute Continuity) *Let the probability measures $\mathbb{P}$ and $\mathbb{Q}$ be defined on the same measurable space $(\Omega, \mathcal{F})$. The probability $\mathbb{P}$ is said to be* absolutely continuous *with respect to $\mathbb{Q}$, denoted by $\mathbb{P} \prec \mathbb{Q}$, if*

$$\mathbb{Q}(A) = 0 \Rightarrow \mathbb{P}(A) = 0, \qquad A \in \mathcal{F}.$$

The two measures are said to be equivalent, *written $\mathbb{P} \sim \mathbb{Q}$, if $\mathbb{P} \prec \mathbb{Q}$ and $\mathbb{Q} \prec \mathbb{P}$.*

Note that equivalence is the same as saying that the two measures have the same null sets, or

$$\mathbb{Q}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0, \qquad A \in \mathcal{F}.$$

We now have some notion of the relations between two measures on the same measurable space. Now we will seek to change between them. In the following, $\mathbf{1}_A$ denotes the indicator function of the set A:

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Theorem 2.1.1 (Radon-Nikodým Derivative) *Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures defined on a common measurable space $(\Omega, \mathcal{F})$. Then $\mathbb{P} \prec \mathbb{Q}$ iff there exists a measurable function $f \geq 0$ such that*

$$\mathbb{P}(A) = \int_A f d\mathbb{Q}, \qquad \forall A \in \mathcal{F}.$$

The function f is called the Radon-Nikodým Derivative *of $\mathbb{P}$ with respect to $\mathbb{Q}$.*

The Radon-Nikodým Derivative is often written as

$$f = \frac{d\mathbb{P}}{d\mathbb{Q}}.$$

Notice that for $\mathbb{P} \sim \mathbb{Q}$,

$$f = \frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{f^{-1}} = 1/\frac{d\mathbb{Q}}{d\mathbb{P}}.$$

The existence of the Radon-Nikodým Derivative is beneficial in two ways. Firstly, it enables the conversion of 'tricky' calculations on one measure into integrals over a simpler measure. This is a technique used in pricing some path-dependent options. Secondly, if we have to change measure, the Radon-Nikodým Derivative enables us to calculate all that we may need on the new measure.

We are now at a position to state the key theorem for our purposes, a proof of which may be found in any text on financial applications of Brownian Motion. In what follows, if a process is a Brownian Motion under a probability $\mathbb{M}$, we call it an $\mathbb{M}$-Brownian Motion.

Theorem 2.1.2 (Girsanov's Theorem) *Let $W_P(t)$ be a $\mathbb{P}$-Brownian Motion, and $\theta(t)$ be an $\mathbb{F}$-adapted process with $\int_0^t \theta(s)ds < \infty$ $\mathbb{P}$-a.s. Then there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that*

$$W_Q(t) = W_P(t) + \int_0^t \theta(s)ds \tag{2.3}$$

is a $\mathbb{Q}$-Brownian Motion.

The Radon-Nikodým Derivative is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(t) = \exp\Big\{ - \int_0^t \theta(s)dW_P(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\Big\}. \tag{2.4}$$

This allows us to change between equivalent measures $\mathbb{P}$ and $\mathbb{Q}$. We will apply Girsanov's theorem throughout to change from the 'real world' measure to a 'risk-neutral' measure where we can price claims.

2.2 Market Model

While we consider specific models below—namely the Black-Scholes model and a Stochastic Volatility model—most of the methods applied derive from a far more general setting. Let us now describe this setting.

We take as our most general model that which is largely described in Karatzas (1997), and replicated in Ben Ameur *et. al.* (1999). Let $\mathcal{M}$ be a market containing $N+1$ *tradable* assets. There may exist other assets in out market, but they may not be traded—that is, they may not be bought or sold for a price. We assume the existence of a risk-free interest rate, $r(t)$, in our market. In practice this is often taken to be the zero rate for money lent for the period $[0, T]$. The first tradable asset describes the deterministic evolution of an amount borrowed or lent at this rate, and its value at time t follows the differential equation

$$dB(t) = B(t) \cdot r(t)\ dt. \tag{2.5}$$

This asset is termed a *bond*. With the value $B(0) = 1$, the solution to this equation enables us to derive the *discount factor*, a multiplicative factor that represents the *present value*, or value at time 0, of \$1 held at time t:

$$D(t) := \frac{1}{B(t)} = e^{-\int_0^t r(s)\ ds}, \qquad t > 0.$$

We assume throughout this paper than any amount may be borrowed or lent at this risk-free rate, and that there are no transaction costs, taxes or other frictions associated with trading in the bond. We further assume throughout this paper that the risk-free rate $r(t) \equiv r,\ \forall t$.

Our other N assets in $\mathcal{M}$ are called *risky* assets. Risky here corresponds to the assumption that the prices of these assets do not evolve deterministically. We assume that their paths evolve according to the following stochastic differential equations under the "real world" probability measure $\mathbb{P}$:

$$dS_i(t) = \mu_i S_i(t)\ dt + S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)\ dW_j^P(t), \qquad i = 1, \ldots, N. \tag{2.6}$$

The vector of constants $\mu_i > 0\ \forall i$, give the *appreciation* rates of the assets $S_i(t)$, which, in the absence of randomness, would be the instantaneous continuously compounded rates of interest, in the same way as r for the bond. The term $\mu_i S_i(t)$ gives the *drift* coefficient of the trajectory of $S_i(t)$.

We assume that our assets—including the bond—do not pay dividends. This we can do without loss of generality, since to include dividends we may let $\mu_i = \mu_i^* - q_i\ ,\ i = 1, \ldots, N$, where $q = \{q_1, \ldots, q_N\}$ is the vector of the dividend yields of our assets, and $\mu^* = \{\mu_1^*, \ldots, \mu_N^*\}$ is the vector of appreciation rates of the asset prices before losses paid in dividends.

The components of $W^P(t) = \{W_1^P(t), \ldots, W_d^P(t)\}$ are independent $\mathbb{P}$-Brownian Motions, and the matrix of *volatility* $(\sigma_{ij}(t))$ gives the $\mathbb{F}$-adapted coefficients of these Brownian Motions. The term $S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)$ is called the *diffusion* coefficient of the trajectory of $S_i(t)$.

It is important to note that the condition that our coefficients be $\mathbb{F}$-adapted prevents the prediction of the future, a desirable property for an asset-price model.

2.3 Portfolios

We assume the existence of a *small investor* in our market. This investor is small in the sense that he or she is free to trade in the assets of the market without affecting their prices.

Our investor invests in a *portfolio*, which is the number $b(t)$ of bonds held or borrowed at time t, plus a vector process $\pi(t) = \{\pi_1(t), \dots, \pi_N(t)\}$, representing the number of each risky asset i, $i = 1, \dots, N$, held at time t. The wealth $X(t)$ of our investor at time t is given by the value of the portfolio at time t:

$$X(t) = b(t)B(t) + \sum_{i=1}^{N} \pi_i(t)S_i(t).$$

A portfolio $\pi(t)$ is called *self-financing* if the wealth process $X(t)$ obeys the following:

$$X(t) = X(0) + \int_0^t b(s)dB(s) + \sum_{i=1}^{N} \int_0^t \pi_i(s)dS_i(s).$$

A self-financing portfolio has the property that all returns from the sale of risky assets are invested instantaneously into the bond and vice versa, and conversely for losses. No more money is added to or taken from the portfolio after the initial wealth $X(0)$.

Definition 2.3.1 (Arbitrage Opportunity) *A self-financing portfolio is called an* arbitrage opportunity *if*

$$X(0) = 0, \qquad \mathbb{P}(X(t) \geq 0) = 1, \qquad \mathbb{P}(X(t) > 0) > 0, \qquad \exists t > 0.$$

This observation is the cornerstone of claim pricing theory. We judge the fair price of a claim to be the price that admits no arbitrage opportunities. A consequence of this is that asset rates of return greater than the risk free rate r cannot be achieved without assumption of risk. To demonstrate how we will use arbitrage in the pricing of assets we have the following theorem:

Theorem 2.3.1 *Let $A(t)$ be the price of an asset at time t, $0 < t < T$, where $A(t)$ is a deterministic function of t. If we allow trading in this asset and the risk free bond, then the price of the asset at time* 0 *that admits no arbitrage is*

$$A(0) = e^{-rT}A(T).$$

Proof. Suppose that $A(0) > e^{-rT}A(T)$. Then at time 0 we form a portfolio by borrowing $A(0)$ dollars at the risk free rate r, and selling short the asset. This is indeed a self-financing portfolio, with wealth $X(0) = 0$. Our wealth at time T is then

$$X(T) = e^{rT}A(0) - A(T) > A(T) - A(T) = 0 \text{ with probability 1.}$$

But this is an arbitrage opportunity. Conversely, if $A(0) < e^{-rT}A(T)$ we borrow $A(0)$ and purchase the asset, giving a return $A(T) - e^{rT}A(0) > 0$ with probability 1. The only other possibility is $A(T) = e^{rT}A(0)$.

□

Necessary and sufficient conditions for our market $\mathcal{M}$ to be arbitrage-free are given by the following, from Karatzas (1997), which we state without proof:

Theorem 2.3.2 (Arbitrage Free Market) *Let there be a market $\mathcal{M}$ described by (2.5) and (2.6).*
(i) If $\mathcal{M}$ is arbitrage-free, then there exists an $\mathbb{F}$-adapted N-dimensional vector process $\theta(t) = \{\theta_i(t) : i = 1, \dots, N\}$, called the market price of risk*, such that*

$$\theta_i(t)\sigma_{ij}(t) = \mu_i - r, \qquad \forall i = 1, \dots, N, \qquad 0 \le t \le T \qquad a.s.$$

(ii) If $\theta(t)$ exists and satisfies condition (i) and the Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T ||\theta(t)||^2 dt\right\}\right] < \infty, \tag{2.7}$$

where $|| \cdot ||$ denotes the Euclidean norm, then $\mathcal{M}$ is arbitrage-free.

Suppose $N \le d$. Then $\mathcal{M}$ is said to be *standard*. If $N > d$ we can write $N - d$ assets as linear combinations of the other d assets, and reduce the number of assets traded in the market to N to force standardization. It is a property of standard markets that there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which $e^{-rt}S_i(t)$ is a Martingale. The process

$$W_i^Q(t) = W_i^P(t) + \int_0^t \theta_i(s)ds \tag{2.8}$$

is a $\mathbb{Q}$-Brownian Motion by Girsanov's Theorem, and

$$dS_i(t) = rS_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}S_i(t)dW_j^Q(t) \tag{2.9}$$

gives the evolution of the price of each risky asset i under $\mathbb{Q}$. We note that under $\mathbb{Q}$ the drift coefficient of the asset price is $rS_i(t)$, hence $\mathbb{Q}$ is called the "risk-neutral" probability measure.

We note that the discounted asset price $e^{-rT}S_i(t)$ is a $\mathbb{Q}$-Martingale, so $\mathbb{Q}$ is also known as the *Equivalent Martingale Measure.*

The Radon-Nikodŷm Derivative is given by

$$Z(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}(t) = \exp\Big\{ - \int_0^t \theta(s)dW^P(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\Big\}. \tag{2.10}$$

2.4 Claims: Pricing & Hedging

Definition 2.4.1 (Contingent Claim) *In a standard market, an $\mathcal{F}_T$-measurable random variable $Y : \Omega \to [0, \infty)$, is the payoff of a* contingent claim *if* $\mathbb{E}^Q[Y] < \infty$.

The holder of a claim recieves a non-negative amount Y at or before time T—the *expiry* of the claim—from the seller, or *writer* of the claim. The holder pays a premium at time 0 for the right to hold the claim, which is given by the following, a proof of which may be found in Karatzas (1997). We state it here without proof:

Theorem 2.4.1 (Contingent Claim Price) *Assume that the conditions of Theorem 2.3.2 are satisfied. The price of a contingent claim with payoff Y is given by*

$$X(0) = e^{-rT}\mathbb{E}^Q[Y|\mathcal{F}_0] = e^{-rT}\mathbb{E}^P\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}(T)Y|\mathcal{F}_0\Big].$$

The problem we face in the general standard market is that the measure $\mathbb{Q}$ is not necessarily unique, and therefore neither is the price.

Definition 2.4.2 (Attainable and Complete) *A claim with payoff Y is called* attainable *if there exists a self-financing portfolio $(b(t), \pi(t))$ such that the wealth process $X(t)$ for the portfolio obeys*

$$X(T) = Y, \qquad \mathbb{Q}\text{-}a.s.$$

If all contingent claims are attainable in a market, then the market is said to be complete.

The following result is central to this thesis. We state the following theorem, found in Karatzas (1997), without proof:

Theorem 2.4.2 *A standard market $\mathcal{M}$ is complete iff $N = d$ and $\sigma(t)$ is non-singular.*

This is equivalent to the assertion that there exists a *unique* equivalent measure $\mathbb{Q}$ under which $e^{-rT}S_i(t)$ is a $\mathbb{Q}$-Martingale $\forall i = 1, 2, \ldots, N$.

The self-financing portfolio that attains value Y at time T is called the *replicating portfolio* or *perfect hedge.* If we hold this hedge, we replicate the value of the claim. For writers of claims—generally financial intermediaries—hedging claims by the construction of replicating portfolios is essential to minimise exposure to unfavorable price movements of underlying risky assets.

An example of a contingent claim is the so-called *option.* An option gives the holder the right, but not the obligation, to buy or sell an asset, called the *underlying* asset, for some predetermined price, called the *strike*, at or before some time of *expiry* or *maturity.* If the holder of the claim uses this right and trades the asset, he is said to have *exercised* the option. Note that the holder of the claim can choose to not exercise in unfavorable economic circumstances, so his losses are minimised to the cost of the claim, paid at purchase.

The standard options, called *vanilla* options, are either *European* or *American.* A European option may only be exercised at the expiry date, while an American option may be exercised at any time before expiration. They come in two main types: A *call* option is the right to buy an asset at the fixed strike price at or before the fixed expiry. A *Put* option is the right to sell at the strike at or before expiry. Claims with more general characteristics are called *exotic* options.

The price of an option is the present value of its expected future cashflow under the risk-neutral measure $\mathbb{Q}$, given by Theorem 2.4.1. In the following, let K denote the strike price, and T be the expiration date. For an underlying asset with price $S(t)$ at time t, the payoffs of European call and put options are as follows:

$$\begin{aligned} Y_{call} &= (S(T) - K)^+ = \max\{S(T) - K, 0\}, \\ Y_{put} &= (S(T) - K)^- = -\min\{S(T) - K, 0\} = \max\{K - S(T), 0\}. \end{aligned}$$

The fair values of these claims at time t, $0 \leq t \leq T$, are given by

$$\begin{aligned} X_{call}(t) &= e^{-r(T-t)}\mathbb{E}^Q[(S(T) - K)^+ | \mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}^Q[(S(T) - K)\mathbf{1}_{\{S(T)>K\}} | \mathcal{F}_t], \\ X_{put}(t) &= e^{-r(T-t)}\mathbb{E}^Q[(S(T) - K)^- | \mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}^Q[(K - S(T))\mathbf{1}_{\{S(T)<K\}} | \mathcal{F}_t]. \end{aligned}$$

One of the key results of European option pricing is the so-called Put-Call Parity relationship.

Theorem 2.4.3 (Put-Call Parity) *Let $X_{call}(t)$ and $X_{put}(t)$ denote the prices at time t of a European call and put option respectively, with the same expiry T and strike K. Then*

$$X_{call}(t) - X_{put}(t) = S(t) - Ke^{-r(T-t)}. \tag{2.11}$$

Proof. The proof simply consists of application of Theorem 2.4.1:

$$\begin{aligned} X_{call}(t) - X_{put}(t) &= e^{-r(T-t)}\mathbb{E}^Q[(S(T)-K)^+ - (S(T)-K)^-|\mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}^Q[S(T)-K|\mathcal{F}_t] \\ &= S(t) - Ke^{-r(T-t)}, \end{aligned}$$

since $e^{-rt}S(t)$ is a $\mathbb{Q}$-Martingale, and K is constant.

□

This enables us to find the price of the put from the call price, and vice versa.

We use the notation X to denote the price of a claim to highlight the fact that the price of a claim is the price to purchase the perfect hedge portfolio at time 0.

Definition 2.4.3 (Partial Hedge) *Suppose we have a claim with payoff Y, and let $A = \{Y > 0\}$. The arbitrage-free price of the claim and the value of the perfect hedge portfolio at time t is $X^A(t) = e^{-r(T-t)}\mathbb{E}^Q[Y\mathbf{1}_A|\mathcal{F}_t]$. Let $B \subset A$. Then the replicating portfolio for the "partial" claim on B has value $X^B(t) = e^{-r(T-t)}\mathbb{E}^Q[Y\mathbf{1}_B|\mathcal{F}_t]$ at time t, $0 \le t \le T$, and is called a* partial hedge.

Note that

$$\begin{aligned} X^B(t) &= e^{-r(T-t)}\mathbb{E}^Q[Y\mathbf{1}_A\mathbf{1}_B|\mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbb{E}^Q[Y\mathbf{1}_A \cdot (1-\mathbf{1}_{A\backslash B})|\mathcal{F}_t] \\ &= X^A(t) - e^{-r(T-t)}\mathbb{E}^Q[Y\mathbf{1}_{A\backslash B}|\mathcal{F}_t]. \end{aligned} \tag{2.12}$$

If $\mathbb{Q}(A\backslash B) > 0$ then $X^B(t) < X^A(t)$.

While this hedge is indeed cheaper than the perfect hedge, the holder of the partial hedge defaults on the set $A\backslash B = \{Y > 0\}\backslash B$. The idea of partially hedging claims is to construct a set B sufficiently smaller than A to recieve the known and acceptable initial gain $X^A(0) - X^B(0)$ and an acceptable real-world probability of default $\mathbb{P}(A\backslash B)$.

In what follows, we do not consider the construction of hedges, perfect or partial. We merely state that if the conditions for the construction of unique perfect hedges are satisfied, then we may price claims and calculate the probabilities of default.

Chapter 3

Literature Review

The celebrated paper of Black and Scholes (1973), together with the combined works of Merton, accumulated in Merton (1990), heralded the birth of modern mathematical finance, by systematically studying the effects on claims pricing of the assumption of mathematical models of asset prices. The models make use of the techniques of stochastic calculus, and its heart, the Brownian Motion process. For comprehensive accounts of these techniques, see Gikhman and Skorokhod (1969), Karatzas and Shreve (1998) and Shreve (1996). Texts of a more practical than theoretical nature include Bingham and Kiesel (1998), Musiela and Rutkowski (1997), Hull (2000), Meyer (2001) and Klebaner (1998).

The initial model assumed for the asset price process was Geometric Brownian Motion, convenient since it is a simple case of the stochastic exponential, and a closed form solution is known. Further, the market of the Black Scholes model is complete, and therefore claims can be priced by discounting their expected payoff under a *unique* risk neutral measure. However, consideration of trading realities has rendered the Black Scholes model largely inaccurate, and its use has been restricted to the foundations, methodologies and points of reference for claim prices under more accurate models.

Hull and White (1987) adapted the Black Scholes approach by considering firstly the volatility parameter[1] σ of Black Scholes to be *deterministic*, not constant, and secondly allowing σ to be a random process. Such models are the so-called *Stochastic Volatility* models. Bingham and Kiesel (1998) list a number of volatility process, proposed in order to increase consistency with trading practice. The sacrifice made for consistency is that often the markets under these models are incomplete, and adjustments must be made to price claims. Karatsas (1997) generalises the approach of Black Scholes to a market

[1] The Black Scholes model is a special case of our market $\mathcal{M}$ with $N = d = 1$. In this case, the volatility matrix (σ_{ij}), $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, d$ is a single parameter σ. Black and Scholes took this σ to be constant for $t \in [0, T]$, that is, over the life of a contingent claim.

with arbitrary numbers of tradable assets and sources of uncertainty, and we adopt his arguments and add tradable assets to 'complete' our stochastic volatility market.

Note that if asset volatility was tradable then we could complete a single asset stochastic volatility model easily. In 1993 the Chicago Board Options Exchange (CBOE) introduced a volatility index on assets in the S&P 100 index, and this index provides a tradable volatility for pricing options on the S&P 100 index. An extension to equities is the obvious purpose and aim of this index. Detemple and Osakwe (1999) and Brenner, Ou and Zhang (2004) look at possible methods and implications of tradable volatility.

Another problem relating to market incompleteness is that of hedging of claims. Equivalent to the existence of a risk neutral measure is the ability to construct perfect hedges, and in incomplete markets, such hedges are not known to exist. Many methodologies have been developed to essentially minimise rather than eliminate the risk of holding an imperfect hedge. Such approaches generally seek to find a hedge that will minimise the mean squared error over the claim price, or they assume a utility function for price returns to somehow optimise hedge construction. See for example Grasselli and Hurd (2004) and Schweitzer (1993). Note that in what follows we do not consider the explicit construction of the hedges we describe.

The calculation of claim prices under stochastic volatility models is generally done by Monte Carlo simulation. This involves approximating the continuous asset price trajectory by restricting consideration to a discrete set of times. The Euler method of approximating differential equations is trivially extended to the case of stochastic differential equations, and higher order approximations are able to be made in the conventional ways. For a thorough study of these ways see Kloedin and Platen (1992), Broadie and Glasserman (1996) and Boyle, Broadie and Glasserman (1997).

The claim price is a function of the asset price trajectory. In the case of European vanilla calls, the function is simply the discounted positive part of the difference between the terminal price and a strike price. The terminal price is taken to be the last value of the approximated trajectory. In what follows we also consider the maximum asset price over the life of the claim, and for this purpose we make use of the results of Beaglehole, Dybvig and Zhou (1997). See also Metwally and Atiya (2002).

Another approach we use to approximate the asset price trajectory involves taking known properties of correlation between estimators of claim prices to induce a variance reduction. The method of Antithetic variates involves using the average of the claim prices under two negatively correlated random drivers. The method of Control variates involves considering the correlation between the price process under stochastic volatility with the known price under Geometric Brownian Motion. See Fouque and Tullie (2002), Lavenberg and Welch (1981) and Nelson (1990).

A further approach to variance reducion uses the double expectation law to write the claim price as the expectation over volatility trajectories of claim prices conditional on fixed volatility. In simulation, this is simply a Monte Carlo average of prices under Geometric Brownian Motion, making use of the existence of closed form solutions to reduce computational requirements. This approach uses the Mixing theorem of Romano and Touzi (1997), and is summarised and implemented in Lewis (2002). This approach seems to work work for payoff functions based on terminal asset prices, but not for those based on price trajectories. In what follows we focus substantially on partial hedging strategies based on these trajectories, so the mixing approach cannot be used for our purposes.

In this study, simulations have been perfomed with the aid of Microsoft Visual Basic for Applications 6.0. General assistance with writing user-defined functions may be found in Benninga (2000) and Jackson and Staunton (2002).

This thesis generally extends and adapts the approach and methodology of the paper of Ben Ameur, Breton and L'Écuyer (1999).

Chapter 4

Partial Hedging Under Geometric Brownian Motion

We initially consider partial hedges and associated risks for a market where $d = N = 1$, and $\sigma(t) \equiv \sigma$. This is the classic model of Black and Scholes (1973). Our single risky asset follows the stochastic differential equation under $\mathbb{P}$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^P(t), \qquad 0 \leq t \leq T. \tag{4.1}$$

This process is known as the *Geometric Brownian Motion*.

We obtain the market price of risk,

$$\theta = \frac{\mu - r}{\sigma}.$$

This indeed satisfies the Novikov condition (2.7), and therefore by Theorems 2.3.2 and 2.4.2 the market is arbitrage free and complete. Therefore there exists a unique equivalent martingale measure $\mathbb{Q}$, and a unique perfect hedge and claim price.

The asset price obeys the following equation under the risk neutral measure $\mathbb{Q}$:

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t), \qquad 0 \leq t \leq T. \tag{4.2}$$

The solution to this equation, which is an example of the *stochastic exponential* is given by

$$S(t) = S(0)\exp\left\{(r - \frac{1}{2}\sigma^2)t + \sigma W^Q(t)\right\}.$$

This allows us to derive the famous Black Scholes price of a European call at time 0:

$$X^{BS}_{call}(K, S(0), T) = S(0)N(d_+) - Ke^{-rT}N(d_-),$$

where

$$d_-(S(0), K, T) = \frac{1}{\sigma\sqrt{T}}\Big[\ln\frac{S(0)}{K} + (r - \frac{1}{2}\sigma^2)T\Big], \qquad d_+ = d_- + \sigma\sqrt{T}.$$

Details of a derivation of this formula are given in Appendix A.

In what follows, we consider only call options. This we may do without loss of generality, since we can recover the put prices by Put-Call Parity (2.11). We will also restrict ourselves to the consideration of European vanilla calls. Recall that the payoff Y for a European vanilla call is

$$Y_{call} = (S(T) - K)^+ = (S(T) - K)\mathbf{1}_{\{S(T)>K\}} = (S(T) - K)\mathbf{1}_A,$$

and for a partially hedged European vanilla call

$$Y^B_{call} = (S(T) - K)\mathbf{1}_B,$$

where $B \subset A$. In considering the price of constructing a partial hedge, we consider only the price *now*, i.e. for $t = 0$. With this in mind use the following notation for brevity:

$$X^B_{call}(K, S(0), T) \equiv X^B(K, S(0))$$

for a partial hedge of a European call on the set B. We omit $(K, S(0))$ often, as it is generally obvious from the context. The set B generally has parameters, and the set will be explicitly denoted $B(\cdot)$ if not obvious from context. The payoff function Y will follow the same notational conventions as X, as will $d_\pm$.

If $B \equiv A$, then we have a perfect hedge, and the Black-Scholes result. We denote

$$X^A \equiv X_{BS}.$$

We consider the cases in detail here. As we will see in later sections, the solutions in the Black Scholes case may easily be applied in a more general framework.

4.1 Partial Hedging on $B_1(a) = \{K < S(T) < a\}$, $K < a$

We first consider a hedge with an upper bound on the asset price at expiration. This is given by the set $B_1(a)$ above. Note that we make no restrictions on $S(t)$ for $0 \le t < T$.

The fair price of a European call partially hedged on $B_1(a)$, at time 0, is

$$\begin{aligned}
X^{B_1(a)}(K) &= e^{-rT}\mathbb{E}^Q[(S(T)-K)\mathbf{1}_{B_1}] = e^{-rT}\mathbb{E}^Q[(S(T)-K)\mathbf{1}_A\mathbf{1}_{B_1}] \\
&= e^{-rT}\mathbb{E}^Q[(S(T)-K)\mathbf{1}_A] - e^{-rT}\mathbb{E}^Q[(S(T)-K)\mathbf{1}_{A\backslash B_1}] \\
&= X_{BS}(K) - X_{BS}(a) - e^{-rT}(a-K)\mathbb{E}^Q[\mathbf{1}_{A\backslash B_1}] \\
&= X_{BS}(K) - X_{BS}(a) - e^{-rT}(a-K)\mathbb{Q}(A\backslash B_1).
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{Q}(A\backslash B_1(a)) &= \mathbb{Q}(S(T)\geq a) = \mathbb{Q}\Big(S(0)\exp\Big\{\Big(r-\frac{1}{2}\sigma^2\Big)T + \sigma W^Q(T)\Big\}\geq a\Big) \\
&= \mathbb{Q}\Big(W^Q(1)\geq \frac{1}{\sigma\sqrt{T}}\Big[\ln\frac{a}{S(0)} - \Big(r-\frac{1}{2}\sigma^2\Big)\Big]\Big) \\
&= \mathbb{Q}(W^Q(1)\geq -d_-(a)).
\end{aligned}$$

Since $W^Q(1)\sim N(0,1)$, we have

$$\mathbb{Q}(A\backslash B_1(a)) = N(d_-(a)).$$

So

$$\begin{aligned}
X^{B_1(a)} &= S(0)N(d_+(K)) - Ke^{-rT}N(d_-(K)) - S(0)N(d_+(a)) \\
&\quad + ae^{-rT}N(d_-(a)) - (a-K)e^{-rT}N(d_-(a)) \\
&= S(0)\big[N(d_+(K)) - N(d_+(a))\big] - Ke^{-rT}\big[N(d_-(K)) - N(d_-(a))\big]. \qquad (4.3)
\end{aligned}$$

This explicit solution tells us the value of the partial hedge for all hedge parameters $a > K$. Note that as $a\to\infty$, $X^{B_1(a)}\to X_{BS}$.

We next find the real-world probability of default:

$$\mathbb{P}(A\backslash B_1(a)) = \mathbb{E}^P[\mathbf{1}_{A\backslash B_1(a)}] = \mathbb{E}^Q\Big[\frac{d\mathbb{P}}{d\mathbb{Q}}(T)\mathbf{1}_{A\backslash B_1(a)}\Big] = \int_{A\backslash B_1(a)}\frac{d\mathbb{P}}{d\mathbb{Q}}(T)d\mathbb{Q}.$$

Note that

$$\begin{aligned}
S(T)\geq a &\Leftrightarrow S(0)\exp\Big\{\Big(r-\frac{1}{2}\sigma^2\Big)T+\sigma W^Q(T)\Big\}\geq a \\
&\Leftrightarrow W^Q(T)\geq\frac{1}{\sigma}\Big[\ln\frac{a}{S(0)} - \Big(r-\frac{1}{2}\sigma^2\Big)T\Big] \\
&\Leftrightarrow W^Q(T)\geq -d_-(a)\sqrt{T},
\end{aligned}$$

and

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(T) = e^{-\frac{1}{2}\theta^2T+\theta W^Q(T)}, \qquad \theta = \frac{\mu-r}{\sigma}.$$

So, since $W^Q(T) \sim \sqrt{T} W^Q(1)$ and $W^Q(1) \sim N(0,1)$,

$$\begin{aligned}
\mathbb{P}(A \backslash B_1(a)) &= \int_{-d_-(a)}^{\infty} e^{-\frac{1}{2}\theta^2 T + \theta\sqrt{T}x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
&= \int_{-d_-(a)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta\sqrt{T})^2} dx \\
&= N(d_-(a) + \theta\sqrt{T}) = N\left(\frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S(0)}{a} + \left(\mu - \frac{1}{2}\sigma^2\right)T\right]\right) \\
&= N(d_-^{\mu}(a)), \qquad (4.4)
\end{aligned}$$

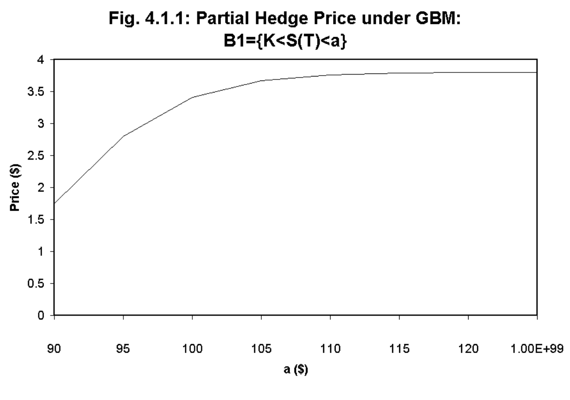
where the superscript μ indicates that we replace the r in the expression for $d_-(a)$ with μ. We find that as $a \to \infty$, $\mathbb{P}(A \backslash B_1) \to 0$. Indeed, in the limit $a \to \infty$ the cost of the partial hedge is the cost of the perfect hedge—i.e. the Black Scholes price, and the hedge has a risk of default equal to 0.

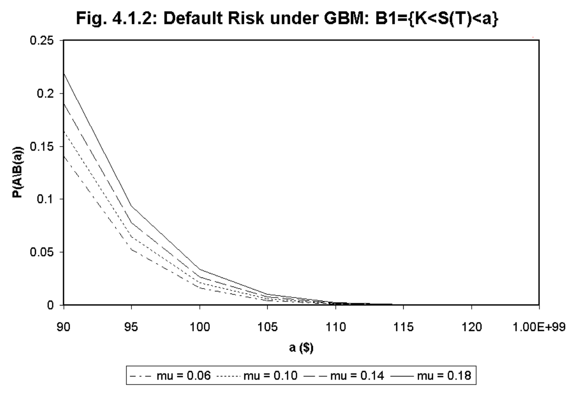
With the expressions for $X^{B_1(a)}$ and $\mathbb{P}(A \backslash B_1(a))$, we can analytically evaluate the consequences of a partial hedging strategy of the kind $B_1(a) = \{K < S(T) < a\}$, where the asset price $\{S(t)\}$ follows the Geometric Brownian Motion.

We consider an asset with $S(0) = 80$ and $\sigma = 0.20$, and a risk free rate $r = 0.06$. We also consider a European call option with strike price $K = 80$ and maturity in 3 months, or $T = 0.25$.

The partial hedge parameter a is allowed to vary, taking values $a = 90, 95, 100, 110, 120$ and ∞. The case $a = \infty$ gives of course, the Black Scholes price. We make our calculations with asset appreciation rates equal to $\mu = 0.06, 0.10, 0.14, 0.18$ under $\mathbb{P}$. We calculate the values for $a = \infty$ using $a = 10^{99}$ for practical reasons.

We obtain the following graphs. Note that the values for $a = 10^{99}$ have been plotted next to those of $a = 120$ for practical reasons. The trend should be obvious since the difference is slight.





Note that Fig. 4.1.2 indicates that $\mathbb{P}(A\backslash B_1(a))$ increases with μ. Note also the figures indicate that $\mathbb{P}(A\backslash B_1(a)) \to 0$ and $X^{B_1(a)} \to X_{BS}$ as $a \to \infty$.

The data table used to produce the above figures may be found in Appendix C.1.

4.2 Partial Hedging on $B_2(a,b) = \{K < S(T) < a, \max_{0\le t\le T} S(t) < b\}$, $K < a < b$

We consider a partial hedge on the set $B_2(a,b)$ defined above, with an upper bound on the asset price at the claim's expiration date, and an upper bound on the maximum of the asset price trajectory.

Recall that under $\mathbb{Q}$,

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t).$$

Then

$$S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W^Q(t)\right\} = S(0)\exp\left\{\sigma\left[\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)t + W^Q(t)\right]\right\}.$$

Let

$$\phi = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right), \qquad W^{Q^*}(t) = \phi t + W^Q(t), \tag{4.5}$$

where $\mathbb{Q}^* \sim \mathbb{Q}$. Then by Girsanov's Theorem, $W^{Q^*}(t)$ is a $\mathbb{Q}^*$-Brownian Motion. We have the Radon-Nikodỳm Derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*}(t) = e^{\phi W^{Q^*}(t) - \frac{1}{2}\phi^2 t}.$$

Our price process follows

$$S(t) = S(0)\exp\left\{\sigma W^{Q^*}(t)\right\},$$

so

$$W^{Q^*}(t) = \frac{1}{\sigma}\ln\frac{S(t)}{S(0)},$$

and our maximum price is

$$\max_{0\le t\le T} S(t) = S(0)\exp\left\{\sigma \max_{0\le t\le T} W^{Q^*}(t)\right\},$$

so

$$\max_{0\le t\le T} W^{Q^*}(t) = \frac{1}{\sigma}\ln\frac{\max_{0\le t\le T} S(t)}{S(0)}.$$

The joint density function of $\big(W^{Q^*}(T), \max_{0\le t\le T} W^{Q^*}(t)\big)$ can be found in Karatzas and Shreve (1998), and is given by

$$\mathbb{Q}^*\Big[W^{Q^*}(T) \in dx, \max_{0\le t\le T} W^{Q^*}(t) \in dy\Big] = \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\Big\{-\frac{(2y-x)^2}{2t}\Big\} dxdy.$$

The price of our partially hedged call is then

$$\begin{aligned} X^{B_2(a,b)} &= e^{-rT}\mathbb{E}^Q\Big[\big(S(T)-K\big)\mathbf{1}_{B_2}\Big] \\ &= e^{-rT}\mathbb{E}^{Q^*}\Big[\big(S(0)e^{\sigma W^{Q^*}(T)}-K\big)\frac{d\mathbb{Q}}{d\mathbb{Q}^*}(T)\mathbf{1}_{B_2}\Big]. \end{aligned}$$

Now, the set

$$\begin{aligned} B_2 &= \big\{K < S(T) < a, \max_{0\le t\le T} S(T) < b\big\} \\ &= \Big\{\frac{1}{\sigma}\ln\frac{K}{S(0)} < W^{Q^*}(T) < \frac{1}{\sigma}\ln\frac{a}{S(0)}, \max_{0\le t\le T} W^{Q^*}(t) < \frac{1}{\sigma}\ln\frac{b}{S(0)}\Big\}. \end{aligned}$$

We will be using the notation

$$\upsilon^* = \frac{1}{\sigma}\ln\frac{\upsilon}{S(0)}, \qquad \upsilon \in \{K, a, b\}.$$

Then

$$
\begin{aligned}
X^{B_2} &= e^{-rT}\int_{B_2(a,b)}\big(S(T)-K\big)\frac{d\mathbb{Q}}{d\mathbb{Q}^*}(T)d\mathbb{Q}^* \\
&= e^{-rT}\int_{K^*}^{a^*}\int_{x}^{b^*}\big(S(0)e^{\sigma x}-K\big)e^{\phi x-\frac{1}{2}\phi^2T}\frac{2(2y-x)}{\sqrt{2\pi T^3}}e^{-\frac{1}{2T}(2y-x)^2}dxdy \\
&= \frac{2}{\sqrt{2\pi T^3}}e^{-rT-\frac{1}{2}\phi^2T}\int_{K^*}^{a^*}\big(S(0)e^{\sigma x}-K\big)e^{\phi x}\Big[\int_{x}^{b^*}(2y-x)e^{-\frac{1}{2T}(2y-x)^2}dy\Big]dx \\
&= \frac{1}{\sqrt{2\pi T}}e^{-rT-\frac{1}{2}\phi^2T}\int_{K^*}^{a^*}\big(S(0)e^{\sigma x}-K\big)e^{\phi x}(e^{-\frac{1}{2T}x^2}-e^{-\frac{1}{2T}(2b^*-x)^2})dx \\
&= \frac{1}{\sqrt{2\pi T}}e^{-rT-\frac{1}{2}\phi^2T}\int_{K^*}^{a^*}\Big[S(0)e^{(\phi+\sigma)x-\frac{1}{2T}x^2}-S(0)e^{(\phi+\sigma)x-\frac{1}{2T}(2b^*-x)^2} \\
&\quad -Ke^{\phi x-\frac{1}{2T}x^2}+Ke^{\phi x-\frac{1}{2T}(2b^*-x)^2}\Big]dx \\
&= S(0)\Big[N\Big(d_+(K)\Big)-N\Big(d_+(b)\Big)\Big] \\
&\quad -S(0)e^{2b^*(\phi+\sigma)}\Big[N\Big(d_+\Big(\frac{KS(0)^2}{b^2}\Big)\Big)-N\Big(d_+\Big(\frac{S(0)^2}{b}\Big)\Big)\Big] \\
&\quad -Ke^{-rT}\Big[N\Big(d_-(K)\Big)-N\Big(d_-(b)\Big)\Big] \\
&\quad +Ke^{-rT+2b^*\phi}\Big[N\Big(d_-\Big(\frac{KS(0)^2}{b^2}\Big)\Big)-N\Big(d_-\Big(\frac{S(0)^2}{b}\Big)\Big)\Big]. \qquad (4.6)
\end{aligned}
$$

The price $\lim_{a\to b}X^{B_2(a,b)}$ gives the price of a European up-and-out barrier call. This is an option that pays the positive part of the difference between the asset price at expiration and the strike price, *conditional* on the asset never reaching the barrier level $b > K$. It is *up* since the asset price must rise if it is to meet the barrier, and it is *out* since the option is worthless if the barrier is reached.

The default risk for the hedge on B_2 is given by the following equality:

$$\mathbb{P}(A\backslash B_2) = \mathbb{E}^P[\mathbf{1}_A] - \mathbb{E}^P[\mathbf{1}_{B_2}].$$

Similar arguments as in section 4.1 lead to

$$\mathbb{E}^P[\mathbf{1}_A] = N(d_-^{\mu}(K)),$$

and the second term is

$$
\begin{aligned}
\mathbb{E}^P[\mathbf{1}_{B_2}] &= \mathbb{E}^{Q^*}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\frac{d\mathbb{Q}}{d\mathbb{Q}^*}\mathbf{1}_{B_2}\right] \\
&= \int_{B_2(a,b)} \frac{d\mathbb{P}}{d\mathbb{Q}}\frac{d\mathbb{Q}}{d\mathbb{Q}^*} d\mathbb{Q}^* \\
&= \int_{K^*}^{a^*}\int_x^{b^*} e^{-\frac{1}{2}\theta^2 T+\theta(x-\phi T)} e^{\phi x - \frac{1}{2}\phi^2 T} \frac{2}{\sqrt{2\pi T^3}}(2y-x)e^{-\frac{1}{2T}(2y-x)^2} dx dy \\
&= \frac{2}{T\sqrt{2\pi T}} e^{-\frac{1}{2}T(\theta+\phi)^2} \int_{K^*}^{a^*} e^{(\theta+\phi)x}\left[\int_x^{b^*}(2y-x)e^{-\frac{1}{2T}(2y-x)^2}dy\right]dx \\
&= \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}T(\theta+\phi)^2} \int_{K^*}^{a^*} e^{(\theta+\phi)x}\left[e^{-\frac{1}{2T}x^2} - e^{-\frac{1}{2T}(2b^*-x)^2}\right]dx
\end{aligned}
$$

$$
= N\Big(d_-^\mu(K)\Big) - N\Big(d_-^\mu(a)\Big) - e^{2b^*(\theta+\phi)}\left[N\Big(d_-^\mu\Big(\frac{KS(0)^2}{b^2}\Big)\Big) - N\Big(d_-^\mu\Big(\frac{aS(0)^2}{b^2}\Big)\Big)\right].
$$

Therefore

$$
\mathbb{P}(A\backslash B_2) = N\Big(d_-^\mu(a)\Big) + e^{2b^*(\theta+\phi)}\left[N\Big(d_-^\mu\Big(\frac{KS(0)^2}{b^2}\Big)\Big) - N\Big(d_-^\mu\Big(\frac{aS(0)^2}{b^2}\Big)\Big)\right]. \tag{4.7}
$$

In this case, $\lim_{a\to b}\mathbb{P}(A\backslash B_2)$ gives the risk of default when partially hedging a written vanilla call by holding an up-and-out barrier call.

We consider the asset and option of section 3.1, with $S(0) = 80$, $\sigma = 0.20$ and $r = 0.06$, strike price $K = 80$ and expiry $T = 0.25$. We vary a and b to measure the effect on the partial hedge price. We also calculate the default risk for different values of the asset appreciation rate μ and the hedge parameter b. For the default risk calculations, we let $a = 100$.

The first graph shows that the prices of X^{B_1} and X^{B_2} are very close for $b = a + 8$. The uppermost line represents the curve of Fig 4.1.1: X^{B_1}. The second graph illustrates the dependence of the default risk on the partial hedge parameter b.

Fig. 4.2.1: Partial Hedge Price under GBM: B2={K<S(T)<a,max S(t)<b}

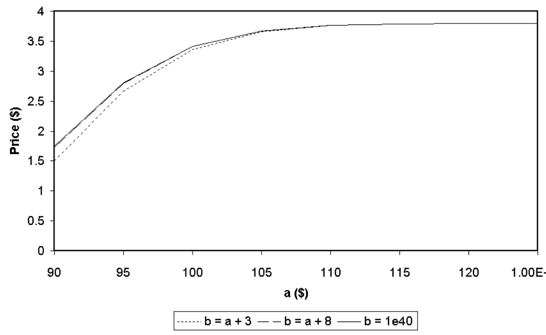
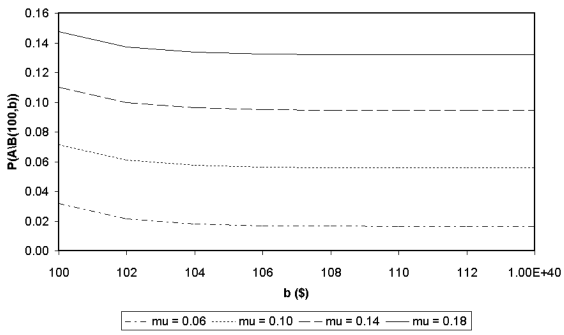


Fig. 4.2.2: Default Risk under GBM: B2={K<S(T)<a,max S(t)<b}



Note that

$$\lim_{b\to\infty} X^{B_2(a,b)} = X^{B_1(a)},$$

and

$$\lim_{b\to\infty} \mathbb{P}(A\backslash B_2(a,b)) = \mathbb{P}(A\backslash B_1(a)).$$

We similarly find that

$$\lim_{a,b\to\infty} X^{B_2(a,b)} = X_{BS},$$

and

$$\lim_{a,b\to\infty} \mathbb{P}(A\backslash B_2(a,b)) = 0.$$

The graph indicates confirmation of intuitive results: higher risk of default for higher real-world appreciation rates and lower b.

As in the previous section, the data tables used to produce Figures 4.2.1 and 4.2.2 may be found in Appendix C.1.

For the following section, let

$$M(T) = \inf\{s : S(s) = \max_{0\le t\le T} S(t)\}$$

be the time at which the maximum asset price on $[0, T]$ is first reached

4.3 Partial Hedging on $B_3(a, b, s) = \{K < S(T) < a, \max_{0\le t\le T} S(t) < b, M(T) < s\}$, $K < a < b$, $0 < s < T$

We consider a hedge on the event $B_3(a, b, s)$ described above.

We use the same measure $\mathbb{Q}^*$ from section 3.2, defined by (4.5). The joint density function for $\big(W^{Q^*}(T), \max_{0\le t\le T} W^{Q^*}(t)\big)$ on the event $\{M(T) < s\}$ can be found in Karatzas and Shreve (1998), and is given by:

$$\begin{aligned}
& \mathbb{Q}^*\Big[W^{Q^*}(T) \in dx, \max_{0\le t\le T} W^{Q^*}(t) \in dy, M(T) < s\Big] \\
= \; & \frac{2}{\sqrt{2\pi T^3}}\Big[N\Big(-\frac{\alpha_+}{\beta}\Big)(2y-x)e^{-\frac{1}{2T}(2y-x)^2} - N\Big(-\frac{\alpha_-}{\beta}\Big)xe^{-\frac{1}{2T}x^2}\Big]dydx \\
:= \; & \psi(x, y; s)dydx,
\end{aligned}$$

where

$$\alpha_{\pm} = y(T-s) \pm s(x-y) \text{ and } \beta = \sqrt{sT(T-s)}.$$

The price of the partial hedge on B_3 is given by

$$\begin{aligned}
X^{B_3(a,b,s)} &= e^{-rT}\Big[\Big(S(T)-K\Big)\frac{d\mathbb{Q}}{d\mathbb{Q}^*}(T)\mathbf{1}_{B_3}\Big] \\
&= e^{-rT}\int_{K^*}^{a^*}\int_{x}^{b^*}\Big(S(0)e^{\sigma x}-K\Big)e^{\phi x-\frac{1}{2}\phi^2 T}\psi(x,y;s)dxdy,
\end{aligned}$$

and the default risk by

$$\begin{aligned}
\mathbb{P}(A\backslash B_3) &= \mathbb{P}(A)-\mathbb{P}(B_3) \\
&= N\Big(d_-^{\mu}(K)\Big)-\mathbb{E}^{Q^*}\Big[\frac{d\mathbb{P}}{d\mathbb{Q}}\frac{d\mathbb{Q}}{d\mathbb{Q}^*}\mathbf{1}_{B_3}\Big] \\
&= N\Big(d_-^{\mu}(K)\Big)-\int_{K^*}^{a^*}\int_{x}^{b^*}e^{-\frac{1}{2}\theta^2 T+\theta(x-\phi T)}e^{\phi x-\frac{1}{2}\phi^2 T}\psi(x,y;s)dxdy \\
&= N\Big(d_-^{\mu}(K)\Big)-e^{-\frac{1}{2}T(\theta+\phi)^2}\int_{K^*}^{a^*}e^{(\theta+\phi)x}\Big[\int_{x}^{b^*}\psi(x,y;s)dy\Big]dx.
\end{aligned}$$

Closed form solutions can not be derived for these integrals, however numerical integration techniques may be used to obtain approximate solutions. The following algorithm was implemented in Microsoft Visual Basic for Applications 6.0, details may be found in Appendix B.1.1.

Let H be the number of steps in our approximation, $\Delta_1 = \frac{a^*-K^*}{H}$ and $\Delta_2 = \frac{b^*-a^*}{H}$. Let there be three sequences of partitions $\tau_H^x = \{x_1, x_2, \ldots, x_H\}$, $\tau_H^y = \{y_1, y_2, \ldots, y_H\}$ and $\tau_H^{y'} = \{y_1', y_2', \ldots, y_H'\}$ such that $x_i = K^* + \frac{2i-1}{2}\Delta_1$, $y_j = K^* + \frac{2j-1}{2}\Delta_1$ and $y_k' = a^* + \frac{2k-1}{2}\Delta_2$, $i,j,k \in \{1,2,\ldots,H\}$. Then

$$\begin{aligned}
X^{B_3} &= e^{-rT-\frac{1}{2}\phi^2 T} \qquad (4.8)\\
&\times \lim_{H\to\infty}\sum_{i=1}^{H}\Big(S(0)e^{\sigma x_i}-K\Big)e^{\phi x_i}\Delta_1\Big[\sum_{j=i}^{H}\psi(x_i,y_j;s)\Delta_1+\sum_{k=1}^{n}\psi(x_i,y_k';s)\Delta_2\Big],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(A\backslash B_3) &= N(d_-^{\mu}(K))-e^{-\frac{1}{2}T(\phi+\theta)^2} \\
&\times \lim_{H\to\infty}\sum_{i=1}^{H}e^{(\phi+\theta)x_i}\Delta_1\cdot\Big[\sum_{j=i}^{H}\psi(x_i,y_j;s)\Delta_1+\sum_{k=i}^{H}\psi(x_i,y_k';s)\Delta_2\Big]. \qquad (4.9)
\end{aligned}$$

We consider the asset of sections 3.1 and 3.2, with $S(0) = 80$, $\sigma = 0.20$ and $r = 0.06$, and a European vanilla call with strike price $K = 80$ and $T = 0.25$. We let the parameter $H = 1000$. Each data point takes approximately 5 minutes to compute for this value of H.

Figure 4.3.1 shows the price of the partial hedge for different values of a and s. We let $b = a + 3$ here and below. Note that using this procedure, the approximation is very inefficient for large a. Most positive contributions to the price algorithm are for $S(T)$ near K, and for very large a, Δ_1 is too large to accurately approximate the price in the important region. We use $a = 1000$ instead of $a = \infty$. Note that even then the approximate price actually drops as a goes from 120 to 1000. However, since

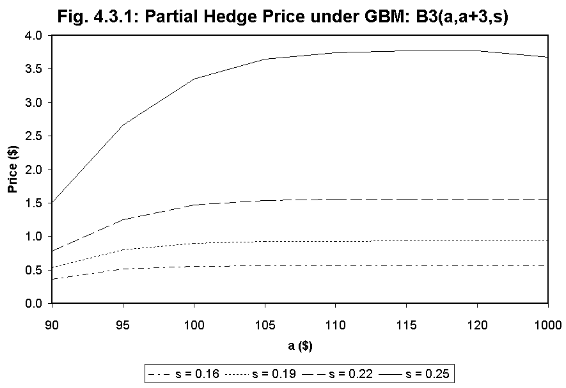
$$B_3(a, b, s) \subset B_3(\infty, \infty, s), \qquad 0 \leq s \leq T,$$

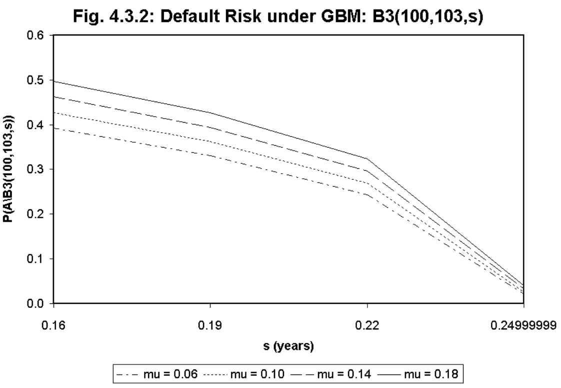
we have the inequality

$$X^{B_3(a,b,s)} < X^{B_3(\infty,\infty,s)}, \qquad 0 \leq s \leq T.$$

Note that the price $X^{B_3(\infty,\infty,s)}$ is the price of an exotic option, paying the difference between the asset price at expiry and a strike price, and becoming worthless if the asset reaches its maximum price on $0 \leq t \leq T$ *after* a given time s, $0 \leq s \leq T$.

Figure 4.3.2 shows the values of default risk associated with the partial hedge on B_3. We let s take the values it takes in Figure 4.3.1, and measure the risk for different asset appreciation rates μ. Here $a = 100$ and $b = 103$.





Tables for the above figures may be found in Appendix C.1.

Chapter 5

Partial Hedging Under Stochastic Volatility

We turn now to the case of stochastic volatility. One of the major practical problems with the Black Scholes description of asset prices is that is doesn't give a sufficiently accurate depiction of reality. One of the key assumptions of Black and Scholes was that the volatility, σ, of an asset is constant throughout the lifetime of a claim written on the asset. Empirical studies of trading practices have shown that the *implied* volatility—that is, the asset volatility that would give the price at which a claim trades if used in the Black Scholes formula—if sketched as a function of strike price appears to be curved. This curve is known as a volatility *smile*, or *smirk*.

Attempts to deal with this trading anomaly have included *stochastic volatility models*, which formulate the volatility as a *process*, governed by stochastic differential equations, which then feed into the asset price process. One of the advantages of the Black Scholes approach is that closed-form solutions may be derived. Stochastic volatility models do not in general possess such known analytical results. Asset prices and volatility are generally simulated along their trajectories, then claim prices are estimated by a Monte Carlo approach.

In this chapter, this approach is extended to the case of partial hedging, and claim prices and default risks are measured as in Chapter 4.

5.1 The Model

We consider claims on an underlying asset, whose price $S_1(t)$ follows the stochastic differential equation under $\mathbb{Q}$:

$$dS_1(t) = rS_1(t)dt + \sqrt{V(t)}S_1(t)dW_1^Q(t),$$

where $W_1^Q(t)$ is a $\mathbb{Q}$-Brownian Motion and r is the risk-free interest rate. The term $V(t)$ is the *stochastic volatility* process, which follows the stochastic differential equation:

$$dV(t) = \alpha(\overline{V} - V(t))dt + \lambda\sqrt{V(t)}dW_V^Q(t),$$

where $W_V^Q(t)$ is a $\mathbb{Q}$-Brownian Motion to be specified below. We call α the *volatility mean reversion rate*, $\overline{V}$ the *long-run average volatility* and λ the *volatility of volatility*. Here $\alpha, \overline{V} > 0$ and $\lambda \geq 0$ are constant for all $t \in [0, T]$—that is, over the life of an option on our asset $S_1(t)$ with this volatility process, expiring at T. The volatility process has the additional condition that it is not a *tradable asset*.

However, our market does possess a second tradable asset, whose price $S_2(t)$ follows the Geometric Brownian Motion

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2^Q(t)$$

under $\mathbb{Q}$, where $W_2^Q(t)$ is a $\mathbb{Q}$-Brownian Motion *independent* of $W_1^Q(t)$ and σ_2 is a positive constant volatility.

Note that for this model we have $N = 2$ and $d = 3$ so the market is not complete, by Theorem 2.4.2. However, we note that if we can completely determine one volatility process as the linear combination of two others, we have $d = 2$ and a complete market.

To do this, we write the stochastic driver of the volatility as

$$dW_V^Q(t) = \rho dW_1^Q(t) + \sqrt{1-\rho^2}dW_2^Q(t),$$

so

$$dV(t) = \alpha(\overline{V} - V(t))dt + \lambda\sqrt{V(t)}\big[\rho dW_1^Q(t) + \sqrt{1-\rho^2}dW_2^Q(t)\big].$$

Then we have

$$\begin{aligned}\mathrm{Cov}\big(W_V^Q(t), W_1^Q(t)\big) &= \mathbb{E}\big[W_V^Q(t)W_1^Q(t)\big] - \mathbb{E}W_V^Q(t)\mathbb{E}W_1^Q(t) = \mathbb{E}\big[W_V^Q(t)W_1^Q(t)\big] \\ &= \rho\mathbb{E}\big(W_1^Q(t)\big)^2 + \rho\sqrt{1-\rho^2}\mathbb{E}W_1^Q(t)W_2^Q(t) = \rho t,\end{aligned}$$

and

$$\mathrm{Var}(W_V^Q(t)) = \rho^2 t + (1-\rho^2)t = t,$$

so

$$\mathrm{Corr}\big(W_V^Q(t), W_1^Q(t)\big) = \frac{\mathrm{Cov}\big(W_V^Q(t), W_1^Q(t)\big)}{\sqrt{\mathrm{Var}(W_V^Q(t)) \cdot \mathrm{Var}(W_1^Q(t))}} = \frac{\rho t}{t} = \rho.$$

We proceed in a similar way to obtain

$$\mathrm{Corr}\big(W_V^Q(t), W_2^Q(t)\big) = \sqrt{1-\rho^2}.$$

Note that for this model we now have $N = d = 2$ and a non-singular volatility matrix, so by Theorem 2.4.2 the market is complete and there exists the market price of risk vector $\theta(t)$ and a measure $\mathbb{P}$ equivalent to $\mathbb{Q}$, which we call the real-world measure, such that

$$\theta_1 = \frac{\mu_1 - r}{\sqrt{V(t)}}, \qquad \theta_2 = \frac{\mu_2 - r}{\sigma_2}, \qquad dW_i^Q(t) = dW_i^P(t) + \theta_i(t)dt, \qquad i = 1, 2.$$

Then our market is arbitrage free. Our complete model under the measure $\mathbb{P}$ is given by

$$\begin{aligned}
dS_1(t) &= \mu_1 S_1(t)dt + \sqrt{V(t)}S_1(t)dW_1^P(t), && (5.1)\\
dV(t) &= \alpha\big(\overline{V} - V(t)\big)dt \\
&\quad +\lambda\sqrt{V(t)}\left[\rho\Big(dW_1^P(t) + \frac{\mu_1 - r}{\sqrt{V(t)}}dt\Big) + \sqrt{1-\rho^2}\Big(dW_2^P(t) + \frac{\mu_2 - r}{\sigma_2}dt\Big)\right] \\
&= \left[\alpha\big(\overline{V} - V(t)\big) + \lambda\rho(\mu_1 - r) + \lambda\sqrt{1-\rho^2}(\mu_2 - r)\frac{\sqrt{V(t)}}{\sigma_2}\right]dt \\
&\quad +\lambda\sqrt{V(t)}\Big[\rho dW_1^P(t) + \sqrt{1-\rho^2}dW_2^P(t)\Big], && (5.2)\\
dS_2(t) &= \mu_2 S_2(t)dt + \sigma_2 S_2(t)dW_2^P(t). && (5.3)
\end{aligned}$$

5.2 Simulation

Despite the fact that we have here a complete market, with an associated unique risk-neutral measure, an analytical closed-from solution for the asset price at expiry is not yet known. Our approach in the next two chapters is to simulate the volatility and price processes using the Euler method. Note that the second asset need not be simulated, since it is only its *parameters* that feature in the volatility process under $\mathbb{P}$ and not its price. A discrete-time version of the price of the first asset and volatility under $\mathbb{Q}$ makes use of a sequence of partitions $\tau_n = \{t_0^{(n)} = t_0, t_1^{(n)} = t_1, \ldots, t_n^{(n)} = t_n\}$, where $t_0 = 0$, $t_n = T$ and $t_i - t_{i-1} = \Delta_n$, $i = 1, 2, \ldots, n$, and is given by the following:

$$\begin{aligned}
S(t_i) - S(t_{i-1}) &= rS(t_{i-1})\Delta_n + \sqrt{V(t_{i-1})}S(t_{i-1})\sqrt{\Delta_n}Z_1^Q(t_{i-1}), \\
V(t) - V(t_{i-1}) &= \alpha\big(\overline{V} - V(t_{i-1})\big)\Delta_n \\
&\quad +\lambda\sqrt{V(t_{i-1})}\sqrt{\Delta_n}\big[\rho Z_1^Q(t_{i-1}) + \sqrt{1-\rho^2}Z_2^Q(t_{i-1})\big], \qquad (5.4)
\end{aligned}$$

where $Z_1^Q(t_{i-1})$ and $Z_2^Q(t_{i-1})$ are independent and identically distributed standard normal random variables, $\forall i = 1, 2, \ldots, n$, and $n \in \mathbb{N}$ is the number of time steps in the simulation.

Under $\mathbb{P}$ the simulation follows

$$\begin{aligned} S(t_i) &= S(t_{i-1}) + \mu_1 S(t_{i-1})\Delta_n + \sqrt{V(t_{i-1})}S(t_{i-1})\sqrt{\Delta_n}Z_1^P(t_{i-1}), \\ V(t) &= \left[\alpha\big(\overline{V} - V(t_{i-1})\big) + \lambda\rho(\mu_1 - r) + \lambda\sqrt{1-\rho^2}(\mu_2 - r)\frac{\sqrt{V(t_{i-1})}}{\sigma_2}\right]\Delta_n \\ &\quad + V(t_{i-1}) + \lambda\sqrt{V(t_{i-1})}\sqrt{\Delta_n}\big[\rho Z_1^P(t_{i-1}) + \sqrt{1-\rho^2}Z_2^P(t_{i-1})\big], \qquad (5.5) \end{aligned}$$

where $Z_1^P(t_{i-1})$ and $Z_2^P(t_{i-1})$ are independent and identically distributed standard normal random variables, $\forall i = 1, 2, \ldots, n$. Note that under $\mathbb{P}$ or $\mathbb{Q}$, during its simulation $V(t)$ can become negative. If this is the case, we in effect reflect the volatility in the origin, and iterate using $|V(t_{i-1})|$.

Our method is to simulate the asset price and volatility m times over the interval $[0, T]$ under $\mathbb{Q}$ and $\mathbb{P}$, and calculate the partially hedged claim price $\hat{X}_{j,n}^B$ and indicator function of default $\hat{\mathbf{1}}_{A\backslash B,j,n}$ respectively, $\forall j = 1, 2, \ldots, m$. We then average their prices and indicators, and calcuate the standard errors of these averages. In other words, we let

$$\hat{X}_{n,m}^B = \frac{1}{m}\sum_{j=1}^m \hat{X}_{j,n}^B, \qquad \hat{\mathbb{P}}(A\backslash B)_{n,m} = \frac{1}{m}\sum_{j=1}^m \hat{\mathbf{1}}_{A\backslash B,j,n},$$

which are unbiased estimators of the Euler method approximation of the price and default probability respectively, given a value of n, and

$$\lim_{m\to\infty}\lim_{n\to\infty}\hat{X}_{n,m}^B = X^B, \qquad \lim_{m\to\infty}\lim_{n\to\infty}\hat{\mathbb{P}}(A\backslash B)_{n,m} = \mathbb{P}(A\backslash B).$$

The errors of our estimators are calculated by the following, assuming normally distributed errors, and a 95% confidence interval:

$$\begin{aligned} \text{err}(\hat{X}_{n,m}^B) &= \frac{1.96}{m}\sqrt{\frac{m\sum_{j=1}^m(\hat{X}^B)^2 - (\sum_{j=1}^m \hat{X}^B)^2}{m-1}}, \\ \text{err}(\hat{\mathbb{P}}(A\backslash B)_{n,m}) &= \frac{1.96}{m}\sqrt{\frac{m\sum_{j=1}^m(\hat{\mathbf{1}}_{A\backslash B})^2 - (\sum_{j=1}^m \hat{\mathbf{1}}_{A\backslash B})^2}{m-1}}, \end{aligned}$$

and we take n and m large enough to make the error arbitrarily small. We consider various methods of reducing the error further in the following chapter.

In order to simulate the maximum of the price process over the period $[0, T]$ we make use of the results of Beaglehole *et. al.* (1997). These results make use of the following definition:

Definition 5.2.1 (The Brownian Bridge) *The process* $W^{BB}(t;t^*,u) = [W(t)|W(t^*) = u]$, $0 \le t \le t^*$, *where* $W(t)$ *is the Brownian Motion, is called the* Brownian Bridge, *connecting* $W(0) = 0$ *and* $W(t^*) = u$. *The process* $W^{BB}(t;1,0)$ *is called the* standard *Brownian Bridge.*

Note that in our discretisation, we use $V(t_{i-1})$ to get $S(t_i)$. This is an assumption of *constant* volatility over the interval $[t_{i-1}, t_i]$. We assume further, then, that in the time interval the price process follows

$$S(t) = S(t_{i-1}) + \sqrt{V(t_{i-1})}W^{BB}(t - t_{i-1}; t_i - t_{i-1}, u_i), \qquad t_{i-1} \le t \le t_i,$$

where $W^{BB}(t - t_{i-1}; t_i - t_{i-1}, u_i)$ is the Brownian Bridge connecting 0 and

$$\frac{S(t_i) - S(t_{i-1})}{\sqrt{V(t_{i-1})}} =: u_i$$

over $t \in [t_{i-1}, t_i]$.

The distribution of the maximum of a Brownian Bridge connecting 0 and u_i is known:

$$\mathbb{P}_0\Big(\max_{t_{i-1}\le t\le t_i} W^{BB}(t - t_{i-1}; t_i - t_{i-1}, u_i) \le x_i\Big) = 1 - e^{\frac{-2x_i(x_i - u_i)}{\Delta_n}}.$$

Knowing that a distribution function lies between 0 and 1, we simulate a variable, F_i, $\forall i = 1, 2, \ldots, n$, from the uniform distribution on $[0, 1]$ and invert the distribution function to get a random draw of the maximum over each interval. We obtain

$$x_i = \frac{1}{2}\Big[u_i + \sqrt{u_i^2 - 2\Delta_n \ln(F_i)}\Big]$$

as a simulation of the maximum of a Brownian Bridge connecting 0 and u_i on $t \in [t_{i-1}, t_i]$. Then the simulation of the maximum of our price process between each time discretization is given by the following:

$$\begin{aligned}\max_{t_{i-1}\le t\le t_i} S(t) &= S(t_{i-1}) + \sqrt{V(t_{i-1})}\max_{t_{i-1}\le t\le t_i} W^{BB}(t - t_{i-1}; t_i - t_{i-1}, u_i) \\ &= S(t_{i-1}) + \frac{1}{2}\sqrt{V(t_{i-1})}\Big[u_i + \sqrt{u_i^2 - 2\Delta_n \ln(F_i)}\Big], \qquad t_{i-1} \le t \le t_i,\end{aligned}$$

where $u_i = \frac{S(t_i) - S(t_{i-1})}{\sqrt{V(t_{i-1}}}$ and $F_i \sim U[0, 1]$.

The time to maximum $M(T)$ is taken to be the midpoint of the time increment in which the maximum occurs.

The code for the simulation may be found in Appendix B.2.

We will also need a method of calculating implied volatilities. The implied volatility is the volatility that, when used in the Black Scholes formula, will give the price of the claim calculated under the stochastic volatility model. Unfortunately, the Black Scholes formula is not invertible in σ, so we use the Newton-Raphson method to iteratively calculate the implied volatility. This method is used to calculate zeros of a function. We note that the value of implied volatility we want, σ^*, is that which solves the equation

$$X_{BS}(\sigma^*) - \hat{X}^A = 0.$$

The algorithm we follow is

$$\sigma_{k+1} = \sigma_k - \frac{X_{BS}(\sigma_k) - \hat{X}^A}{\frac{d}{d\sigma_k} X_{BS}(\sigma_k)}, \qquad \sigma_0 = \sqrt{V(0)}.$$

Noting that

$$\begin{aligned} \frac{d}{d\sigma} X_{BS}(\sigma) &= \frac{1}{\sqrt{2\pi}} S(0) e^{-\frac{1}{2}d_+^2} \frac{dd_+}{d\sigma} - \frac{1}{\sqrt{2\pi}} K e^{-rT} e^{-\frac{1}{2}(d_+ - \sigma\sqrt{T})^2} \Big(\frac{dd_+}{d\sigma} - \sqrt{T} \Big) \\ &= \frac{1}{\sqrt{2\pi}} S(0) e^{-\frac{1}{2}d_+^2} \Big(\frac{dd_+}{d\sigma} - \frac{dd_+}{d\sigma} + \sqrt{T} \Big) \\ &= \frac{1}{\sqrt{2\pi}} S(0) \sqrt{T} e^{-\frac{1}{2}d_+^2}, \qquad d_+ = \frac{1}{\sigma\sqrt{T}} \Big[\ln \frac{S(0)}{K} + \big(r + \frac{1}{2}\sigma^2 \big) T \Big], \end{aligned}$$

we use

$$\sigma_{k+1} = \sigma_k - \frac{\sqrt{2\pi}}{S(0)\sqrt{T}} e^{\frac{1}{2}d_+^2} (X_{BS}(\sigma_k) - \hat{X}^A), \tag{5.6}$$

where $\sigma_k \to \sigma^*$ as $k \to \infty$. We stop iterating when $|\sigma_{k+1} - \sigma_k| < 0.0001$.

Note that we can of course calculate implied volatilities for partial hedges, but we would have to adjust the Newton-Raphson algorithm by using $\hat{X}^B - X^B(\sigma^*) = 0$ and by differentiating the expressions in Chapter 4 for $X^B(\sigma^*)$. No more insight is gained, since the implied volatilities depend on the *model* and not the *hedge*, so without loss of generality we consider only the full hedge A because it simplifies calculation.

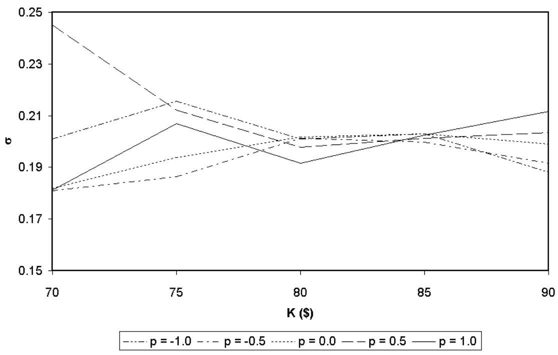
5.3 Results

In what follows we examine how changing the parameters of the partial hedge and the correlation ρ effect the price and default risk of a partial hedging strategy under stochastic volatility. We also compare the results to the analytic solutions of the previous chapter, under the Geometric Brownian Motion.

Our parameters for the simulation take the following values: $S(0) = 80$, $V(0) = 0.04 = 0.2^2$, $\alpha = 2$, $\overline{V} = 0.04$, $\lambda = 0.1$, $r = 0.06$, $T = 0.25$, $\mu_1 = 0.1$, $\mu_2 = 0.12$ and $\sigma_2 = 0.15$. We simulate $n = 90$ time steps, roughly the number of days in $T = 0.25$ years, and we do $m = 4000$ simulations.

The first simulation estimates the volatility smile for our stochastic volatility model. This enables us to gauge the difference between the claim price under our stochastic volatility model, as opposed to under the Geometric Brownian Motion, and also the effect of the correlation parameter ρ on the volatility process. We obtain the following graph:

Fig. 5.3.1: Volatility Smile for Stochastic Volatility



The typical error for the claim price ranges from around 10% for $K = 70$ to around 2% for $K = 90$. For $K = 80$ it is approximately 5%, which translates to a difference in implied volatility of about 0.015. Most of the data is clustered within bounds this wide. One notable thing is that for $K = 90$ the implied volatility increases with correlation. The error for claim price at $K = 90$ translates to approximately 0.0075 for implied volatility, making the estimates close to significantly different. This will be examined further when we consider variance reduction techniques in the next chapter.

We table the simulated asset price $\hat{X}^A$, simulation error $\text{err}(\hat{X}^A)$ and implied volatility σ^* vs strike K for different levels of correlation ρ in Appendix C.2. The set A is of course the set of the perfect hedge, where $a = b = \infty$, $s = T$.

Now we move to partial hedges. In the remainder of this chapter, let $K = 80$. We consider the effect of varying a in a stochastic volatility setting. Let $b = a + 3$, $s = 0.23$. We obtain the following graphs.

With consideration of the approximately 6% errors, we are hestitant about claiming the existence of any trends at this stage, but we do note that for low a, price seems to decrease as ρ increases.

Fig. 5.3.2: Partial Hedge Price under Stochastic Volatility: B3(a,a+3,0.23)

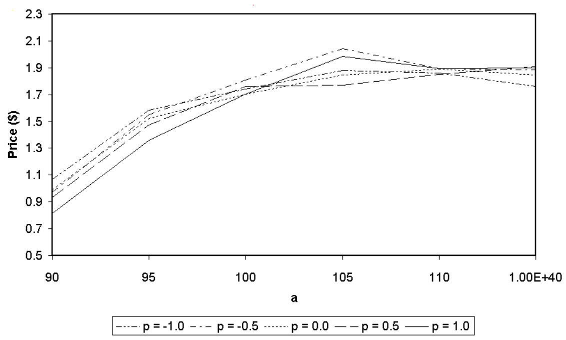
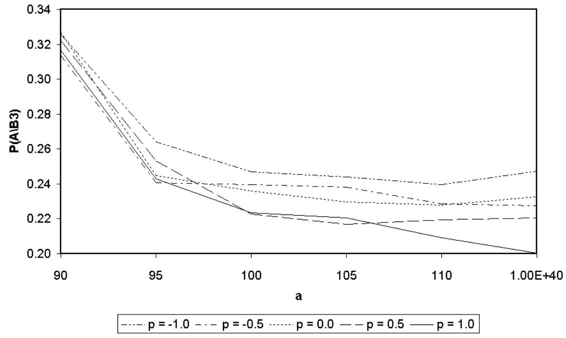


Fig. 5.3.3: Default Risk under Stochastic Volatility: B3(a,a+3,0.23)



Typical errors for default risk are approximately 5.5%, and while the default risk curves are not particularly regular, the differences between values for different ρ are often significant. It seems that default risk increases for decreasing ρ. This remaining errors bear further investigation below, when measurements are more accurate.

Consider now varying parameter b. Let $a = 100$ and $s = 0.23$. We note for the following graphs that in this case the default risks and prices are rarely distinguishable, especially so when the typical 5% error is accounted for. The default risk perhaps follows what we noted above about Fig. 5.3.2 and 5.3.3, but the estimates are far too inaccurate to be conclusive.

Tables of our estimators and associated errors may be found in Appendix C.2.

Fig. 5.3.4: Partial Hedge Price under Stochastic Volatility: B3(100,b,0.23)

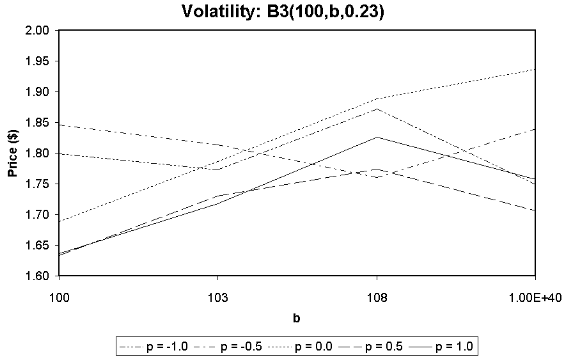
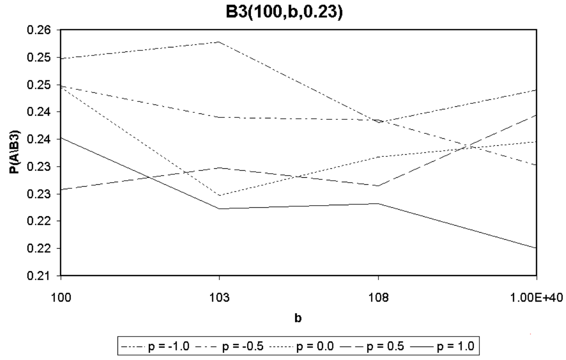
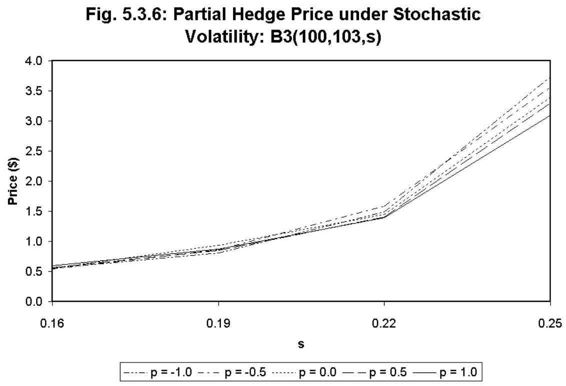
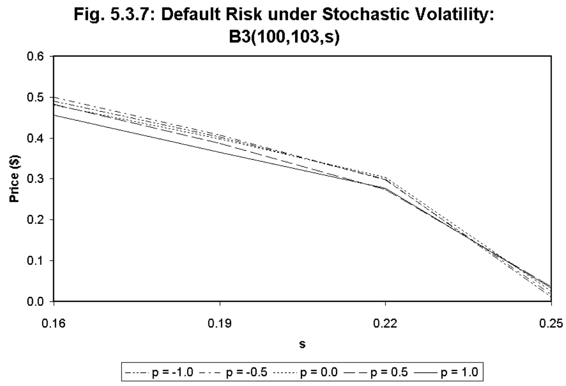


Fig. 5.3.5: Default Risk under Stochastic Volatility: B3(100,b,0.23)



Because the volatility is mean reverting and the asset drift coefficient is positive, the maximum asset price is most likely to occur near expiry. This means the s term will most likely be the major contributor to the risk in the partial hedge. We will consider this idea now, by varying the parameter s. Let $a = 100$, $b = 103 = a + 3$. The following graphs result:





The prices and risks are heavily dependent on the value of s. For prices, typical errors are in the range 4% to 9%, with higher error for lower s. For low s, the prices are indistinguishable given the size of the errors, yet for $s = 0.22$ and $s = 0.25 = T$ most of the differences are significant for different ρ. It is notable that for high s there seems to be a trend of low price for high ρ. We examine this further in the next chapter, with more accurate estimators.

Similarly, for the default risks, for low values of s the typical errors of 4% are generally too high to distinguish between the estimates. However, it is notable for further investigation that in the case of $s = 0.16$, the estimates of default risk for $\rho = -1$ and $\rho = +1$ *are* significantly different, with the estimate for $\rho = -1$ greater than that for $\rho = +1$. The estimates for $s = 0.25$ are mostly significantly different, and the trend there is the other way, with the estimate for $\rho = -1$ *less* than that for $\rho = +1$. We note the graph of the point estimates seems to show that the lines cross somewhere between $s = 0.22$ and $s = 0.25$. This we will examine further below.

Chapter 6

Variance Reduction Techniques

The measurements of the preceeding chapter suffered most prominently from lack of accuracy. In this chapter we demonstrate how variance reduction techniques, namely *Antithetic Variates* and *Control Variates*, may be used to greatly decrease the errors of our partial hedge price and default risk estimators, whilst maintaining an efficient compuational budget.

In what follows we build on the work by Ben Ameur *et. al.* (1999) by applying their methods more generally.

6.1 Antithetic Variates

Consider an unbiased estimator $\hat{\eta}$ of a parameter η. In the method of Antithetic Variates, we construct another unbiased estimator $\hat{\eta}_a$, which has a negative covariance with $\hat{\eta}$. Then $\hat{\eta}_* = \frac{1}{2}(\hat{\eta} + \hat{\eta}_a)$ is an unbiased estimator of η, and

$$\mathrm{Var}(\hat{\eta}_*) = \frac{1}{4}\mathrm{Var}(\hat{\eta}) + \frac{1}{4}\mathrm{Var}(\hat{\eta}_a) + \frac{1}{2}\mathrm{Cov}(\hat{\eta}, \hat{\eta}_a).$$

If we choose $\hat{\eta}_a$ to have a similar variance to $\hat{\eta}$, the variance of $\hat{\eta}_*$ will be smaller than that of $\hat{\eta}$ and $\hat{\eta}_a$.

The estimators $\hat{X}^B$ and $\hat{\mathbb{P}}(A\backslash B)$ are functions of independent and identically normally distributed random variables, taken in our iterative simulation above in (5.4) and (5.5). Without loss of generality, consider the case $\hat{X}^B$ over one simulation run:

$$\hat{X}^B = f(Z_1^Q(t_0), Z_1^Q(t_1), \ldots, Z_1^Q(t_{n-1}), Z_2^Q(t_0), Z_2^Q(t_1), \ldots, Z_2^Q(t_{n-1})).$$

Then if we take the estimator

$$\hat{X}_a^B = f(-Z_1^Q(t_0), -Z_1^Q(t_1), \ldots, -Z_1^Q(t_{n-1}), Z_2^Q(t_0), Z_2^Q(t_1), \ldots, Z_2^Q(t_{n-1})),$$

following Ben Ameur *et. al.* (1999), we induce negative covariance and therefore variance reduction. Note that we only take the negative of the $Z_1^Q(t_i)$'s. The simulation proceeds under $\mathbb{Q}$ as follows:

$$\begin{aligned}
S(t_i) - S(t_{i-1}) &= rS(t_{i-1})\Delta_n + \sqrt{V(t_{i-1})}S(t_{i-1})\sqrt{\Delta_n}Z_1^Q(t_{i-1}) \\
V(t) - V(t_{i-1}) &= \alpha\big(\overline{V} - V(t_{i-1})\big)\Delta_n \\
&\quad +\lambda\sqrt{V(t_{i-1})}\sqrt{\Delta_n}\big[\rho Z_1^Q(t_{i-1} + \sqrt{1-\rho^2}Z_2^Q(t_{i-1})\big] \\
S_a(t_i) - S_a(t_{i-1}) &= rS_a(t_{i-1})\Delta_n + \sqrt{V_a(t_{i-1})}S_a(t_{i-1})\sqrt{\Delta_n}(-Z_1^Q(t_{i-1})) \\
V_a(t) - V_a(t_{i-1}) &= \alpha\big(\overline{V} - V_a(t_{i-1})\big)\Delta_n \\
&\quad +\lambda\sqrt{V_a(t_{i-1})}\sqrt{\Delta_n}\big[\rho(-Z_1^Q(t_{i-1}) + \sqrt{1-\rho^2}Z_2^Q(t_{i-1})\big].
\end{aligned}$$

We simulate maxima and time to maxima as per the preceeding chapter for both unbiased estimators, and calculate a separate estimate of X^B given each set of inputs. We have chosen two estimators in such a way that their average guarantees variance reduction. We take $\hat{X}_*^B = \frac{1}{2}(\hat{X}^B + \hat{X}_a^B)$, where the subscript a indicates the estimator of X^B using $-Z_1^Q(t_i)$, $i = 0, 1, \ldots, n-1$, and $\hat{X}_*^B$ is the antithetic variates estimator of X^B.

6.2 Control Variates

Suppose we have calculated an estimator, $\hat{X}^B$, of X^B over one simulation run. Suppose we also calculate another estimator $\hat{C}^B$ of a random variable C^B with known expectation $\mathbb{E}C^B$, that is positively correlated with X^B. The most effective choice of C^B is the one with most positive correlation with X^B. For our purposes, we take C^B to be one simulation of the partial hedge price under constant volatility, using the same random variables as $\hat{X}^B$ to drive the simulation. We then use the analytical results of Chapter 4 to calculate its known expectation $\mathbb{E}C^B$.

In the method of Control Variates we let

$$X_c^B = X^B - \beta(C^B - \mathbb{E}C^B),$$

where

$$\beta = \frac{\mathrm{Cov}(X^B, C^B)}{\mathrm{Var}(C^B)}.$$

Then

$$\mathrm{Var}(X_c^B) = \mathrm{Var}(X^B) - \beta^2\mathrm{Var}(C^B) = \Big[1 - \big(\mathrm{Corr}(X^B, C^B)\big)^2\Big]\mathrm{Var}(X^B).$$

Since we use the same random sample from the normal distribution throughout the simulation of X^B and C^B, we have $\mathrm{Corr}(X^B, C^B) > 0$, and so $\mathrm{Var}(X_c^B) < \mathrm{Var}(X^B)$, and we have variance reduction.

Proceeding in the conventional way, we run a Monte Carlo simulation by averaging m trials of X_c^B. The Control Variates estimator $\overline{\hat{X}_c^B}$ of X^B is

$$\overline{\hat{X}_c^B} = \overline{\hat{X}^B} - \hat{\beta}\Big(\overline{\hat{C}^B} - \mathbb{E}C^B\Big),$$

where $\overline{(\cdot)}$ denotes the sample mean over the Monte Carlo simulation:

$$\overline{\hat{X}^B} = \frac{1}{m}\sum_{j=1}^{m} \hat{X}_j^B, \qquad \overline{\hat{C}^B} = \frac{1}{m}\sum_{j=1}^{m} \hat{C}_j^B.$$

6.3 Integration of Variance Reduction Techniques

For the purposes of presenting results, combine the two variance reduction approaches by constructing antithetic estimators of X_*^B and C_*^B in the way described in section 6.1, and then applying the method of control variates. We have

$$\overline{\hat{X}_{*c}^B} = \overline{\hat{X}_*^B} - \hat{\beta}\Big(\overline{\hat{C}_*^B} - \mathbb{E}C^B\Big),$$

where

$$\hat{\beta} = \frac{\mathrm{Cov}(\hat{X}_{*j}^B, \hat{C}_{*j}^B)}{\mathrm{Var}(\hat{C}_{*j}^B)},$$

and

$$\hat{X}_{*j}^B = \frac{1}{2}\Big(\hat{X}_j^B + \hat{X}_{aj}^B\Big), \qquad \hat{C}_{*j}^B = \frac{1}{2}\Big(\hat{X}_j^B + \hat{X}_{aj}^B\Big), \qquad j = 1, 2, \ldots, m,$$

are the estimators of the partial hedge prices over one simulation run. As previously, $\overline{(\cdot)}$ indicates the Monte Carlo simulation over m trials.

We calculate $\hat{\beta}$ using the following formula:

$$\hat{\beta} = \frac{\sum_{j=1}^{m} \hat{X}_{*j}^B \hat{C}_{*j}^B - \frac{1}{m}\sum_{j=1}^{m} \hat{X}_{*j}^B \sum_{j=1}^{m} \hat{C}_{*j}^B}{\sum_{j=1}^{m} (\hat{C}_{*j}^B)^2 - \frac{1}{m}(\sum_{j=1}^{m} \hat{C}_{*j}^B)^2}.$$

And the errors of our estimators are calclated as follows, giving the upper and lower bounds of a 95% confidence interval, assuming normally distributed errors:

$$\mathrm{err}(\overline{\hat{X}_{*c}^B}) = \frac{1.96}{m}\sqrt{\frac{m\sum_{j=1}^{m}(\hat{X}_{*j}^B)^2 - (\sum_{j=1}^{m} \hat{X}_{*j}^B)^2}{m-1} - \hat{\beta}^2 \frac{m\sum_{j=1}^{m}(\hat{C}_{*j}^B)^2 - (\sum_{j=1}^{m} \hat{C}_{*j}^B)^2}{m-1}}.$$

The complete code for the calculation of the integrated variance reduction technique may be found in Appendix B.3.

6.4 Results

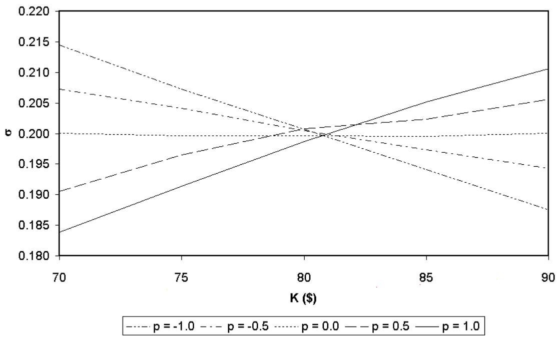
In what follows we examine how changing the parameters of the partial hedge and the correlation ρ effect the price and default risk of a partial hedging strategy.

Recall that C_*^B is the antithetic estimator of the claim price under the Geometric Brownian Motion. Therefore, its expected value $\mathbb{E}C_*^B$ is given by the analytic formulae of chapter 3. The particular formula depends on the parameters of the hedge $B(a, b, s)$. In general, we use the algorithm of section 4.3 and Appendix B.1.1 to calculate $\mathbb{E}C_*^B$. If a or $b = \infty$, the algorithm will not be accurate, as noted above, and we take $a = 1000$ or $b = 1000$ instead. Note however that if $s = T$ we may use the results of section 3.2. If in addition to this $b = \infty$ we may use the results of section 3.1, and if in addition also $a = \infty$, we have the perfect hedge, and the Black Scholes formula may be used to calculate $\mathbb{E}C_*^A$.

In fact, we do this in the first case, when we consider the effects of correlation ρ on the volatility smile. Recall that the volatility smile is a graph of implied volatility σ^* vs strike price K. We again use the Newton-Raphson algorithm (5.6) to calculate σ^* from our estimated prices.

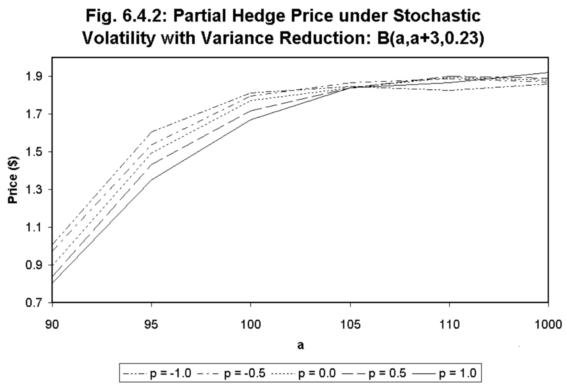
We find that the errors are significantly reduced using the variance reduction techniques, however, computationally the simulation takes far longer than in the case of the previous chapter. The typical errors of the prices are approximately 0.3%. This equates to an error of approximately 0.0005 in implied volatility. This is a large reduction from the 5% typical error and 0.0075 volatility error for the standard simulation. All values are statistically significant. The following graph indicates a strong dependence between the volatility smile and the correlation ρ:

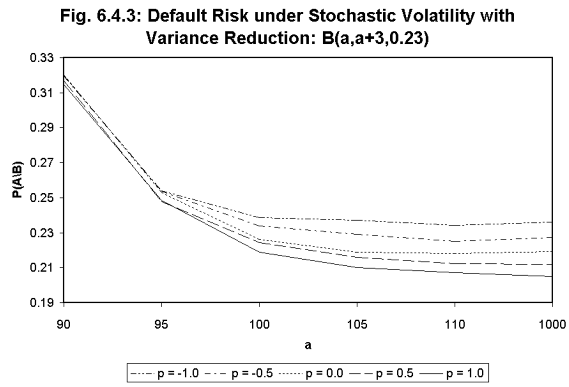
Fig. 6.4.1: Volatility Smile under Stochastic Volatility with Variance Reduction



Our results show that for positive ρ, the implied volatility increases as strike K increases, and conversely for negative ρ. If $\rho = 0$, the curve is slight and not significantly different from the flat line of the Black Scholes volatility "smile".

Following Chapter 5, we let $K = 80$ and first vary the parameter a, fixing $b = a + 3$ and $s = 0.23$. We obtain the following graphs of partial hedge prices and default risks, which indicate the relationship between partial hedge prices and default risks for different levels of correlation ρ:



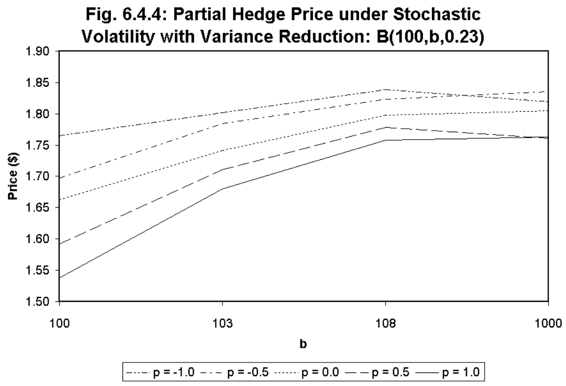


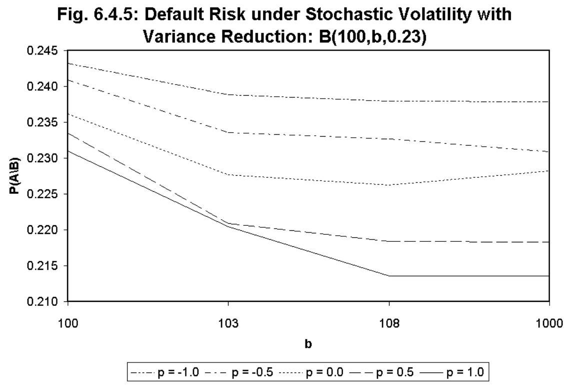
The errors are approximately 1% of the claim prices, so the prices are significantly different for $a \leq 100$. We see that generally for low values of a, the more positive the correlation, the lower the partial hedge price. But around $a = 105$ a threshold level seems to be reached, where the relationship between prices and correlation inverts. Note that, as we mentioned above and in chapter 3, the algorithm generating the expected value of the control variates estimator is inefficient for large a, and we limit ourself to $a = 1000$ for this reason. Despite this, we certainly have a graphical indication of some possible change in the character of the relationship between X^B and ρ moving from low a to high a. This may be caused by the remaining error, however it may also be the case that the afore-mentioned threshold level exists.

A qualitative analysis considers that in the case of positive ρ, the underlying asset price will move up or down with volatility. This means that for asset price increases, volatility increases, and for asset falls, volatilty falls. The overall impact is that high prices are more likely with high correlation, and this partial hedge *defaults* for high asset prices. When the value of a is large enough, however, larger payoffs are possible, so the hedge costs more with positive ρ. The converse holds for negative ρ, and thus we suppose that in for $\rho < 0$, the hedge price as a function of a appears more concave than in the $\rho > 0$ case. This results in a higher claim price for negative ρ when a is small, and a higher claim price for positive ρ when a is large.

We similarly find errors for the default risk of around 1%, giving mostly significant differences for $a \geq 100$. We find here a regular pattern indicating higher default risk for lower correlation. If we are to assume that greater risks correspond to greater expected returns, this would indicate that for low correlation we should get higher prices, which we do until around $a = 105$, as we mentioned above. Our results are consistent with this principle for $a = 100$ and $a = 105$, but suffer from overwhelming errors outside these two values. A study involving large values for the trials m and increments n would perhaps shed more light on this problem.

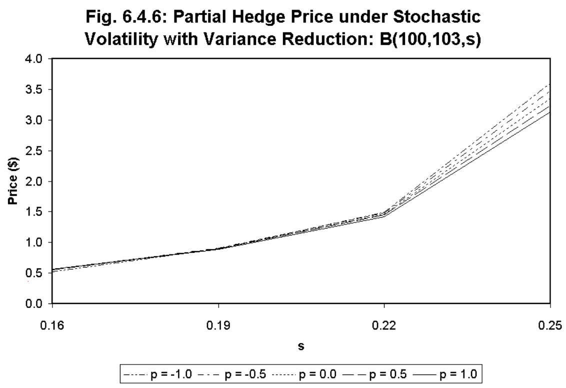
Now consider varying the parameter b. Let $a = 100$, $s = 0.23$ to obtain the following:

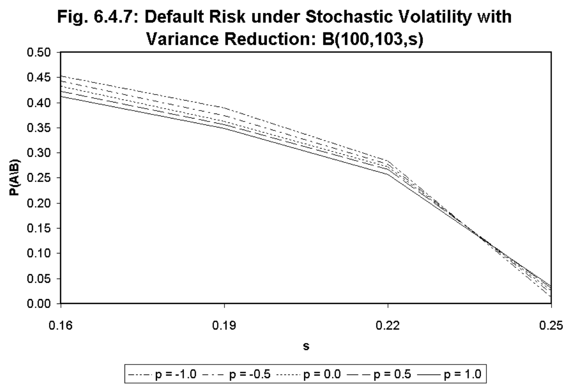




The typical errors are around 1%, giving significant differences between estimators in almost every case. These graphs provide a strong indication that high correlation corresponds to low returns and risks, considering the size of the errors. We have the risk flattening out as $b \to \infty$, which we expect.

Lastly, let $a = 100$ and $b = 103$, and vary the parameter s. The following graphs are produced:





The errors are all around 0.02 for prices and 0.002 for default risks, giving statistically significant differences in almost every case again. As noted in the previous chapter, s has the largest effect on the price and risk. The observations cluster close and generally display the regular behaviour of lower price and risk for more positive correlation, however this *ad hoc* rule is violated in the case $s = 0.25 = T$. This irregularity could be caused by the fact that the expectation of the control variates estimator is calculated using the formula for B_3 in chapter 2 for $s < 0.25$, and by numerical integration in the case $s = 0.25$. A problem is that the $s = T$ case has analytic results, and is therefore more accurate! Indeed, we ran the simulation using $s = 0.24999999$, and the results tabulated below indicate the same behaviour for high levels of s:

Table 6.4.1: Default Risk under Stochastic Volatility with Variance Reduction: $B(100, 103, 0.24999999)$

ρ	-1.0	-0.5	0.0	0.5	1.0
$\mathbb{P}(A\backslash B)$	0.0129	0.0209	0.0274	0.0326	0.0364
err	0.0016	0.0016	0.0015	0.0019	0.0023

Indeed, the estimators are significantly different, and we conclude that there exists a value of s such that the risk is close to the same for all values of correlation ρ. The absense of scaling and presence of error leaves it a little unclear, but it seems to show that for $s = 0.23$ risk is higher for more negative ρ and for s near T risk is higher for more positive ρ. They all seem to intersect—give or take errors—at one or more values of $s \in (0.22, 0.24999999)$. A possible qualitative explanation is that since the maximum of a process with positive drift is most likely to occur at the end of a given period, the positive correlation case is more likely to hit the upper barrier at b near expiry T. However, the negative correlation case has a less positive drift for reasons explained above, and is more likely to reach its maximum earlier. The negative correlation case then, while less likely to default because of reaching the upper barrier b, is more likely to default by reaching its maximum at a time less than s.

The tables used to produce the graphs in this chapter are reproduced in Appendix C.3, with associated errors.

Chapter 7

Conclusion

The partial hedging of contingent claims has practical application for investors needing to profitably hedge their exposure to written claims. In this thesis we have presented analytic formulae or Monte Carlo simulation procedures to calculate the costs of and risks attached to partial hedges of written European vanilla options. If such risks are appropriate to take, the partial hedge presents a way of reducing losses due to trading costs of rebalancing 'perfect' hedging strategies.

We examined partial hedges based on final asset prices, maximum asset prices over the life of the claim and the time this maximum was reached, and we investigated how changing the parameters of our partial hedges effects the costs and default risks of the partial hedge.

The costs and default risks of partial hedging strategies based on final asset prices and maximum asset prices over the life of the claim admit closed form solutions for the Geometric Brownian Motion model. These solutions were found, and a numerical integration procedure was proposed to evaluate costs and risks of partial hedges based on the time to the asset price maximum over the life of the claim.

Using a mean-reverting square root stochastic volatility model, the Monte Carlo simulation of a Euler approximation of the asset price path results in errors for our estimates which are too large given our computational budget. With application of the variance reduction techniques of Antithetic Variates and Control Variates, more accurate estimates of the partial hedge price and default risk were developed. It was found that the Control and Antithetic approaches gave consistently lower errors for given parameters.

Graphs and tables were produced to investigate the effect of the stochastic volatility model on implied volatility. The volatility smiles produced showed that the relationship between the implied volatility and strike price changes for different levels of the correlation between the random drivers of the asset price and volatility processes.

Graphs and tables were produced for different levels of the hedge parameters and the correlation coefficient. Some dependence between the partial hedge parameters and the correlation was also noted in section 6.4 which require deeper investigation. Notably, it was found that the relationship between claim price and correlation changes for high values of the hedge parameter a, and that the relationship between default risk and correlation changes for values of s near expiry T.

Proposals for further study would include measuring dependence of the partial hedge on the volatility and real world measure parameters, and the construction of hedges themselves.

The partial hedging strategies noted above also form perfect hedges for exotic options in themselves, and therefore the analysis presented is applicable also to the pricing and hedging of exotic options. A general framework was presented that allows the formulation of pricing functions for exotic options of any required characteristics.

Appendix A

A Derivation of the Black Scholes Formula

We consider the market of Chapter 4 with trading in two assets: a risky asset and the risk free bond. The single risky asset follows the stochastic differential equation under $\mathbb{P}$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^P(t), \qquad 0 \leq t \leq T.$$

By Itô's formula,

$$\begin{aligned} d(e^{-rt}S(t)) &= (\mu - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW^P(t) \\ &= \sigma e^{-rt}S(t)\Big[\frac{\mu - r}{\sigma}dt + dW^P(t)\Big] \\ &= \sigma e^{-rt}S(t)dW^Q(t), \end{aligned}$$

which is a $\mathbb{Q}$-Martingale. The market price of risk is given by $\theta = \frac{\mu - r}{\sigma}$.

Now,

$$\begin{aligned} d(e^{-rt}S(t)) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ \Rightarrow dS(t) &= rS(t)dt + \sigma S(t)dW^Q(t) \\ \Rightarrow S(t) &= S(0) \cdot \exp\Big\{\Big(r - \frac{1}{2}\sigma^2\Big)t + \sigma W^Q(t)\Big\}. \end{aligned}$$

We have that

$$dW^Q(t) = \frac{\mu - r}{\sigma}dt + dW^P(t) \Rightarrow W^Q(t) = \frac{\mu - r}{\sigma}t + W^P(t)$$

is a $\mathbb{Q}$-Brownian Motion by Girsanov's Theorem. The measure $\mathbb{Q}$ is therefore an Equivalent Martingale Measure, and claims with payoff $Y \geq 0$ may be priced at time $t < T$ as the discounted expectation of the payoff over $\mathbb{Q}$:

$$X(t) = e^{-r(T-t)}\mathbb{E}^{Q}[Y|\mathcal{F}_t].$$

The Radon-Nikodỳm Derivative is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(t) = \exp\bigg\{-\frac{1}{2}\theta^2 - \theta W^P(t)\bigg\}.$$

Consider a European call option, which pays $(S(T) - K)^+$ at time T. The price of this claim at time 0, $X_{BS}(0)$, is given by

$$X_{BS}(0) = e^{-rT}\mathbb{E}^{Q}[(S(T) - K)^+]$$

$$\begin{aligned}
&= e^{-rT}\mathbb{E}^{Q}\Bigg[\bigg(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma W^Q(T)} - K\bigg)\mathbf{1}_{\{S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma W^Q(T)}>K\}}\Bigg] \\
&= e^{-rT}\mathbb{E}^{Q}\Bigg[\bigg(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}W^Q(1)} - K\bigg)\mathbf{1}_{\{W^Q(1)>\frac{1}{\sigma\sqrt{T}}[\ln\frac{K}{S(0)}-(r-\frac{1}{2}\sigma^2)T]\}}\Bigg].
\end{aligned}$$

Let

$$\frac{1}{\sigma\sqrt{T}}\bigg[\ln\frac{K}{S(0)} - (r - \frac{1}{2}\sigma^2)T\bigg] = -d_-$$

and

$$\{W^Q(1) > -d_-\} = \{W^Q(1) < d_-\} = A.$$

Then

$$X_{BS}(0) = e^{-rT}\int_A \bigg(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}W^Q(1)} - K\bigg)d\mathbb{Q}.$$

But $W^Q(1) \sim N(0, 1)$. So

$$\begin{aligned}
X_{BS}(0) &= e^{-rT}\int_{-\infty}^{d_-}\bigg(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} - K\bigg)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \\
&= S(0)\int_{-\infty}^{d_-}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2}dx - Ke^{-rT}\int_{-\infty}^{d_-}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \\
&= S(0)\cdot N(d_+) - Ke^{-rT}\cdot N(d_-),
\end{aligned}$$

where $N(\cdot)$ denotes the standard normal distribution function and $d_+ = d_- + \sigma\sqrt{T}$. This is indeed the Black Scholes formula.

Appendix B

Microsoft VBA 6.0 Code

B.1 Functions for Chapter 4

The functions *ph_1*, *barrier* and *PA* calculate the analytic fomulae given in Chapter 4: X^{B_1} under GBM, X^{B_2} under GBM and $\mathbb{P}(B_2)$ under GBM, respectively.

```
Function ph_1(S, v, r, T, K, a)
'Calculates XB_1 under GBM

    Dim dK, da As Double

    dK = (Log(S / K) + (r + v ^ 2 / 2) * T) / (v * Sqr(T))
    da = (Log(S / a) + (r + v ^ 2 / 2) * T) / (v * Sqr(T))

ph_1 = S * (Application.NormSDist(dK) - Application.NormSDist(da)) - _
K * Exp(-r * T) * (Application.NormSDist(dK - v * Sqr(T)) - _
Application.NormSDist(da - v * Sqr(T)))

End Function

Function barrier(T, r, s0, v, K, a, b)
'Calculates XB_2 under GBM

Dim th, KQ, bQ, aQ As Double
    th = (r - 0.5 * v ^ 2) / v
    KQ = Log(K / s0) / v
    bQ = Log(b / s0) / v
```

```
    aQ = Log(a / s0) / v

Dim C As Double
    C = Exp(-r * T - 0.5 * th ^ 2 * T)

Dim A1, A2, A3, A4 As Double
    A1 = s0 * Exp(0.5 * T * (th + v) ^ 2) * (Application.NormSDist(( _
     aQ - T * (th + v)) / Sqr(T)) - Application.NormSDist((KQ - T _
     * (th + v)) / Sqr(T)))
    A2 = -s0 * Exp(0.5 * T * (th + v) ^ 2 + 2 * bQ * (th + v)) * _
     (Application.NormSDist((aQ - 2 * bQ - T * (th + v)) / Sqr(T)) _
     - Application.NormSDist((KQ - 2 * bQ - T * (th + v)) / Sqr(T)))
    A3 = -K * Exp(0.5 * T * th ^ 2) * (Application.NormSDist((aQ - T _
     * th) / Sqr(T)) - Application.NormSDist((KQ - T * th) / Sqr(T)))
    A4 = K * Exp(0.5 * T * th ^ 2 + 2 * bQ * th) * _
     (Application.NormSDist((aQ - 2 * bQ - T * th) / Sqr(T)) - _
     Application.NormSDist((KQ - 2 * bQ - T * th) / Sqr(T)))

barrier = C * (A1 + A2 + A3 + A4)

End Function

Function PA(T, mu, s0, v, K, a, b)
'Calculates P(B_2) under GBM

Dim th, KQ, bQ, aQ As Double
    th = (mu - 0.5 * v ^ 2) / v
    KQ = Log(K / s0) / v
    bQ = Log(b / s0) / v
    aQ = Log(a / s0) / v

PA = (Application.NormSDist((aQ - T * th) / Sqr(T)) - _
Application.NormSDist((KQ - T * th) / Sqr(T))) - Exp(2 * bQ * th) * _
(Application.NormSDist((aQ - 2 * bQ - T * th) / Sqr(T)) - _
Application.NormSDist((KQ - 2 * bQ - T * th) / Sqr(T)))

End Function
```

B.1.1 Numerical Integration for Section 4.3

Recall that under GBM the partial hedge on the set
$B_3 = \{K < S(T) < a, \max_{0 \le t \le T} S(t) < b, M^S(T) < s\}$ has price

$$X^{B_3} = e^{-rT} \int_{K^*}^{a^*} \int_{x}^{b^*} \Big(S(0)e^{\sigma x} - K\Big) e^{\phi x - \frac{1}{2}\phi^2 T} \psi(x, y; s) dy dx$$

and default risk

$$\mathbb{P}(A \backslash B_3) = N\Big(d_-^{\mu}\big(K\big)\Big) - e^{-\frac{1}{2}T(\theta+\phi)^2} \int_{K^*}^{a^*} e^{(\theta+\phi)x} \int_{x}^{b^*} \psi(x, y; s) dy dx.$$

This leads us to the numerical procedures

$$\begin{aligned} X^{B_3} &= e^{-rT - \frac{1}{2}\phi^2 T} \\ &\times \lim_{H \to \infty} \sum_{i=1}^{H} \Big(S(0)e^{\sigma x_i} - K\Big) e^{\phi x_i} \Delta_1 \left[\sum_{j=i}^{H} \psi(x_i, y_j; s)\Delta_1 + \sum_{k=1}^{H} \psi(x_i, y_k'; s)\Delta_2 \right] \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(A \backslash B_3) &= N(d_-^{\mu}(K)) - e^{-\frac{1}{2}T(\phi+\theta)^2} \\ &\times \lim_{H \to \infty} \sum_{i=1}^{H} e^{(\phi+\theta)x_i} \Delta_1 \cdot \left[\sum_{j=i}^{H} \psi(x_i, y_j; s)\Delta_1 + \sum_{k=i}^{H} \psi(x_i, y_k'; s)\Delta_2 \right]. \end{aligned}$$

This is implemented using the following iterative procedure:

```
Function XB3GBM(s0, v, r, T, K, a, b, s, n)
'Numerical Integration for XB_3

Dim phi, KQ, aQ, bQ As Double
phi = (r - 0.5 * v ^ 2) / v
KQ = Log(K / s0) / v
aQ = Log(a / s0) / v
bQ = Log(b / s0) / v

Dim dt, ds As Double
'Interval length
dt = (aQ - KQ) / n
ds = (bQ - aQ) / n

Dim f, fi As Variant
```

```
f = 0

Dim x, y As Variant

For i = 1 To n
'Integration over x
    x = KQ + (2 * i - 1) * dt / 2
    fi = 0
    For j = i + 1 To n
    'First integration over y, given x
        y = KQ + (2 * j - 1) * dt / 2
        fi = fi + (2 / Sqr(2 * Application.Pi() * T ^ 3)) * _
            (Application.NormSDist((y * (s - T) + s * (y - x)) / _
            Sqr(s * T * (T - s))) * (2 * y - x) * Exp(-(2 * y - x) _
            ^ 2 / (2 * T)) - Application.NormSDist((y * (s - T) - _
            s * (y - x)) / Sqr(s * T * (T - s))) * x * Exp(-(x ^ 2) _
            / (2 * T))) * dt
    Next j
    For j = 1 To n
    'Second integration over y, given x
        y = aQ + (2 * j - 1) * ds / 2
        fi = fi + (2 / Sqr(2 * Application.Pi() * T ^ 3)) * _
            (Application.NormSDist((y * (s - T) + s * (y - x)) / _
            Sqr(s * T * (T - s))) * (2 * y - x) * Exp(-(2 * y - x) _
            ^ 2 / (2 * T)) - Application.NormSDist((y * (s - T) - _
            s * (y - x)) / Sqr(s * T * (T - s))) * x * Exp(-(x ^ 2) _
            / (2 * T))) * ds
    Next j
    f = f + fi * (s0 * Exp(v * x) - K) * Exp(phi * x) * dt
Next i

XB3GBM = f * Exp(-r * T - phi ^ 2 * T / 2)

End Function

Function PAB3GBM(s0, v, r, T, K, a, b, s, mu, n)
'Numerical Integration for P(B_3)

Dim PhiTh, KQ, aQ, bQ As Double
PhiTh = (mu - 0.5 * v ^ 2) / v
KQ = Log(K / s0) / v
```

```
aQ = Log(a / s0) / v
bQ = Log(b / s0) / v

Dim dt, ds As Double
'Interval length
dt = (aQ - KQ) / n
ds = (bQ - aQ) / n

Dim g, gi As Variant
g = 0

Dim x, y As Variant

For i = 1 To n
'Integration over x
    x = KQ + (2 * i - 1) * dt / 2
    gi = 0
    For j = i + 1 To n
    'First integration over y, given x
        y = KQ + (2 * j - 1) * dt / 2
        gi = gi + (2 / Sqr(2 * Application.Pi() * T ^ 3)) * _
            (Application.NormSDist((y * (s - T) + s * (y - x)) _
            / Sqr(s * T * (T - s))) * (2 * y - x) * Exp(-(2 * y - x) _
            ^ 2 / (2 * T)) - Application.NormSDist((y * (s - T) - _
            s * (y - x)) / Sqr(s * T * (T - s))) * x * Exp(-(x ^ 2) _
            / (2 * T))) * dt
    Next j
    For j = 1 To n
    'Second integration over y, given x
        y = aQ + (2 * j - 1) * ds / 2
        gi = gi + (2 / Sqr(2 * Application.Pi() * T ^ 3)) * _
            (Application.NormSDist((y * (s - T) + s * (y - x)) _
            / Sqr(s * T * (T - s))) * (2 * y - x) * Exp(-(2 * y - x) _
            ^ 2 / (2 * T)) - Application.NormSDist((y * (s - T) - _
            s * (y - x)) / Sqr(s * T * (T - s))) * x * Exp(-(x ^ 2) _
            / (2 * T))) * ds
    Next j
    g = g + gi * Exp(PhiTh * x) * dt
Next i

PAB3GBM = Application.NormSDist(Log(s0 / K) / (v * Sqr(T)) + PhiTh * _
```

```
Sqr(T)) - g * Exp(-0.5 * T * PhiTh ^ 2)

End Function
```

B.2 Chapter 5 Simulations

Consider the iterative procedure in chapter 5. The function *SVNX* gives $\hat{X}^B$ and *SVNP* gives $\hat{\mathbb{P}}(A\backslash B)$. The function *impvol* gives the implied volatility of a perfect hedge price under stochastic volatility. Note that the simulations of prices and default probabilities do not use the same randomly generated variables, so are independent.

```
Function SVNX(s0, v0, al, av, lmb, rho, r, K, T, n, m, A, b, s)
'Monte Carlo simulation of X^B under stochastic volatility

Dim dt As Double
'Interval length
    dt = T / n

Dim F, St, G, X, Mx, Tx, V, Q1, Q2 As Variant

Dim Ssum, Ssum2 As Variant
    Ssum = 0
    Ssum2 = 0

For j = 1 To m
'Runs simulation m times

    St = s0
    V = v0
    Mx = s0
    Tx = 0

    For i = 1 To n
    'Euler method with n intervals

        Q1 = Application.NormSInv(Rnd)
        F = St
        St = St + r * St * dt + Sqr(V) * St * Sqr(dt) * Q1
```

```
        G = (St - F) / Sqr(V)
        X = F + Sqr(V) * 0.5 * (G + Sqr(G ^ 2 - 2 * Log(Rnd) * dt))
        If Mx < X Then Mx = X: Tx = (i - 0.5) * dt
        Q2 = Application.NormSInv(Rnd)
        V = V + al * (av - V) * dt + lmb * Sqr(V) * Sqr(dt) * _
         (rho * Q1 + Sqr(1 - rho ^ 2) * Q2)
        If V < 0 Then V = -V
    Next i

    'The following gives 0 if outside the partial hedge, and _
     Y^B otherwise
    If St < K Then
        St = 0
    ElseIf St > A Then
        St = 0
    ElseIf Mx > b Then
        St = 0
    ElseIf Tx > s Then
        St = 0
    Else: St = Exp(-r * T) * (St - K)
    End If

    Ssum = Ssum + St
    Ssum2 = Ssum2 + St ^ 2

Next j

Dim outX(1, 0) As Double
'Gives the point estimate and standard error over m trials
    outX(0, 0) = Ssum / m
    outX(1, 0) = 1.96 * Sqr((m * Ssum2 - Ssum ^ 2) / (m - 1)) / m

SVNX = outX

End Function

Function SVNP(s0, v0, al, av, lmb, rho, r, K, T, mu1, mu2, vol2, _
n, m, A, b, s)
'Monte Carlo simulation of P(B) under stochastic volatility

Dim dt As Double
```

```
'Interval length
    dt = T / n

Dim F, St, G, X, Mx, Tx, V, P1, P2 As Variant

Dim Isum As Variant
    Isum = 0

For j = 1 To m
'Runs simulation m times

    St = s0
    V = v0
    Mx = s0
    Tx = 0

    For i = 1 To n
    'Euler method with n intervals
        P1 = Application.NormSInv(Rnd)
        F = St
        St = St + mu1 * St * dt + Sqr(V) * St * Sqr(dt) * P1
        G = (St - F) / Sqr(V)
        X = F + Sqr(V) * 0.5 * (G + Sqr(G ^ 2 - 2 * Log(Rnd) * dt))
        If Mx < X Then Mx = X: Tx = (i - 0.5) * dt
        P2 = Application.NormSInv(Rnd)
        V = V + (al * (av - V) + lmb * rho * (mu1 - r) + lmb * _
         Sqr(1 - rho ^ 2) * Sqr(V) * (mu2 - r) / vol2) * dt + _
         lmb * Sqr(V) * Sqr(dt) * (rho * P1 + Sqr(1 - rho ^ 2) * P2)
        If V < 0 Then V = -V
    Next i

'The following gives 0 if the claim is hedged and 1 otherwise
    If St > A Then
        IND = 1
    ElseIf Mx > b Then
        IND = 1
    ElseIf Tx > s Then
        IND = 1
    Else: IND = 0
    End If
```

```
    Isum = Isum + IND

Next j

Dim outP(1, 0) As Double
'Gives point estimate and standard error over m trials
    outP(0, 0) = Isum / m
    outP(1, 0) = 1.96 * Sqr((m * Isum - Isum ^ 2) / (m - 1)) / m

SVNP = outP

End Function
Function impvol(s0, v0, r, T, K, XA)
'Algorithm for implied volatility

    Dim A, V, dplus As Variant
    V = v0

    Do
        dplus = (Log(s0 / K) + (r + 0.5 * V ^ 2) * T) / (V * Sqr(T))

        A = V
        V = V - (s0 * Application.NormSDist(dplus) - K * Exp(-r * T) _
         * Application.NormSDist(dplus - V * Sqr(T)) - XA) / _
         (s0 * Sqr(T) * Application.NormDist(dplus, 0, 1, False))

    Loop While (A - V) ^ 2 > 0.00000001

impvol = V

End Function
```

B.3 Chapter 6 Simulations

Consider the Variance Reduction techniques employed in chapter 6. The function *SVACX* gives the variance reduced estimator of the partially hedged claim price under stochastic volatility, and *SVACP* gives the default risk. Note that the statements calculating $\mathbb{E}C_*^B$ refer to functions defined previously in Chapter 4.

```
Function SVACX(s0, v0, al, av, lmb, rho, r, K, T, n, m, a, b, S, H)
```

```
'Estimates X^B under stochastic volatility with variance reduction

Dim dt As Double
'Interval length
    dt = T / n

Dim EC As Double
'Expected value of price with constant volatility (GBM)
'Refers to calculation formulae for chapter 4
    If S = T Then
        If b > 500 Then
            If a > 500 Then
                EC = s0 * Application.NormSDist((Log(s0 / K) + T * _
                 (r + 0.5 * v0)) / Sqr(v0 * T)) - K * Exp(-r * T) * _
                 Application.NormSDist((Log(s0 / K) + T * (r - 0.5 _
                 * v0)) / Sqr(v0 * T))
            Else: EC = ph_1(S, Sqr(v0), r, T, K, a)
            End If
        Else: EC = barrier(T, r, s0, Sqr(v0), K, a, b)
        End If
    Else: EC = XB3GBM(s0, Sqr(v0), r, T, K, a, b, S, H)
    End If

Dim Q1, Q2, U As Variant

Dim F, St, G, X, Mx, Tx, v As Variant
Dim Fa, Sta, Ga, Xa, Mxa, Txa, Va As Variant

Dim FC, StC, GC, XC, MxC, TxC As Variant
Dim FCa, StCa, GCa, XCa, MxCa, TxCa As Variant

Dim SA, CA As Variant

Dim Ssum, Ssum2 As Variant
    Ssum = 0
    Ssum2 = 0

Dim Csum, Csum2 As Variant
    Csum = 0
    Csum2 = 0
```

```
Dim SCsum As Variant
    SCsum = 0

For j = 1 To m
'Monte Carlo simulation with m trials
    St = s0
    v = v0
    Mx = s0
    Tx = 0

    Sta = s0
    Va = v0
    Mxa = s0
    Txa = 0

    StC = s0
    MxC = s0
    TxC = 0

    StCa = s0
    MxCa = s0
    TxCa = 0

    For i = 1 To n
    'Euler method with n intervals
        Q1 = Application.NormSInv(Rnd)
        Q2 = Application.NormSInv(Rnd)
        U = Rnd

        'Stochastic vol
        F = St
        St = St + r * St * dt + Sqr(v) * St * Sqr(dt) * Q1
        G = (St - F) / Sqr(v)
        X = F + Sqr(v) * 0.5 * (G + Sqr(G ^ 2 - 2 * Log(U) * dt))
        If Mx < X Then Mx = X: Tx = (i - 0.5) * dt
        v = v + al * (av - v) * dt + lmb * Sqr(v) * Sqr(dt) * (rho * _
         Q1 + Sqr(1 - rho ^ 2) * Q2)
        If v < 0 Then v = -v

        'Antithetic with stochastic vol
        Fa = Sta
```

```
        Sta = Sta + r * Sta * dt + Sqr(Va) * Sta * Sqr(dt) * (-Q1)
        Ga = (Sta - Fa) / Sqr(Va)
        Xa = Fa + Sqr(Va) * 0.5 * (Ga + Sqr(Ga ^ 2 - 2 * Log(U) * dt))
        If Mxa < Xa Then Mxa = Xa: Txa = (i - 0.5) * dt
        Va = Va + al * (av - Va) * dt + lmb * Sqr(Va) * Sqr(dt) * _
         (rho * (-Q1) + Sqr(1 - rho ^ 2) * Q2)
        If Va < 0 Then Va = -Va

        'Constant vol
        FC = StC
        StC = StC + r * StC * dt + Sqr(v0) * StC * Sqr(dt) * Q1
        GC = (StC - FC) / Sqr(v0)
        XC = FC + Sqr(v0) * 0.5 * (GC + Sqr(GC ^ 2 - 2 * Log(U) * dt))
        If MxC < XC Then MxC = XC: TxC = (i - 0.5) * dt

        'Antithetic with constant vol
        FCa = StCa
        StCa = StCa + r * StCa * dt + Sqr(v0) * StCa * Sqr(dt) * (-Q1)
        GCa = (StCa - FCa) / Sqr(v0)
        XCa = FCa + Sqr(v0) * 0.5 * (GCa + Sqr(GCa ^ 2 - 2 * Log(U) _
         * dt))
        If MxCa < XCa Then MxCa = XCa: TxCa = (i - 0.5) * dt
    Next i

'Payoff of partial hedge - stochastic vol
    If St < K Then
        St = 0
    ElseIf St > a Then
        St = 0
    ElseIf Mx > b Then
        St = 0
    ElseIf Tx > S Then
        St = 0
    Else: St = Exp(-r * T) * (St - K)
    End If

    'Payoff of partial hedge - antithetic with stochastic vol
    If Sta < K Then
        Sta = 0
    ElseIf Sta > a Then
        Sta = 0
```

```
ElseIf Mxa > b Then
    Sta = 0
ElseIf Txa > S Then
    Sta = 0
Else: Sta = Exp(-r * T) * (Sta - K)
End If

'Payoff of partial hedge - constant vol
If StC < K Then
    StC = 0
ElseIf StC > a Then
    StC = 0
ElseIf MxC > b Then
    StC = 0
ElseIf TxC > S Then
    StC = 0
Else: StC = Exp(-r * T) * (StC - K)
End If

'Payoff of partial hedge - antithtic with constant vol
If StCa < K Then
    StCa = 0
ElseIf StCa > a Then
    StCa = 0
ElseIf MxCa > b Then
    StCa = 0
ElseIf TxCa > S Then
    StCa = 0
Else: StCa = Exp(-r * T) * (StCa - K)
End If

'Average payoffs for stochastic and constant vol
SA = 0.5 * (St + Sta)
CA = 0.5 * (StC + StCa)

Ssum = Ssum + SA
Ssum2 = Ssum2 + SA ^ 2

Csum = Csum + CA
Csum2 = Csum2 + CA ^ 2
```

```
    SCsum = SCsum + (SA * CA)

Next j

Dim beta As Double
    beta = (SCsum - Ssum * Csum / m) / (Csum2 - (Csum ^ 2) / m)

Dim outXCa(1, 0) As Double
'Monte Carlo estimators of price and standard error
'under Control and Antithetic variates method
    outXCa(0, 0) = Ssum / m - beta * (Csum / m - EC)
    outXCa(1, 0) = 1.96 * Sqr((m * Ssum2 - Ssum ^ 2) / (m - 1) - _
(beta ^ 2) * (m * Csum2 - Csum ^ 2) / (m - 1)) / m

SVACX = outXCa

End Function

Function SVACP(s0, v0, al, av, lmb, rho, r, K, T, mu1, mu2, vol2, _
n, m, a, b, S, H)
'Estimates P(B) under stochastic volatility with variance reduction

Dim dt As Double
    dt = T / n

Dim EC As Double
'Calculates prob of default with constant vol (GBM):
'refers to functions in chapter 4
    If S = T Then
        If b > 500 Then
            If a > 500 Then
                EC = 0
            Else: EC = Application.NormSDist((Log(s0 / a) + T * (mu1 _
             - 0.5 * v0)) / Sqr(v0 * T))
            End If
        Else: EC = Application.NormSDist((Log(s0 / K) + T * (mu1 - _
         0.5 * v0)) / Sqr(v0 * T)) - PA(T, r, s0, Sqr(v0), K, a, b)
        End If
    Else: EC = PAB3GBM(s0, Sqr(v0), r, T, K, a, b, S, mu1, H)
    End If
```

```
Dim P1, P2, U As Variant

Dim F, St, G, X, Mx, Tx, v As Variant
Dim Fa, Sta, Ga, Xa, Mxa, Txa, Va As Variant

Dim FC, StC, GC, XC, MxC, TxC As Variant
Dim FCa, StCa, GCa, XCa, MxCa, TxCa As Variant

Dim SA, CA As Variant

Dim Ssum, Ssum2 As Variant
    Ssum = 0
    Ssum2 = 0

Dim Csum, Csum2 As Variant
    Csum = 0
    Csum2 = 0

Dim SCsum As Variant
    SCsum = 0

For j = 1 To m

    St = s0
    v = v0
    Mx = s0
    Tx = 0

    Sta = s0
    Va = v0
    Mxa = s0
    Txa = 0

    StC = s0
    MxC = s0
    TxC = 0

    StCa = s0
    MxCa = s0
    TxCa = 0
```

```
For i = 1 To n
    P1 = Application.NormSInv(Rnd)
    P2 = Application.NormSInv(Rnd)
    U = Rnd

    'Stochastic vol
    F = St
    St = St + mu1 * St * dt + Sqr(v) * St * Sqr(dt) * P1
    G = (St - F) / Sqr(v)
    X = F + Sqr(v) * 0.5 * (G + Sqr(G ^ 2 - 2 * Log(U) * dt))
    If Mx < X Then Mx = X: Tx = (i - 0.5) * dt
    v = v + (al * (av - v) + lmb * rho * (mu1 - r) + lmb * Sqr(1 _
     - rho ^ 2) * Sqr(v) * (mu2 - r) / vol2) * dt + lmb * Sqr(v) _
     * Sqr(dt) * (rho * P1 + Sqr(1 - rho ^ 2) * P2)
    If v < 0 Then v = -v

    'Antithetic under stochastic vol
    Fa = Sta
    Sta = Sta + mu1 * Sta * dt + Sqr(Va) * Sta * Sqr(dt) * (-P1)
    Ga = (Sta - Fa) / Sqr(Va)
    Xa = Fa + Sqr(Va) * 0.5 * (Ga + Sqr(Ga ^ 2 - 2 * Log(U) * dt))
    If Mxa < Xa Then Mxa = Xa: Txa = (i - 0.5) * dt
    Va = Va + (al * (av - Va) + lmb * rho * (mu1 - r) + lmb * _
     Sqr(1 - rho ^ 2) * Sqr(Va) * (mu2 - r) / vol2) * dt + lmb * _
     Sqr(Va) * Sqr(dt) * (rho * (-P1) + Sqr(1 - rho ^ 2) * P2)
    If Va < 0 Then Va = -Va

    'Constant vol
    FC = StC
    StC = StC + mu1 * StC * dt + Sqr(v0) * StC * Sqr(dt) * P1
    GC = (StC - FC) / Sqr(v0)
    XC = FC + Sqr(v0) * 0.5 * (GC + Sqr(GC ^ 2 - 2 * Log(U) * dt))
    If MxC < XC Then MxC = XC: TxC = (i - 0.5) * dt

    'Antithetic with constant vol
    FCa = StCa
    StCa = StCa + mu1 * StCa * dt + Sqr(v0) * StCa * Sqr(dt) * _
     (-P1)
    GCa = (StCa - FCa) / Sqr(v0)
    XCa = FCa + Sqr(v0) * 0.5 * (GCa + Sqr(GCa ^ 2 - 2 * Log(U) * _
     dt))
```

```
    If MxCa < XCa Then MxCa = XCa: TxCa = (i - 0.5) * dt
Next i

'Indicator of default - stochastic vol
If St > a Then
    St = 1
ElseIf Mx > b Then
    St = 1
ElseIf Tx > S Then
    St = 1
Else: St = 0
End If

'Indicator of default - antithetic with stochastic vol
If Sta > a Then
    Sta = 1
ElseIf Mxa > b Then
    Sta = 1
ElseIf Txa > S Then
    Sta = 1
Else: Sta = 0
End If

'Indicator of default - constant vol
If StC > a Then
    StC = 1
ElseIf MxC > b Then
    StC = 1
ElseIf TxC > S Then
    StC = 1
Else: StC = 0
End If

'Indicator of default - antithetic with constant vol
If StCa > a Then
    StCa = 1
ElseIf MxCa > b Then
    StCa = 1
ElseIf TxCa > S Then
    StCa = 1
Else: StCa = 0
```

```
    End If

'Average of estimates under stochastic and constant vol
    SA = 0.5 * (St + Sta)
    CA = 0.5 * (StC + StCa)

    Ssum = Ssum + SA
    Ssum2 = Ssum2 + SA ^ 2

    Csum = Csum + CA
    Csum2 = Csum2 + CA ^ 2

    SCsum = SCsum + (SA * CA)

Next j

Dim beta As Double
    beta = (SCsum - Ssum * Csum / m) / (Csum2 - (Csum ^ 2) / m)

Dim outPCa(1, 0) As Double
'Monte Carlo point estimate and standard error of P(B)
'using Antithetic and Control variates method
    outPCa(0, 0) = Ssum / m - beta * (Csum / m - EC)
    outPCa(1, 0) = 1.96 * Sqr((m * Ssum2 - Ssum ^ 2) / (m - 1) _
     - (beta ^ 2) * (m * Csum2 - Csum ^ 2) / (m - 1) ) / m

SVACP = outPCa

End Function
```

Appendix C

Results Tables

C.1 Chapter 4 Tables

The following tables were produced for the results of partial hedging strategies under the Geometric Brownian Motion. The data here was used to produce the graphs found in chapter 4.

Table C.1.1: Partial Hedge Price & Default Risk under GBM: $B_1 = \{K < S(T) < a\}$.

a		90	95	100	110	120	∞
X_1^B		1.7481	2.8013	3.4115	3.7642	3.7959	3.7975
$\mathbb{P}(A \backslash B_1)$	$\mu = 0.06$	0.1406	0.0528	0.0165	0.0010	0.0000	0
	$\mu = 0.10$	0.1641	0.0644	0.0211	0.0014	0.0001	0
	$\mu = 0.14$	0.1900	0.0780	0.0267	0.0020	0.0001	0
	$\mu = 0.18$	0.2183	0.0937	0.0335	0.0027	0.0001	0

Table C.1.2: Partial Hedge Price under GBM: $B_2 = \{K < S(T) < a, \max S(t) < b\}$.

a	90	95	100	105	110	115	120	∞
$X^{B_2(a+3)}$	1.5089	2.6714	3.3628	3.6600	3.7607	3.7889	3.7957	3.7975
$X^{B_2(a+8)}$	1.7322	2.7952	3.4098	3.6740	3.7642	3.7897	3.7959	3.7975
$X^{B_2(\infty)}$	1.7481	2.8013	3.4115	3.6744	3.7642	3.7897	3.7959	3.7975

Table C.1.3: Default Risk under GBM: $B_2 = \{K < S(T) < a, \max S(t) < b\}$.

b	100	102	104	106	108	110	112	∞
$\mu = 0.06$	0.0319	0.0217	0.0181	0.0169	0.0166	0.0165	0.0165	0.0165
$\mu = 0.10$	0.0714	0.0611	0.0575	0.0564	0.0561	0.0560	0.0560	0.0560
$\mu = 0.14$	0.1100	0.0998	0.0962	0.0950	0.0947	0.0946	0.0946	0.0946
$\mu = 0.18$	0.1475	0.1373	0.1337	0.1325	0.1322	0.1321	0.1321	0.1321

Table C.1.4: Partial Hedge Price under GBM:
$B_3 = \{K < S(T) < a, \max S(t) < a + 3, M^S(T) < s\}$.

a	90	95	100	105	110	115	120	1000
$X^{B_3(0.16)}$	0.3672	0.5120	0.5544	0.5623	0.5634	0.5635	0.5635	0.5634
$X^{B_3(0.19)}$	0.5311	0.7994	0.9002	0.9258	0.9306	0.9314	0.9315	0.9311
$X^{B_3(0.22)}$	0.7795	1.2523	1.4683	1.5374	1.5545	1.5579	1.5585	1.5579
$X^{B_3(0.25)}$	1.5074	2.6666	3.3539	3.6474	3.7451	3.7709	3.7755	3.6719

Table C.1.5: Default Risk under GBM:
$B_3 = \{K < S(T) < a, \max S(t) < a + 3, M^S(T) < s\}$.

s	0.16	0.19	0.22	0.25
$\mu = 0.06$	0.3921	0.3307	0.2440	0.0208
$\mu = 0.10$	0.4268	0.3619	0.2695	0.0261
$\mu = 0.14$	0.4619	0.3940	0.2960	0.0326
$\mu = 0.18$	0.4971	0.4265	0.3235	0.0402

C.2 Chapter 5 Tables

The following tables were used to produce the graphs in Chapter 5, for partially hedged claim prices and default risks under Stochastic Volatility.

Table C.2.1: Volatility Smile for Stochastic Volatility

K		70	75	80	85	90
$\rho = -1.0$	$\hat{X}^A$	11.2724	7.2269	3.8138	1.7755	0.5517
	err	0.2329	0.2013	0.1534	0.1069	0.0583
	σ^*	0.2011	0.2157	0.2010	0.2030	0.1882
$\rho = -0.5$	$\hat{X}^A$	11.1836	6.8981	3.8167	1.7267	0.5834
	err	0.2335	0.2031	0.1616	0.1091	0.0626
	σ^*	0.1810	0.1863	0.2012	0.1997	0.1916
$\rho = 0.0$	$\hat{X}^A$	11.1870	6.9805	3.8244	1.7753	0.6542
	err	0.2338	0.2098	0.1672	0.1158	0.0709
	σ^*	0.1818	0.1939	0.2017	0.2030	0.1989
$\rho = +0.5$	$\hat{X}^A$	11.5399	7.1862	3.7627	1.7501	0.7005
	err	0.2420	0.2172	0.1673	0.1193	0.0772
	σ^*	0.2451	0.2122	0.1978	0.2013	0.2036
$\rho = +1.0$	$\hat{X}^A$	11.1843	7.1269	3.6661	1.7686	0.7821
	err	0.2375	0.2208	0.1755	0.1217	0.0814
	σ^*	0.1812	0.2070	0.1916	0.2025	0.2215
X_{BS}		11.2670	7.0476	3.7975	1.7311	0.6647

Table C.2.2: Partial Hedge Price under Stochastic Volatility: $B_3(a, a+3, 0.23)$

a		90	95	100	105	110	∞
$\rho = -1.0$	$\hat{X}^{B_3}$	1.0693	1.5842	1.7436	1.8810	1.8592	1.7592
	err	0.0682	0.0942	0.1049	0.1108	0.1100	0.1067
$\rho = -0.5$	$\hat{X}^{B_3}$	0.9707	1.5491	1.8079	2.0407	1.8926	1.8865
	err	0.0644	0.0931	0.1074	0.1194	0.1119	0.1136
$\rho = 0.0$	$\hat{X}^{B_3}$	0.9920	1.5196	1.7011	1.8445	1.8888	1.8482
	err	0.0654	0.0921	0.1036	0.1137	0.1157	0.1133
$\rho = +0.5$	$\hat{X}^{B_3}$	0.9322	1.4727	1.7621	1.7700	1.8498	1.9068
	err	0.0626	0.0920	0.1076	0.1103	0.1162	0.1188
$\rho = +1.0$	$\hat{X}^{B_3}$	0.8175	1.3615	1.7028	1.9869	1.8951	1.8976
	err	0.0590	0.0879	0.1061	0.1219	0.1189	0.1228

Table C.2.3: Default Risk under Stochastic Volatility: $B_3(a, a+3, 0.23)$

a		90	95	100	105	110	∞
$\rho = -1.0$	$\mathbb{P}(A \backslash B_3)$	0.3258	0.2640	0.2468	0.2440	0.2398	0.2473
	err	0.0145	0.0137	0.0134	0.0133	0.0132	0.0134
$\rho = -0.5$	$\mathbb{P}(A \backslash B_3)$	0.3138	0.2405	0.2398	0.2383	0.2285	0.2275
	err	0.0144	0.0132	0.0132	0.0132	0.0130	0.0130
$\rho = 0.0$	$\mathbb{P}(A \backslash B_3)$	0.3268	0.2448	0.2358	0.2298	0.2278	0.2325
	err	0.0145	0.0133	0.0132	0.0130	0.0130	0.0131
$\rho = +0.5$	$\mathbb{P}(A \backslash B_3)$	0.3220	0.2533	0.2228	0.2168	0.2195	0.2205
	err	0.0145	0.0135	0.0129	0.0128	0.0128	0.0128
$\rho = +1.0$	$\mathbb{P}(A \backslash B_3)$	0.3165	0.2430	0.2235	0.2205	0.2090	0.2005
	err	0.0144	0.0133	0.0129	0.0128	0.0126	0.0124

Table C.2.4: Partial Hedge Price under Stochastic Volatility: $B_3(100, b, 0.23)$

b		100	103	108	∞
$\rho = -1.0$	$\hat{X}^{B_3}$	1.7986	1.7727	1.8719	1.7489
	err	0.1041	0.1044	0.1103	0.1021
$\rho = -0.5$	$\hat{X}^{B_3}$	1.8463	1.8136	1.7606	1.8401
	err	0.1074	0.1098	0.1068	0.1105
$\rho = 0.0$	$\hat{X}^{B_3}$	1.6885	1.7862	1.8882	1.9367
	err	0.1020	0.1073	0.1105	0.1141
$\rho = +0.5$	$\hat{X}^{B_3}$	1.6331	1.7300	1.7735	1.7062
	err	0.1012	0.1070	0.1092	0.1052
$\rho = +1.0$	$\hat{X}^{B_3}$	1.6362	1.7173	1.8262	1.7575
	err	0.1016	0.1057	0.1107	0.1105

Table C.2.5: Default Risk under Stochastic Volatility: $B_3(100, b, 0.23)$

b		100	103	108	∞
$\rho = -1.0$	$\mathbb{P}(A \backslash B_3)$	0.2498	0.2528	0.2380	0.2440
	err	0.0134	0.0135	0.0132	0.0133
$\rho = -0.5$	$\mathbb{P}(A \backslash B_3)$	0.2448	0.2390	0.2385	0.2303
	err	0.0133	0.0132	0.0132	0.0130
$\rho = 0.0$	$\mathbb{P}(A \backslash B_3)$	0.2445	0.2248	0.2318	0.2345
	err	0.0133	0.0129	0.0131	0.0131
$\rho = +0.5$	$\mathbb{P}(A \backslash B_3)$	0.2258	0.2298	0.2265	0.2395
	err	0.0130	0.0130	0.0130	0.0132
$\rho = +1.0$	$\mathbb{P}(A \backslash B_3)$	0.2353	0.2223	0.2233	0.2150
	err	0.0131	0.0129	0.0129	0.0127

Table C.2.6: Partial Hedge Price under Stochastic Volatility: $B_3(100, 103, s)$

s		0.16	0.19	0.22	0.25= T
$\rho = -1.0$	$\hat{X}^{B_3}$	0.5543	0.8051	1.4879	3.7342
	err	0.0556	0.0673	0.0967	0.1477
$\rho = -0.5$	$\hat{X}^{B_3}$	0.5433	0.8643	1.5809	3.5624
	err	0.0546	0.0712	0.1014	0.1460
$\rho = 0.0$	$\hat{X}^{B_3}$	0.5492	0.9334	1.4502	3.3936
	err	0.0559	0.0758	0.0953	0.1456
$\rho = +0.5$	$\hat{X}^{B_3}$	0.5655	0.8471	1.4011	3.2955
	err	0.0575	0.0716	0.0941	0.1436
$\rho = +1.0$	$\hat{X}^{B_3}$	0.5925	0.8673	1.3953	3.0971
	err	0.0617	0.0725	0.0949	0.1404

Table C.2.7: Default Risk under Stochastic Volatility: $B_3(100, 103, s)$

a		0.16	0.19	0.22	0.25= T
$\rho = -1.0$	$\mathbb{P}(A \backslash B_3)$	0.4910	0.4033	0.2995	0.0108
	err	0.0155	0.0152	0.0142	0.0032
$\rho = -0.5$	$\mathbb{P}(A \backslash B_3)$	0.5005	0.4080	0.2978	0.0175
	err	0.0155	0.0152	0.0142	0.0041
$\rho = 0.0$	$\mathbb{P}(A \backslash B_3)$	0.4810	0.3980	0.3038	0.0250
	err	0.0155	0.0152	0.0143	0.0048
$\rho = +0.5$	$\mathbb{P}(A \backslash B_3)$	0.4818	0.3865	0.2748	0.0358
	err	0.0155	0.0151	0.0138	0.0058
$\rho = +1.0$	$\mathbb{P}(A \backslash B_3)$	0.4560	0.3650	0.2778	0.0328
	err	0.0154	0.0149	0.0139	0.0055

C.3 Chapter 6 Tables

The following tables were used to produce the graphs in Chapter 6, for partially hedged claim prices and default risks under Stochastic Volatility with Variance Reduction. Note the sizes of the errors in comparison to those of Chapter 5 and Appendix C.2.

Table C.3.1: Volatility Smile for Stochastic Volatility with Variance Reduction

K		70	75	80	85	90
$\rho = -1.0$	$\hat{X}^A$	11.3437	7.1299	3.8074	1.6448	0.5455
	err	0.0054	0.0034	0.0042	0.0046	0.0021
	σ^*	0.2145	0.2073	0.2006	0.1941	0.1875
$\rho = -0.5$	$\hat{X}^A$	11.3040	7.0936	3.8044	1.6922	0.6094
	err	0.0066	0.0077	0.0088	0.0082	0.0065
	σ^*	0.2072	0.2041	0.2004	0.1973	0.1943
$\rho = 0.0$	$\hat{X}^A$	11.2673	7.0435	3.7914	1.7247	0.6652
	err	0.0066	0.0086	0.0097	0.0091	0.0081
	σ^*	0.2000	0.1996	0.1996	0.1996	0.2000
$\rho = +0.5$	$\hat{X}^A$	11.2229	7.0084	3.8091	1.7651	0.7209
	err	0.0071	0.0087	0.0486	0.0085	0.0069
	σ^*	0.1905	0.1965	0.2007	0.2023	0.2056
$\rho = +1.0$	$\hat{X}^A$	11.1950	6.9528	3.7764	1.8080	0.7720
	err	0.0055	0.0033	0.0046	0.0056	0.0024
	σ^*	0.1839	0.1914	0.1986	0.2052	0.2106
X_{BS}		11.2670	7.0476	3.7975	1.7311	0.6647

Table C.3.2: Partial Hedge Price under Stochastic Volatility with Variance Reduction: $B_3(a, a+3, 0.23)$

a		90	95	100	105	110	1000
$\rho = -1.0$	$\hat{X}^{B_3}$	1.0096	1.6078	1.8131	1.8489	1.8257	1.8619
	err	0.0178	0.0308	0.0278	0.0246	0.0277	0.0224
$\rho = -0.5$	$\hat{X}^{B_3}$	0.9743	1.5399	1.7964	1.8668	1.8870	1.8699
	err	0.0181	0.0243	0.0215	0.0185	0.0172	0.0194
$\rho = 0.0$	$\hat{X}^{B_3}$	0.8981	1.4930	1.7727	1.8408	1.8947	1.8798
	err	0.0187	0.0225	0.0170	0.0201	0.0181	0.0200
$\rho = +0.5$	$\hat{X}^{B_3}$	0.8397	1.4330	1.7187	1.8391	1.9025	1.8910
	err	0.0190	0.0235	0.0230	0.0186	0.0196	0.0170
$\rho = +1.0$	$\hat{X}^{B_3}$	0.8066	1.3530	1.6711	1.8412	1.8687	1.9221
	err	0.0163	0.0247	0.0305	0.0311	0.0237	0.0247

Table C.3.3: Default Risk under Stochastic Volatility with Variance Reduction: $B_3(a, a+3, 0.23)$

a		90	95	100	105	110	1000
$\rho = -1.0$	$\mathbb{P}(A \backslash B_3)$	0.3197	0.2539	0.2385	0.2373	0.2341	0.2361
	err	0.0032	0.0030	0.0030	0.0029	0.0027	0.0028
$\rho = -0.5$	$\mathbb{P}(A \backslash B_3)$	0.3196	0.2536	0.2338	0.2291	0.2252	0.2274
	err	0.0030	0.0028	0.0025	0.0023	0.0022	0.0024
$\rho = 0.0$	$\mathbb{P}(A \backslash B_3)$	0.3201	0.2530	0.2264	0.2188	0.2180	0.2191
	err	0.0030	0.0027	0.0024	0.0024	0.0024	0.0022
$\rho = +0.5$	$\mathbb{P}(A \backslash B_3)$	0.3168	0.2478	0.2243	0.2159	0.2123	0.2119
	err	0.0029	0.0030	0.0025	0.0022	0.0021	0.0022
$\rho = +1.0$	$\mathbb{P}(A \backslash B_3)$	0.3145	0.2484	0.2188	0.2101	0.2070	0.2050
	err	0.0029	0.0031	0.0029	0.0026	0.0026	0.0025

Table C.3.4: Partial Hedge Price under Stochastic Volatility with Variance Reduction: $B_3(100, b, 0.23)$

b		100	103	108	1000
$\rho = -1.0$	$\hat{X}^{B_3}$	1.7647	1.8018	1.8389	1.8196
	err	0.0299	0.0271	0.0253	0.0273
$\rho = -0.5$	$\hat{X}^{B_3}$	1.6970	1.7843	1.8234	1.8360
	err	0.0255	0.0264	0.0206	0.0248
$\rho = 0.0$	$\hat{X}^{B_3}$	1.6623	1.7416	1.7979	1.8045
	err	0.0257	0.0165	0.0232	0.0202
$\rho = +0.5$	$\hat{X}^{B_3}$	1.5917	1.7110	1.7786	1.7605
	err	0.0252	0.0243	0.0232	0.0227
$\rho = +1.0$	$\hat{X}^{B_3}$	1.5382	1.6802	1.7580	1.7629
	err	0.0288	0.0258	0.0307	0.0242

Table C.3.5: Default Risk under Stochastic Volatility with Variance Reduction: $B_3(100, b, 0.23)$

b		100	103	108	1000
$\rho = -1.0$	$\mathbb{P}(A \backslash B_3)$	0.2432	0.2389	0.2380	0.2379
	err	0.0030	0.0029	0.0029	0.0030
$\rho = -0.5$	$\mathbb{P}(A \backslash B_3)$	0.2409	0.2335	0.2327	0.2309
	err	0.0029	0.0026	0.0025	0.0027
$\rho = 0.0$	$\mathbb{P}(A \backslash B_3)$	0.2362	0.2276	0.2263	0.2282
	err	0.0026	0.0025	0.0025	0.0024
$\rho = +0.5$	$\mathbb{P}(A \backslash B_3)$	0.2335	0.2209	0.2184	0.2183
	err	0.0028	0.0025	0.0024	0.0024
$\rho = +1.0$	$\mathbb{P}(A \backslash B_3)$	0.2310	0.2204	0.2136	0.2136
	err	0.0031	0.0028	0.0026	0.0026

Table C.3.6: Partial Hedge Price under Stochastic Volatility with Variance Reduction: $B_3(100, 103, s)$

s		0.16	0.19	0.22	0.25= T
$\rho = -1.0$	$\hat{X}^{B_3}$	0.5219	0.9032	1.4896	3.6049
	err	0.0189	0.0221	0.0260	0.0369
$\rho = -0.5$	$\hat{X}^{B_3}$	0.5494	0.9120	1.4882	3.4762
	err	0.0142	0.0194	0.0231	0.0327
$\rho = 0.0$	$\hat{X}^{B_3}$	0.5662	0.8935	1.4622	3.3542
	err	0.0133	0.0177	0.0178	0.0287
$\rho = +0.5$	$\hat{X}^{B_3}$	0.5523	0.8969	1.4473	3.2394
	err	0.0127	0.0184	0.0197	0.0302
$\rho = +1.0$	$\hat{X}^{B_3}$	0.5589	0.8882	1.4244	3.1301
	err	0.0150	0.0234	0.0287	0.0352

Table C.3.7: Default Risk under Stochastic Volatility with Variance Reduction: $B_3(100, 103, s)$

a		0.16	0.19	0.22	0.25= T
$\rho = -1.0$	$\mathbb{P}(A \backslash B_3)$	0.4525	0.3890	0.2841	0.0122
	err	0.0030	0.0034	0.0029	0.0016
$\rho = -0.5$	$\mathbb{P}(A \backslash B_3)$	0.4426	0.3743	0.2775	0.0208
	err	0.0029	0.0028	0.0026	0.0017
$\rho = 0.0$	$\mathbb{P}(A \backslash B_3)$	0.4324	0.3630	0.2725	0.0258
	err	0.0029	0.0028	0.0028	0.0016
$\rho = +0.5$	$\mathbb{P}(A \backslash B_3)$	0.4225	0.3558	0.2670	0.0300
	err	0.0027	0.0028	0.0024	0.0018
$\rho = +1.0$	$\mathbb{P}(A \backslash B_3)$	0.4122	0.3486	0.2573	0.0349
	err	0.0031	0.0029	0.0032	0.0022

Bibliography

[1] Beaglehole, D., Dybvig, P. and Zhou, G., 1997. *Going to Extremes: Correcting Simulation Bias in Exotic Option Valuation*. Financial Analysts Journal, Vol 53-1, Jan/Feb, pp. 62-68.

[2] Ben Ameur, Ha., Breton, M. and L'Écuyer, P., 1999. *Partial Hedging for Options based on Extreme Values and Passage Times.*

[3] Benninga, S., 2000. *Financial Modeling, 2nd edition*. Massachusetts: MIT Press.

[4] Bingham, N. and Kiesel R., 1998. *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*. London: Springer-Verlag.

[5] Black, F. and Scholes, M., 1973. *The Pricing of Options and Corporate Liabilities*. The Journal of Political Economy, Vol 81-3, May/June, pp. 637-654.

[6] Boyle, P., Broadie, M. and Glasserman, P., 1997. *Monte Carlo Methods for Sewcurity Pricing*. Journal of Economic Dynamics and Control, Vol 21, pp. 1267-1321.

[7] Brenner, M., Ou, E. and Zhang, J., 2004. *Hedging Volatility Risk*.

[8] Broadie, M. and Glasserman, P., 1996. *Estimating Security Price Derivatives Using Simulation*. Management Science, Vol 42-2, Feb, pp. 269-285.

[9] Detemple, J. and Osakwe, C., 1999. *The Valuation of Volatility Options*. Cirano, Scientific Series, 99s-43, Dec.

[10] Fouque, J. and Tullie, A., 2002. *Variance Reduction for Monte Carlo Simulation in a Stochastic Volatility Environment*. Quantitive Finance, Vol 2, pp. 24-30.

[11] Gikhman, I. and Skorokhod, A., 1969. *Introduction to the Theory of Random Processes*. Philadelphia: W. B. Saunders.

[12] Grasselli, M. and Hurd, T., 2004. *A Monte Carlo Method For Exponential Hedging of Contingent Claims.*

[13] Hull, J., 2000. *Options, Futures and Other Derivatives. 4th Edition*. Upper Saddle River: Prentice Hall.

[14] Hull, J. and White, A., 1987. *The Pricing of Options on Assets with Stochastic Volatilities*. Journal of Finance, Vol 42-2, June, pp. 281-300.

[15] Jackson, M. and Staunton, M., 2002. *Advanced Modelling in Finance using Excel and VBA*. West Sussex: John Wiley & Sons.

[16] Karatzas, I., 1997. *Lectures on the Mathematics of Finance*. Rhode Island: American Mathematical Society.

[17] Karatzas, I. and Shreve, S., 1998. *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag.

[18] Klebaner, F., 1998. *Introduction to Stochastic Calculus with Applications*. London: Imperial College Press.

[19] Kloeden, P. and Platen, E., 1992. *Numerical Solution of Stochastic Differential Equations*. Berlin: Springer-Verlag.

[20] Lavenberg, S. and Welch, P., 1981. *A Perspective on the Use of Control Variables to Increase the Efficiency of Monte Carlo Simulations*. Management Science, Vol 27-3, Mar, pp. 322-335.

[21] Lewis, A., 2002. *The Mixing Approach to Stochastic Volatility and Jump Models.*

[22] Merton, R., 1990. *Continuous-Time Finance*. Oxford: Basil Blackwell.

[23] Metwally, S. and Atiya, A., 2002. *Using Brownian Bridge for Fast Simulation of Jump-Diffusion Processes and Barrier Options*. Journal of Derivatives, Fall, pp. 43-54.

[24] Meyer, Michael., 2001. *Continuous Stochastic Calculus with Applications to Finance*. Florida: Chapman & Hall/CRC.

[25] Musiela, M. and Rutkowski, M., 1997. *Martingale Methods in Financial Modelling*. Berlin: Springer-Verlag.

[26] Nelson, B., 1990. *Control Variate Remedies*. Operations Research, Vol 38-6, Nov-Dec, pp. 972-992.

[27] Romano, M. and Touzi, N., 1997. *Contingent Claims and Market Completenessin a Stochastic Volatility Model*. Mathematical Finance, Vol 7-4, October, pp. 399-410.

[28] Schweizer, M., 1993. *Variance-Optimal Hedging in Discrete Time*. Mathematics of Operations Research, vol 20, Feb, pp. 1-32.

[29] Shreve, S., 1996. *Stochastic Calculus and Finance.*