Non-Archimedean Analysis and its Applications in Tropical Geometry

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Abstract

Rigid analytic geometry provides a framework for studying algebraic geometry over a non-Archimedean field $k$ by providing a notion of analysis compatible the totally disconnected topology induced by $k$. We provide an introduction to these analytic techniques, with a view towards studying two results by Einsiedler, Kaprnaov and Lind applying this framework to tropical geometry.
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Chapter 1

Basics

The basic object of study in tropical geometry is a polyhedral subset \( T(f) \in \mathbb{R}^n \) associated to a Laurent polynomial \( f \in k[x_1^\pm, \ldots, x_n^\pm] \) called the \textit{tropical hypersurface} of \( f \). A definition of \( T(f) \) can be given directly as system of linear inequalities coming from the terms of \( f \), as in Speyer, [9]. Here it is clear that \( T(f) \) is a polyhedral subset. However the language of valuation theory provides a more elegant language for expressing these defining relations. Namely, for any infinite non-Archimedean non-trivially valued field \( k \), we look at the zeros of \( f \) over \( k^n \), and map these points to \( \mathbb{R}^n \) by the valuation of \( k \); in fact it is a consequence of the elementary theory of valuations that this set is equal to the tropical hypersurface \( T(f) \), and is independent of the choice of \( k \) (see [7]).

This connection to algebraic geometry over non-Archimedean fields suggests that problems about tropical varieties can be tackled by lifting to the affine scheme itself. However, due to the non-Archimedean triangle inequality, the topology on any non-Archimedean field \( k \) is totally disconnected, and so many of the tools of classical analysis are not available to us. The study of rigid analytic spaces, developed by Tate in [10] provides a comprehensive program for analysis on non-Archimedean varieties.

In this thesis, we follow the paper by Einsiedler, Kapranov and Lind [5], which explores two elementary results obtained this way (in [6], this program is taken further): our basic reference for the necessary details of non-Archimedean analysis according to [2] which we stick closely to. First we will look at the Tate algebras, which we will think of as the analytic functions on the unit ball. This will give us a “local” result about tropical varieties concerning the equivalence of the tropical variety to a polyhedral set defined in a different way.

Our “global” result concerns the connectedness of tropical varieties. Our treatment of rigid analytic geometry will be quite narrow since our focus is the non-trivial notion of connectedness afforded by rigid analytic geometry rather than any of the broader algebraic geometric implications. For ex-
ample we will ignore many finer points about morphisms in the categories of affinoid analytic varieties. However, we will attempt to motivate the development by occasionally mentioning the parallels with complex analysis.

Another important technique in non-Archimedean analysis which we will not see directly is the reduction functor which sends the field \( k \) to the residue field
\[
\bar{k} = \{|x| \leq 1\}/\{|x| < 1\};
\]
wherever possible, proofs relying on this technique have been rewritten. However in a few cases this, due to considerations of brevity, the reader is instead referred to [2] for a proof. Similarly, theorems relying on sheaf cohomology are also given without proof, and the reader is again referred to [2].

1.1 Non-Archimedean norms

**Definition 1.1.** A norm on a ring \( A \) is a function \(|\cdot| : A \to \mathbb{R}_+\) with values in the set \( \mathbb{R}_+ \) of non-negative real numbers with the following properties:

1. \(|a| = 0 \) if and only if \( a = 0 \);
2. \(|1| = 1\)
3. \(|ab| = |a||b|\);
4. \(|a + b| \leq |a| + |b|\),

for all \( a, b \in A \).

**Definition 1.2.** A norm on a ring \( A \) is called non-Archimedean if for all \( a, b \in A \) it satisfies the additional condition
\[
|a + b| \leq \max(|a|, |b|).
\]

Sometimes non-Archimedean norm are called *ultrametric*. Note that this condition implies the triangle inequality 1.1(3). The following easy consequence of the axioms will sometimes come in useful.

**Proposition 1.3.** Let \( A \) be a non-Archimedean ring, and let \(|a| \neq |b| \in A\). Then \(|a + b| = \max(|a|, |b|)|.

**Proof.** Assume \(|a| > |b|\). Then \(|a| \leq \max(|a + b|, |b|) \leq \max(|a|, |b|) = |a|\), and hence \(\max(|a + b|, |b|) = |a|\). Then since we are assuming \(|a| \neq |b|\), we must have \(|a + b| = |a|\). \(\square\)

Note that some of the early results (such as the proposition above) are true for normed groups in general, however for simplicity, and since it’s what we will need later on, all results will be stated in terms of rings.
Example 1.4. The usual absolute value on the field \( \mathbb{Q} \) of rational numbers 
\[ |a| = \max(a, -a) \] gives a norm. However there is another way to define a norm on \( \mathbb{Q} \). Let \( p \in \mathbb{N} \) be prime. We can write each rational number \( a \in \mathbb{Q} \) uniquely in the form \( a = p^n u/v \) so that \( p \) doesn’t divide \( u \) or \( v \); we then define a norm \( |\cdot|_p \) by
\[
|a|_p := \begin{cases} 
\left(\frac{1}{p}\right)^n & \text{when } x \neq 0, \\
0 & \text{when } x = 0.
\end{cases}
\]
This is often called the \( p \)-adic norm. The completion \( \mathbb{Q}_p \) of the rational numbers with respect to this valuation is called the \( p \)-adic numbers. These arise naturally in number theory; from a number-theoretic point of view (i.e., ignoring topology), the \( p \)-adic norm is just as natural as the usual absolute value.

Non-Archimedean norms are so-called because (unlike say \( \mathbb{Q} \), \( \mathbb{R} \), or \( \mathbb{C} \)) they fail to satisfy the Archimedean axiom of Euclidean geometry, which states that given any \( a \) with \( |a| \leq 1 \), there exists an \( n \in \mathbb{N} \) such that the sum \( \sum_{i=1}^{n} a \) has norm greater than 1. In fact it is a trivial consequence of Definition 1.2 that \( |na| \leq |a| \) for all \( n \).

In the case of \( \mathbb{Q}_p \), this can be seen explicitly by writing \( a \) in its base \( p \) expansion (note that \( a \) has no fractional part since this would force \( |a| > 1 \)):
\[
a = a_0 + a_1 p + \cdots a_m p^m \quad 0 \leq a_i \leq p
\]
Then \( |a| = p^{-m} \), and we have
\[
|na| = |a| \quad \text{if } n \cdot a_m < p \\
|na| < |a| \quad \text{if } n \cdot a_m \geq p
\]

Definition 1.5. A valuation on a ring \( A \) is a function \( v : k \to R \cup \{\infty\} \) satisfying

1. \( v(a) = \infty \) if and only if \( a = 0 \);
2. \( v(ab) = v(a) + v(b) \);
3. \( v(a + b) \geq \min(v(a), v(b)) \),

for all \( a, b \in A \).

There is a one-to-one correspondence between norms and valuations. Fix a real number \( \rho > 1 \). Then given a norm \( |\cdot| \), we have a valuation defined by \( v(a) = -\log_\rho(|a|) \), and given a valuation \( v \), we have a norm defined by \( |a| = \rho^{v(a)} \). Note that this correspondence will depend on the choice of \( \rho \), since if we choose another \( \tau > 1 \), the norm defined by \( |a| = \tau^{v(a)} \) will not in general be equivalent to the norm defined by \( |a| = \rho^{v(a)} \).
Example 1.6. A very important example will be the spectral norm on an algebraic field extension of a non-Archimedean normed field \( \mathbb{L} \supset k \), defined as follows: For \( l \in \mathbb{L} \), let \( p(x) = x^m + a_1 x^{m-1} + \cdots + a_m, a_i \in k \) be its minimal polynomial. Then we define

\[
|l| = \max_{1 \leq i \leq m} |a_i|^{1/i}.
\]

Of particular interest will be the extension of \( k \) to its algebraic closure \( \overline{k} \). We make the following observations about the valuation \( v(\mathbb{L}) \) corresponding to the spectral norm:

1. If the norm \( |\cdot| \) is non-trivial, then there exists a non-zero \( a \in k \) with \( |a| = \alpha \neq 1 \). We can assume that \( \alpha > 1 \), since if \( \alpha < 1 \), we have \( |1| = |a \cdot a^{-1}| = |a| \cdot |a^{-1}| \) and hence \( |a^{-1}| = \alpha^{-1} \geq 1 \). Then the value group \( v(\mathbb{L}) \) contains the subset

\[
\left\{ \frac{\log(\alpha)m}{n} : m, n \in \mathbb{Z} \right\} \subset \mathbb{R}
\]

since for any \( m, n \in \mathbb{Z} \) we have \( v(a^{m/n}) = \log(|a^{m/n}|) = \log(|a|m/n) = \log(|a|)m/n = \log(\alpha)m/n \). Hence \( v(\mathbb{L}) \) is a dense subgroup of \( \mathbb{R} \).

2. Let \( l \in \mathbb{L} \). By the definition of the spectral norm, \( v(l) = \log(|a|^{1/k}) \) for some \( a \in k \), and hence \( v(a) = kv(l) \). That is,

\[
v(\mathbb{L}) \subset \left\{ \frac{a}{b} \in \mathbb{Q} : a \in v(k), b \in \mathbb{Z} \right\}.
\]

Given a normed ring \( A \) we have a natural induced metric space structure by taking as a basis the open balls \( \mathbb{B}^-_\epsilon(a) = \{ x \in A : |x - a| < \epsilon \} \) for each \( a \in A \). As a result of Non-Archimedean triangle inequality of Definition 1.2, this topology is very different to that induced by an Archimedean norm. For example, let \( \mathbb{B}^-_\epsilon(a) \) be such a ball. Oddly, every point in \( \mathbb{B}^-_\epsilon(a) \) lies at the "center" of the ball: let \( a' \in \mathbb{B}^-_\epsilon(a) \) be any interior point. For any \( b \in \mathbb{B}^-_\epsilon(a) \), we also have \( b \in \mathbb{B}^-_\epsilon(a') \), since by the non-Archimedean triangle inequality we have \( |a' - b| = |(a' - a) + (a - b)| \leq \max(\epsilon, \epsilon) = \epsilon \), and so \( \mathbb{B}^-_\epsilon(a) = \mathbb{B}^-_\epsilon(a') \). By the same argument every point in the "closed ball" \( \mathbb{B}^+_\epsilon(a) = \{ x \in A : |x - a| \leq \epsilon \} \) is the center of the ball (this includes boundary points!) In particular, the ball \( \mathbb{B}^+_\epsilon(a) \) is open, since for any \( a' \in \mathbb{B}^+_\epsilon(a) \) we have \( \mathbb{B}^-_\epsilon(a') \subset \mathbb{B}^+_\epsilon(a) \).

In fact, if \( a' \in S_\epsilon(a) = \{ x \in A : |x - a| = \epsilon \} \) we have \( \mathbb{B}^-_\epsilon(a') \subset S_\epsilon(a) \), since for \( b \in \mathbb{B}^-_\epsilon(a') \) we have \( |b - a| < |a' - a| = \epsilon \), and so by Proposition 1.3, we have \( |b - a'| < |(b - a') + (a' - a)| = \max(|b - a'|, |a' - a|) = \epsilon \). So \( S_\epsilon(a) \) is open in \( k \). Since we have \( \mathbb{B}^-_\epsilon(a) = \mathbb{B}^+_\epsilon(a) \setminus S_\epsilon(a) \), i.e. as a closed set minus an open set, we have that \( \mathbb{B}^-_\epsilon(a) \) is closed as well as open.
Proposition 1.7. Every non-Archimedean ring $A$ is totally disconnected. That is, for all $a \in A$, the only connected set containing $a$ is the singleton set $\{a\}$ containing only $a$.

Proof. Let $a \in k$, and let $T$ be the largest connected set containing $a$. Now $T$ cannot contain any set $U$ which is both closed and open, since then we would have $T = U \cup (T \setminus U)$, with both $U$ and $T \setminus U$ open in the subspace topology on $T$, and $U \cap (T \setminus U) = \emptyset$. But then $T$ cannot contain any $\mathbb{B}^{-}(a)$, and so we have $T = \{x\}$. □

The non-Archimedean triangle inequality also gives an easy way to determine if a subgroup is dense.

Proposition 1.8. Let $A$ be a normed ring and let $H$ be an additive subgroup of $A$ which is “$\epsilon$-dense” in $A$ in the following sense: there exists an $\epsilon < 1$ such that for each $a \in A$, there exists an $h \in H$ with $|a + h| \leq \epsilon|a|$.

Then $H$ is dense in $A$.

Proof. If $\epsilon = 0$ then there is nothing to prove, so assume $\epsilon > 0$. Suppose there is some $a \in A$ with $|a, H| := \inf_{y \in H} |a + y| > 0$. Since $\epsilon^{-1}$ is greater than 1, by the definition of inf, we can choose an $h_1 \in H$ with $|a + h_1| < \epsilon^{-1}|a, H|$. By hypothesis, there is an $h_2 \in H$ with $|(a + h_1) + h_2| \leq \epsilon|a + h_1|$. Combining the two equations we get

$|a + (h_1 + h_2)| = |(a + h_1) + h_2| \leq \epsilon|a + h_1| < \epsilon(\epsilon^{-1}|a, H|) = |a, H|.$

But $H$ is a subgroup, and so $h_1 + h_2 \in H$, and so we have $|a, H| \leq |a + (h_1 + h_2)| < |a, H|$, a contradiction. □

1.2 Tropical Varieties

From now on $k$ will be a field with a nontrivial non-Archimedean norm, and we will assume $k$ is complete with respect to this norm.

A tropical variety is essentially the logarithmic image of an algebraic variety; the standard definition uses a valuation to do this. For the tropical variety to be a well-defined, since $v(0) = \infty$ for any valuation $v$, we need to avoid taking the valuation at the origin. For this reason we will work in the punctured affine space $(k^{\times})^n = (k \setminus \{0\})^n$.

This has co-ordinate ring $\mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$: To see that the maximal ideals of this ring do in fact correspond to points of $(k^{\times})^n$, let

$M_{(a_1, \ldots, a_n)} = \langle (x_1 - a_1), \ldots, (x_n - a_n) \rangle$

be an arbitrary maximal ideal in $\mathbb{k}[x_1, \ldots, x_n]$. Let $M'_{(a_1, \ldots, a_n)} \subset \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the corresponding ideal generated by the images of the $(x_i - a_i)$ under the canonical inclusion. Then we have $1 \in M'_{(a_1, \ldots, a_n)}$ if and only if $a_i = 0$
for each $i$; in this case $M' = k[x^\pm_1, \ldots, x^\pm_n]$ is not a maximal ideal. Otherwise $M'$ is again a maximal ideal.

Given a valuation $v$ on a ring $k$, define the map

$$\text{val} : (k^n)^* \rightarrow R^n$$

by $(a_1, \ldots, a_n) \mapsto (v(a_1), \ldots, v(a_n))$. Let $X = V(I) \subset (k^n)^*$ be the variety defined by an ideal $I \subset k[x^\pm_1, \ldots, x^\pm_n]$. The variety $X$ is defined as zeros of the ideal $I$. But given any field extension $L \supset k$, we can view $I$ as an subset of the ring $L[x^\pm_1, \ldots, x^\pm_n]$; we define the set of "$L$-points" of $X$ to be

$$X(L) = \{(a_1, \ldots, a_n) \in L^n : f(a_1, \ldots, a_n) = 0, \text{ for all } f \in I\}.$$

**Definition 1.9.** The tropical variety of a scheme $X = k[x^\pm_1, \ldots, x^\pm_n]/I$ is defined as the closure of val$(X(k))$ in $R^n$.

Another way of associating a subset of $\mathbb{R}^n$ to a variety $X$ over a non-Archimedean field is due to Bieri and Groves [1].

**Definition 1.10.** Let $A = k[X] = k[x^\pm_1, \ldots, x^\pm_n]/I$ be the coordinate ring of $X$. Set $W(A)$ to be the set of all ring valuations on $A$ extending $v$ on $k$. Let $\beta : W(A) \rightarrow \mathbb{R}^n$ be defined by $w \mapsto (w(x_1), \ldots, w(x_n))$. The Bieri-Groves set associated to $X$ is defined as

$$BG(X) = \beta(W(A)).$$

We call a set $\Delta \subset \mathbb{R}^n$ a convex polyhedron in $\mathbb{R}^n$ if $\Delta$ is defined by a finite system of linear inequalities:

$$b_{11}x_1 + \cdots + b_{1n}x_n \geq c_1$$

$$b_{21}x_1 + \cdots + b_{2n}x_n \geq c_2$$

$$\vdots$$

$$b_{r1}x_1 + \cdots + b_{rn}x_n \geq c_r.$$  

We say that $\Delta$ is $v(\overline{k})$-rational if the inequalities can be chosen so that each of the $b_{ij}$ is an integer, and each $c_i$ is in the value group $v(\overline{k})$. We have that $BG(X)$ and $T(X)$ are both finite unions of $v(\overline{k})$-rational polyhedra, by [1] and [7] respectively.

**Remark.** Although the set $BG(X)$ seems to be larger than the tropical variety $T(X)$, we will see in Section 2 that the two sets are in fact equal. The inclusion $T(X) \subset BG(X)$ is immediate: Let $z \in X(\overline{k})/\Gamma$ with $\Gamma = \text{Gal}(\overline{k}/k)$. The the function $w : k[X] \rightarrow \mathbb{R} \cup \{\infty\}$ defined by $w_z(f) = v(f(z))$ is a valuation satisfying $\beta(w_z(f)) = v(z)$. Since $BG(X)$ is a polyhedral set, in particular it is closed; then since the $v(z) \in BG(X)$ form a dense subset of $T(X)$, we have that $T(X) \subset BG(X)$.

7
Chapter 2

A local result

Since we want to perform analysis over $k$, we will need a notion of which functions we will call “analytic.” Since $k$ is totally disconnected, smooth functions cannot in general be defined locally. For example $\chi_{B_{\epsilon}}$, the characteristic function of the ball of radius $\epsilon < 1$, is everywhere smooth since $B_{\epsilon}$ is both closed and open. However inside $B_{\epsilon}$, $\chi_{B_{\epsilon}}$ is locally defined by the constant Taylor series 1. So the Taylor expansion at a point is not enough to recover the function. There are simply too many functions possessing a convergent Taylor expansion at each point. Therefore we start in some sense at the opposite extreme, by only concerning ourselves with the functions which are defined over the entire unit ball by a single convergent power series. By the non-Archimedean triangle inequality, in 2.1 we get a very lenient convergence condition, so in fact we see that there are a lot functions over $k$ of this type than over $C$.

2.1 The Tate algebras

Lemma 2.1. Let $\{a_n\}$ be a sequence in $k$. Then the sum $\sum a_n$ converges if $|a_n| \to 0$ as $n \to \infty$.

Proof. Let $a_k$ have largest norm among the $\{a_n\}$. By the non-Archimedean triangle inequality, for all $N \geq k$, each partial sum has norm $|S_N| = |\sum_{1}^{N} a_n| \leq |a_k|$. Since $\{|S_N|\}_{N \geq k}$ is a decreasing sequence in $\mathbb{R}$, and bounded by 0, it has a limit. Hence the sequence $\{S_N\}_{N \geq k}$ is Cauchy, and has a limit since $k$ is complete.

Definition 2.2. The free Tate algebra in $n$ indeterminates

$$T_n = k(x_1, \ldots, x_n)$$

consists of the formal power series $f(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n \geq 0} c_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$, $a_{i_1 \ldots i_n} \in k$, where $|c_{i_1 \ldots i_n}| \to 0$ as $i_1 + \cdots + i_n \to \infty$. Often we will write $c_i x^i$ as an abbreviation for $c_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$.
Note that $T_n$ is complete with respect to the Gauss norm, where the norm of an element $f \in T_n$ is equal by the norm of the coefficient of $f$ with the largest norm:

$$|f| = \left| \sum c_i x^i \right| = \sup_i |c_i|.$$ 

Also note that for the condition $|a_i| \to 0$ implies that if $f = \sum c_i x^i \in T_n$, then the norm is realized for some $|c_i| = |f|$.

We refer the reader to [2], Section 5.1 for the elementary but lengthy proof of the following result.

**Proposition 2.3.** Let $f \in T_n$ with $|f| = 1$. Then $f$ is a unit in $T_n$ if and only if $|f(0)| = 1$ and $|f - f(0)| < 1$ (i.e., the constant term of $f$ has norm 1, and all other terms have norm less than 1).

**Lemma 2.4.** Let $f \in T_n$ with $|f| = 1$. Then there is an element $c \in k$ with $|c| = 1$ such that $f + c$ is not a unit in $T_n$.

**Proof.** If $|f(0)| = 1$ then we can just take $c = f(0)$, since $f - f(0)$ has constant term equal to zero, and so is certainly not a unit. On the other hand, suppose $|f(0)| < 1$. Then $|f| = 1$ implies $|f - f(0)| = 1$, i.e., one of the non-constant terms of $f$ has norm equal to 1. Then we can take $c = 1$, since $1 + f$ will still have a non-constant term with norm equal to 1, and hence by the Proposition above, is not a unit.

**Theorem 2.5.** Every $k$-algebra homomorphism $\varphi : T_n \to T_m$ is a contraction.

**Proof.** Suppose at there is some $f \in T_n$ with $|\varphi(f)| > |f|$. Without loss of generality, we may assume that $|\varphi(f)| = 1$ (since $\varphi$ is $k$-linear, we can always scale it such that this is the case). Choose $c \in k$ with $|c| = 1$ so that $\varphi(f) + c$ is not a unit in $T_m$. Now $c + f$ is certainly a unit in $T_n$, since its constant term has norm 1, and $|f| < 1$ means its non-constant terms must all have norm less than 1. Since $\varphi$ is a homomorphism we must have $\varphi(f + c) = \varphi(f) + c$ a unit in $T_m$, a contradiction.

**Corollary 2.6.** Every $k$-algebra isomorphism is an isometry.

### 2.2 Weierstrass polynomials

An important technique for studying the structure of $T_n$ uses Weierstrass polynomials to view $T_n$ as a finitely generated $T_{n-1}$-algebra. This will allow us to use induction from $k$ to prove results in $T_n$.

**Definition 2.7.** A strictly convergent power series $g = \sum_{i=1}^{\infty} g_i(x_1, \ldots, x_{n-1})x_n^i$ is $x_n$-distinguished of degree $s$ if
1. $g_s$ is a unit in $T_{n-1}$ and

2. $|g_s| = |g|, \quad |g_s| > |g_i|$ for all $i > s$.

**Theorem 2.8** (Weierstrass Division Theorem). Let $g \in T_n$ be $x_n$-distinguished of degree $s$. Then for each $f \in T_n$ there exist uniquely determined elements $q \in T_n$ and $r \in T_{n-1}[X_n]$ with $\deg(r) < s$ and such that

$$f = qg + r,$$

and the following estimates hold:

$$|f| = \max(|q||g|, |r|): \quad \text{i.e.,}$$

$$|q| \leq \frac{|g|}{|f|} \quad \text{and} \quad |r| \leq |f|.$$  

**Proof.** Suppose $|g| = c$. Then $g/c$ has norm 1, and if we can calculate $q$ and $r$ so that $f/c = qg/c + r$, with $r \in T_{n-1}[x_n]$, $\deg(r) < s$, and $q \in T_n$ then we have $f = g + cr$, where $cr$ is still in $T_{n-1}[x_n]$. So without loss of generality, we can assume $|g| = 1$. Assume that we have $f = qg + r$ with $q$ and $r$ as above. Let $c = \max(|qg|, |r|) = \max(|q|, |r|)$. Then $|f| \leq c$. In fact this is an equality. Let $qg = \sum h_i x_i^n$, with $h_i \in T_{n-1}$. Since $q$ is $x_n$-distinguished of degree $s$, its $x_n^s$ coefficient has norm 1, and so the norm of $qg$ is realized by $h_i$ with $i$ at least $s$. Since $r$ is of degree less than $s$, no cancelling can occur, and so $|qg + r| = \max(|qg|, |r|)$, and so we have $|f| = \max(|qg|, |r|)$.

Uniqueness is easy now. Suppose that $f = qg + r = q'g + r'$, $r \in T_{n-1}[x_n]$, $\deg(r) < s$, $q \in T_n$. Then $0 = (q - q')g + (r - r')$, and the estimates give $q - q' = r - r' = 0$.

For the existence part we first restrict to $T_n^\circ := \{f \in T_n : |f| \leq 1\}$, and pass to a polynomial quotient where we can use Euclid’s division algorithm; we then show that this is “good enough” in the sense of Proposition 1.8.

Let $B = \{qg + r : r \in T_{n-1}[x_n], \deg(r) < s, q \in T_n\}$. By the inequalities above, we have that $B$ is a closed subgroup of $T_n$. We claim that $B$ is all of $T_n$. Writing $g = \sum g_i x_i^n$, with $g_i \in T_{n-1}$, set $\epsilon = \max_{i > s}(|g_i|)$. Note that $\epsilon < 1$. Set

$$\mathbb{k}_\epsilon = \{k \in \mathbb{k} : |k| \leq 1\}/\{k \in \mathbb{k} : |k| \leq \epsilon\}.$$

Then we have a natural ring homomorphism $\tau_\epsilon : T_n^\circ \rightarrow \mathbb{k}_\epsilon[x_1, \ldots, x_n]$ with kernel $\{f \in T_n^\circ : |f| \leq \epsilon\}$, and since we are assuming $|g| = 1$, we have that $t_\epsilon(g)$ is a unitary polynomial in $x_n$ of degree $s$. Therefore we can perform Euclid’s division algorithm in the ring $(\mathbb{k}_\epsilon[x_1, \ldots, x_{n-1}])[x_n]$ with respect to $\tau_\epsilon(g)$. That is, for every $f \in T_n^\circ$, we can find $q \in T_n^\circ$, $r \in T_n^\circ$, $\deg r < s$ with $|f - (qg + r)| \leq \epsilon$. But for an arbitrary $f \in T_n$, we have $f/|f| \in T_n^\circ$, and so we have some $b \in B$ with $|f/|f| - B| \leq \epsilon$, and hence $|f - b| \leq |f| \epsilon$. So we see that $B$ is $\epsilon$-dense in $T_n$, and hence by Proposition 1.8 is dense. Since $B$ is closed, we get $B = T_n$. $\square$
Definition 2.9. A Weierstrass polynomial (in $x_n$) is a monic polynomial $\omega \in T_{n-1}[x_n]$ with $|\omega| = 1$.

Theorem 2.10 (Weierstrass Preparation Theorem). Let $g \in T_n$ be $x_n$-distinguished of degree $s$. Then there are a Weierstrass polynomial $\omega \in T_{n-1}[x_n]$ of degree $s$, and a unit $e \in T_n$ with $g = \omega \cdot e$.

Proof. Applying the Weierstrass Division Theorem 2.8 to the monomial $x_n^s$, we have that there exist $e' \in T_n$ and $r' \in T_{n-1}[x_n]$ with $\deg r' < s$ such that $x_n^s = e'g + r$. Define $\omega = x_n^s - r'$. Then $\omega$ is a monic polynomial in $T_{n-1}[x_n]$ of degree $s$, and $\omega = e'g$.

To complete the theorem, we just have to show that $e'$ is a unit. Since $|r'| \leq |x_n^s| = 1$, we see that $|\omega| = 1$ and that $\omega$ is $x_n$-distinguished of degree $s$. By scaling if necessary, we may assume $|g| = 1$. Then $|e'| = |\omega|/|g| = 1$.

Also, suppose that, viewing $e'$ as a power series in $T_{n-1}(x_n)$, we have a term $a_rx^r$ with $|a_r| \geq 1$. Then, since $g$ is $x_n$-distinguished of degree $s$, the term $b_{s+r}x_n^{s+r}$ in the product $e'g$ will have $|b_{s+r}| \geq 1$. Since we know that $\omega$ and $g$ are both $x_n$-distinguished of degree $s$, we conclude that $r = 0$, and so by Proposition 2.3 we have that $e'$ is a unit in $T_{n+1}$; then certainly $e'$ is a unit in $T_n$.

Proposition 2.11. Let $0 \neq f_n \in T_n$. Then there is a $k$-algebra automorphism $\sigma$ of $T_n$ such that $\sigma(f)$ is $x_n$-distinguished (and hence has an associated Weierstrass polynomial).

Proof. We may assume $|f| = 1$ since the map $g \mapsto g/|g|$ is an automorphism of $T_n$. Let $f = \sum_i a_ix^i$. Under the lexicographic ordering, let $m = (m_1, \ldots, m_n)$ be the maximal $n$-tuple such that $|a_m| = 1$. Take $t \in \mathbb{N}$ so that for any $|a_{(i_1, \ldots, i_n)}|$ we have $t \geq i_1, \ldots, i_n$. We define an automorphism $\sigma : T_n \rightarrow T_n$ as follows: Set $\sigma(x_i) = x_i + x_n^2$ for $i = 1, \ldots, n-1$, and $\sigma(x_n) = x_n$ where the exponents $c_i$ are defined recursively by

$$c_{n-j} = 1 + \sum_{d=0}^{j-1} c_{n-1}.$$

So, for example, we have $c_n = 1$, $c_{n-1} = 1 + t$, $c_{n-2} = 1 + t + (1 + t) = t^2 + 2t + 1$, etc. We claim that $\sigma(f)$ is $x_n$-distinguished of order $s = \sum_{i=1}^{n} c_im_i$.

Firstly, note that for all $i = (i_1, \ldots, i_n)$ such that $|a_i| \leq 1$, and $i \neq m$, we have $\sum_{l=1}^{n} c_ili < s$: Since $i < m$ under the lexicographic ordering, we have

$$i = (i_1 = m_n, \ldots, i_{p-1} = m_{p-1}, \ i_p < m_p , \ i_{p+1}, \ldots, i_n)$$

for some index $p$ possibly equal to 0. The entries $i_{p+1}, \ldots, i_n$ are arbitrary, but they are all bounded by $t$. Note also that since $i_p$ and $m_p$ are integers, $i_p < m_p$ implies $i_p \leq m_p - 1$. Then we have

$$(\ast) \quad \sum_{l=1}^{n} c_ili \leq \sum_{l=1}^{p-1} c_lm_l + c_p(m_p - 1) + \sum_{l=p+1}^{n} c_l.$$

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But from the definition of \( c_p \), we have
\[
\sum_{l=p+1}^{n} c_l t = t \sum_{d=0}^{n-p} c_{n-d} = c_p - 1,
\]
and so \((*)\) becomes
\[
\sum_{l=1}^{n} c_l t \leq \sum_{l=1}^{p} c_l m_l - 1 < \sum_{i=1}^{n} c_l m_i = s,
\]
and so our claim is justified.

Now computing \( \sigma(f) \), we have
\[
\sigma(f) = \sum_{i} a_i (x_1 + x_n)_{i_1} \cdots (x_{n-1} + x_{n-1})_{i_{n-1}} x_n^{i_n},
\]
and we that any term in the expansion with an \( a_i \) coefficient will have \( x_n \)-degree at most \( (c_1 i_1 + \cdots + c_n i_n) \) which we can write as \( (c_1 i_1 + \cdots + c_n i_n) \) since \( c_n = 1 \). Then by the above claim, if \( |a_i| = 1 \), then any term with an \( a_i \) coefficient will have \( x_n \)-degree at most \( s \). Furthermore, as a power series in \( T_{n-1}(x_n) \), the \( x_n \) term in the above expansion has coefficient \( a_m + \sum_k a_k g(x_1, \ldots, x_n) \), where each \( |a_k| \) is less than 1. Since by assumption \( |a_m| = 1 \), by Proposition 2.3, we have that \( a_m + \sum_k a_k g(x_1, \ldots, x_n) \) is a unit in \( T_{n-1} \), and hence we have that \( \sigma(f) \) is \( x_n \)-distinguished of degree \( s \).

**Definition 2.12.** Let \( I \) be an ideal of \( T_n \). Then \( T_n/I \) is a normed ring via the residue norm
\[
|\overline{f}| = \inf_{y \in I} |f + y|,
\]
for \( f \in T_n/I \).

**Proposition 2.13.** let \( \omega \) be a Weierstrass polynomial of degree \( s \) in \( X_n \). Then
1. \( T_n/\omega T_n \) is a finite free \( T_{n-1} \)-module;
2. \( T_{n-1}[x_n]/\omega T_{n-1}[x_n] \cong T_n/\omega T_n \).

**Proof.** Consider the following diagram of \( T_{n-1} \) homomorphisms
where $i$ is the natural inclusion map, $j$ is the injection defined by

$$(t_0, \ldots, t_{s-1}) \mapsto t_0 + t_1 x_n + \cdots + t_{s-1} x_n^{s-1},$$

$\psi$ and $\pi$ are the natural quotient maps and $\bar{i}$ and $\bar{j}$ are induced by $i$ and $j$ respectively, that is $\bar{i}$ and $\bar{j}$ are the unique maps making the diagram commute.

First we show that $\bar{j}$ is a bijection (the case for $\bar{i}$ is similar). Since $j$ is a bijection onto its image $\text{im}(j) = \{ f \in T_{n-1}[x_n] : \deg f < s \}$, we need only show that $\psi$ restricted to $\text{im}(j)$ is bijective. Let $ar{f} \in T_{n-1}[x_n]/\omega T_{n-1}[x_n]$. By the existence part of the Weierstrass Division Theorem, the coset representative $f$ can be written in the form $f = q \omega + r$ with $\deg r < s$. So we have $\bar{j}(r) = [f]$, and so $\bar{j}$ is onto. Now suppose $\bar{j}(f) = 0$ for some $f \in T_{n-1}[x_n]$, with $\deg f < s$. Then $f = q \cdot \omega + 0$ for some $q \in T_{n-1}[x_n]$. Since we also have $f = 0 \cdot \omega + f$, by the uniqueness part of Weierstrass Division, we conclude that $f = 0$, and hence $\bar{j}$ is injective.

Now we just have to show that $\bar{i}$ and $\bar{j}$ are isometries under the residue norm (we provide $T_{n-1}^s$ with the norm induced by the “inclusion” $j$). For $\bar{j}$, let $\bar{f} \in T_{n-1}[x_n]/\omega T_{n-1}[x_n]$. Firstly we note that since $T_{n-1}[x_n] \subset t_n$, we have $|\bar{f}| = \inf_{y \in \omega T_{n-1}[x_n]} |f + y| \geq \inf_{y \in \omega T_{n}} |f + y| = |\bar{j}(f)|$. Now say $|\bar{j}(f)| = c$. Let $\epsilon > 0$. By the definition of the residue norm, there exists an $y \in \omega T_n$ with $|f + y| < c + \epsilon/2$. Since $\omega T_{n-1}[x_n]$ is dense in $T_n$, we can choose $y_2 \in T_{n-1}[x_n]$ with $|f + y_2| < \epsilon/2$. Then $|\bar{f}| = \inf_{y \in \omega T_{n-1}[x_n]} |f + y| < |f + y_2| < c + \epsilon$. Hence $|\bar{j}(\bar{f})| = |\bar{f}|$.

Since $\bar{j}$ is a $k$-algebra homomorphism, we have that it is a contraction. If it is not an isometry, then there must exist a tuple $(t_0, \ldots, t_{n-1}) \in T_{n-1}^s$ and a polynomial $q \in T_{n-1}[x_n]$ such that $f = q \omega + \sum_{i=0}^{s-1} t_i x_i$ satisfies

$$|\bar{f}| < \max |t_i| = \left| \sum_{i=0}^{s-1} t_i x_i \right|.$$ 

But by the Division Theorem, we have

$$|f| = \max \left( q \omega, \left| \sum_{i=0}^{s-1} t_i x_i \right| \right),$$

and so $|f| \geq \left| \sum_{i=0}^{s-1} t_i x_i \right|$, a contradiction. So $\bar{j}$ is an isometry.

Remark. For an easy illustration of what can happen if $\omega$ fails to be monic, let $\omega = x_1 x_2 \in T_1[x_2]$. Then the residue class $\overline{x_2} = x_2 + \omega T_2$ is a torsion element in $T_2/\omega T_2$, viewed as a module over $T_1$, since $x_1 \cdot \overline{x_2} = x_1 x_2 + \omega T_3 = \overline{0}$. Hence $T_2/\omega T_2$ is not free over $T_1$.

Recall that a $k$-algebra map $\varphi : A \to B$ is called finite if $B$ is finitely generated as an $A$-module via the map $\varphi$. That is, there exists a generating
set \{b_d, \ldots, b_d\} such that for every \(b \in B\), there exists \(a_i \in A\) with
\[
b = \sum_{i=1}^{d} \varphi(a_i) \cdot b_i.
\]

**Proposition 2.14** (Weierstrass Finiteness Theorem). Let \(A\) be a \(k\)-Banach algebra, let \(\varphi : T_n \rightarrow A\) be a finite \(k\)-algebra homomorphism (i.e., \(A\) is finitely generated as a \(T_n\)-module via the map \(\varphi\)) and let \(\omega \in T_{n-1}[x_n]\) be a Weierstrass polynomial contained in the kernel of \(\varphi\). Then the map \(\varphi' : T_{n-1} \rightarrow A\) defined by \(\varphi|_{x_n}\) is also finite.

**Proof.** The trick will be to factor through the algebra \(T_n/\omega T_n\) which we have just seen is a finite \(T_n-1\)-module. Consider the following commutative diagram:

\[
\begin{array}{ccc}
T_{n-1} & \xrightarrow{\epsilon} & T_n \\
\downarrow{\bar{\epsilon}} & & \downarrow{\bar{\pi}} \\
T_n/\omega T_n & & A \\
\end{array}
\]

where \(\epsilon\) is the natural inclusion of \(T_{n-1}\) into \(T_n\) and \(\pi\) is the natural quotient map. Again \(\bar{\epsilon}\) and \(\bar{\varphi}\) are the unique maps required to make the diagram commute. Since \(\varphi\) is finite, so is \(\bar{\varphi}\). By the previous proposition, \(T_n/\omega T_n\) is a finite \(T_{n-1}\) module via the map \(\bar{\epsilon}\), so we have that the map \(\bar{\epsilon}\) is finite. Hence \(\varphi' = \varphi \circ \epsilon = \bar{\varphi} \circ \bar{\epsilon}\) is finite.

\(\square\)

### 2.3 Affinoid Algebras and Noether Normalisation

We now introduce a more general type of Tate algebra, the affinoid algebras, \(A = T_n/I\) where \(I\) is an ideal in \(T_n\).

**Definition 2.15.** A \(k\)-Banach algebra \(A\) is called \(k\)-affinoid (or just affinoid) if there exists a continuous surjection \(\alpha : T_n \rightarrow A\) for some \(n\).

**Lemma 2.16.** Let \(\varphi : B \rightarrow A\) be a continuous homomorphism between \(k\)-Banach algebras. Let \(f_1, \ldots, f_n\) be elements in \(A\) with \(|f| \leq 1\); let \(z_1, \ldots, z_n\) be indeterminants. Then there exists a unique continuous homomorphism \(\varphi : B[z_1, \ldots, z_n] \rightarrow A\) such that \(\varphi|_B = \varphi\) and \(\varphi(x_i) = f_i\) for each \(i\).

**Proof.** Since we require that \(\varphi\) is a \(k\)-algebra homomorphism, the condition \(\varphi(z_i) = f_i\) together with \(k\)-linearity implies that for any \(p = \sum_{0}^{N} a_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}\) in the polynomial algebra \(B[z_1, \ldots, z_n]\), if \(\varphi'\) is an extension of \(\varphi\) to \(B[z_1, \ldots, z_n]\), we must have
\[
\varphi'(p) = \sum_{0}^{N} a_{i_1, \ldots, i_n} f_1^{i_1} \cdots f_n^{i_n}.
\]
Thus we see that the extension of $\varphi$ to $B[z_1, \ldots, z_n]$ is uniquely defined. Then for $h = \sum_0^\infty a_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n} \in A(z_1, \ldots, z_n)$, we define $\varphi$ by the rule

$$\varphi(h) = \sum_0^\infty a_{i_1 \ldots i_n} f_1^{i_1} \cdots f_n^{i_n}.$$ 

Since each $|f_i| \leq 1$, we have $a_{i_1 \ldots i_n} f_1^{i_1} \cdots f_n^{i_n} \leq a_{i_1 \ldots i_n} f_1 \cdots f_n \leq a_{i_1 \ldots i_n} \to 0$, and hence

$$a_{i_1 \ldots i_n} f_1^{i_1} \cdots f_n^{i_n} \to 0.$$ 

Then by an analogous argument to the proof of Lemma 2.1, this series converges to a well-defined element of $B$. Now we have that $B[z_1, \ldots, z_n]$ is dense in $B(z_1, \ldots, z_n)$, since the sequence

$$(h_N)_{N \in \mathbb{N}} = \left( \sum_0^N a_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n} \right)_{N \in \mathbb{N}}$$

converges to $h$. Then since we require that $\varphi$ be continuous, we must have that $\varphi'(h_N) = \varphi(h_N) \to \varphi(h)$. Since the limit of the $\varphi'(h_N)$ is exactly equal to $\varphi(h_N)$ as we have defined it, we have that $\varphi$ is the unique continuous extension of $\varphi$ to $B(z_1, \ldots, z_n)$.

\[\blacksquare\]

**Proposition 2.17.** Let $B$ be an affinoid algebra and let $A$ be a $\mathbb{k}$-Banach algebra. Let $\varphi : B \to A$ be a continuous, finite homomorphism. Then $A$ is affinoid.

**Proof.** If $B$ is affinoid via the continuous map $\psi : T_n \to B$, we have that $\psi$ is finite, since $B \cong T_n/\ker(\psi)$ is generated by the images $\psi(x_1), \ldots, \psi(x_n)$ of the indeterminants $x_1, \ldots, x_n$ of $T_n$. Then the composition $\varphi \circ \psi$ is also continuous and finite, and so by replacing $\varphi$ with $\varphi \circ \psi$ if necessary, we may assume that $B = T_n$ for some $n$. By assumption, there are elements $a_1, \ldots, a_m \in A$ such that $A = \sum_{i=1}^m \varphi(T_n) a_i$. Then by replacing $a_i$ with $a_i/|a_i|$, by Lemma 2.16, the map $\varphi$ extends to a continuous homomorphism $T_{n+m} : \varphi : T_n(z_1, \ldots, z_m) \to A$ such that $\varphi(z_i) = a_i$. Then $\varphi$ is surjective, and so $A$ is affinoid via $\varphi$. \[\blacksquare\]

**Theorem 2.18.**

1. Let $A$ be a non-zero $\mathbb{k}$-affinoid algebra. For every finite homomorphism $\alpha : T_n \to A$, there exists an automorphism $\sigma$ of $T_n$ and an integer $d \geq 0$ such that $\alpha \circ \sigma_{\mathbb{k} \{x_1, \ldots, x_d\}}$ is finite and injective.

2. Let $\varphi : B \to A$ be a finite homomorphism between non-zero $\mathbb{k}$-affinoid algebra. Then there exists a homomorphism $\psi : T_d \to B$ for some $d \geq 0$ such that $\varphi \circ \psi : T_d \to B \to A$ is finite and injective.

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Proof. First we reduce the second assertion to the first one. Since $B$ is $k$-affinoid, let $\alpha : T_n \to B$ be onto and finite. Applying the first statement to the map $\varphi \circ \alpha : T_n \to A$, we get a subalgebra $T_d \subset T_n$ such that $(\varphi \circ \alpha)|_{T_d} : T_d \to A$ is finite and injective. Then we can just define $\psi$ to be $\alpha$ restricted to $T_d$.

For the first assertion, we use induction on $n$. If $n = 0$, then since $k$ is field, $\alpha : T_0 = k \to A$ must be either the zero map, or an isomorphism. But by assumption $A$ is non-zero, so we must have that $\alpha$ itself is injective. Now let $n \geq 1$, and let $\alpha : T_n \to A$ be finite. If $I = \ker(\alpha) = 0$ then we can again simply take $d = n$. Otherwise let $f \in I$. Then by Proposition 2.11, we have an automorphism $\sigma$ so that $\sigma(f)$ is $x_n$-distinguished. Hence by Theorem 2.10, we have $f = \omega e$, for Weierstrass $\omega \in T_{n-1}[x_n]$ and unit $e \in T_n$, and since $I$ is an ideal, we have $\omega \in I$. Then $\alpha$ induces a finite homomorphism $\bar{\alpha} : T_n/\omega T_n \to A$. By the Weierstrass Finiteness Theorem 2.14, the natural inclusion $T_{n-1} \to T_n$ induces a finite injection $\beta : T_{n-1} \to T_n/\omega T_n$. Then also $\bar{\alpha} \circ \beta$ is finite, so by the induction hypothesis, there is some $d$ such that $\bar{\alpha} \circ \beta|_{k(x_1,\ldots,x_d)} : T_{n-1} \to A$ is finite and injective. But $\bar{\alpha} \circ \beta$ is just $\alpha|_{k(x_1,\ldots,x_{n-1})}$, and so in fact we have that $\alpha|_{k(x_1,\ldots,x_d)}$ is injective, and we are done.

Corollary 2.19 (Noether Normalization Lemma). For every $k$-affinoid algebra $A \neq 0$ there exists an injection $\varphi : T_d \to A$ for some $d \geq 0$ such that $A$ is finitely generated as a $T_d$ module via $\varphi$.

Proof. Apply part (ii) of the Theorem above to any surjection $\sigma : T_n \to A$ realising $A$ as an affinoid algebra.

\[\Box\]

2.4 Affinoid varieties

Definition 2.20. Let $A$ be an affinoid algebra. Then the affinoid variety associated to $A$ is the set $\text{Max}(A)$ of maximal ideals in $A$.

Let $\overline{K}$ denote the algebraic closure of $k$. Just as algebraic varieties are defined as subsets of the affine space $\overline{K}^n$ defined by ideals in $\overline{K}[x_1,\ldots,x_n]$, affinoid varieties can be thought of as subsets of the “affinoid” space $\mathbb{B}_n^\circ(k_a)/\Gamma$ where $\Gamma = \text{Gal}(\overline{K}/k)$.

Let $a = (a_1,\ldots,a_n) \in \mathbb{B}_n^\circ(k_a)$. Let $L_a = k(a_1,\ldots,a_n)$ be the field extension generated by the $a_i$. Since $k$ is complete, the sum $\sum c_i a^i$ converges, so we can define an evaluation homomorphism $\eta_a : T_n \to L_a$. Then $\eta_a$ induces an isomorphism $T_n/\ker(\eta_a) \to L_a$, and since $L_a$ is a field, then so is $T_n/\ker(\eta_a)$. So $\ker(\eta_a)$ must be a maximal ideal of $T_n$, and we call $M_a = \ker(\eta_a)$ the maximal ideal associated to the point $a$.

Conversely, let $M \in T_n$ be any maximal ideal. The natural residue map $T_n \to T_n/M$ makes $T_n/M$ into an affinoid algebra, and so we can apply
Corollary 2.19 to get an injective \( k \)-algebra homomorphism \( \varphi : T_d \to T_n/M \) for some \( d \), such that \( T_n/M \) is finitely generated as a \( T_d \) module via the map \( \varphi \). Using the fact that \( \varphi \) is injective and identifying \( T_d \) with its image in \( T_n/M \) under \( \varphi \), we can think of \( T_n/M \) as a finite ring-extension of \( T_d \). Since \( M \) is a maximal ideal, \( T_n/M \) is a field. Then we must have that \( d = 0 \) and \( T_d = k \), since if \( d \geq 1 \), then any field extending \( T_d \) must contain inverses \( u_i = (x_1i)^{-1} \) for each power of \( x_1 \); the set \{\( u_i \)\} is \( T_d \) linearly independent, since multiplication by \( T_d \) can only ever 'emphasize the "power" of \( u_i \) (i.e., viewing \( u_i \) as a Laurent series, \( \deg(fu_i) \geq -i \) for all \( f \in T_d \)) and so the extension cannot be finite.

Hence we have that \( T_n/M \) is a finite extension of \( k \). So let \( T_n/M = k(b_1, \ldots, b_m) \), with \( b_i \in T_n/M \). Suppose one of the \( b_i \) is not algebraic over \( k \). Then \( b_i \) is not the root of any polynomial over \( k \), and in particular for each \((c_0, c_1, \ldots, c_m) \in k^{m+1}\)

\[
(*) \quad c_mb_1^m + \cdots + c_1b_1 + c_0 \neq 0.
\]

But \( T_n/M = k(b_1, \ldots, b_k) \) is a \( k \)-dimensional vector space, and so the \( k + 1 \) \( b_i \) must be linearly dependent, so the equation \((*)\) is impossible, and hence each of the \( b_i \) is algebraic over \( k \). Then we can embed \( T_n/M \) in \( \overline{k} \), although any embedding will only be unique up to the action of the Galois group. Then since \( i \) is continuous, \( i \) gives rise to a continuous map \( \varphi : T_n \to \overline{k} \) with kernel \( M \). If \( \sigma = a_1, \ldots, a_n \) denotes the images of the indeterminants of \( T_n \) under \( \varphi \), then by continuity, \( \varphi \) coincides with \( \eta_n : T_n \to k_n \). Hence \( M = M_n \).

Finally we note that for \( f \in T_n \), and \( M \in \text{Max}(T_n) \), we can define \( f(M) \) to be equal to

\[
|f|_{T_n/M} = \inf_{g \in M} |f + g|,
\]

the residue norm of the class of \( f \) in \( T_n/M \). Hence \( f \) defines a function \( \text{Max}(T_n) \to k \), and thus we see that the sets \( \text{Max}(T_n) \) and \( B^n(k) \) are essentially the same from the point of view of affinoid functions; we call the varieties \( B^n(k) = \text{Max}(T_n) \) "unit balls."

Ideals in \( T_n \) correspond to subsets defined by the zeros of affinoid functions in the way that you'd expect:

**Proposition 2.21.** If \( A = T_n/I \) then \( \text{Max}(A) \) is identified with the set

\[
\{z \in \text{Max}(T_n) : f(z) = 0 \text{ for all } f \in I\}.
\]

**Proof.** The canonical surjection \( \sigma : T_n \to A \) induces an injection \( \sigma : \text{Max}(A) \to \text{Max}(T_n) \) by \( M \mapsto \sigma^{-1}(M) \), which is a bijection since the maximal ideals of \( A \) are exactly the maximal ideals of \( T_n \) containing \( I \). \( \square \)

This gives us a natural way to define maps between affinoid varieties. Namely for any \( k \)-algebra homomorphism \( \sigma : B \to A \), for any \( x \in \text{Max}(A) \), the map \( \sigma \) induces an injection

\[
B/\sigma^{-1}(M_x) \hookrightarrow A/M_x,
\]

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since \( \ker(\sigma) = \sigma^{-1}(0) \subset \sigma^{-1}(M_x) \). Then since \( A/M_x \) is finite algebraic over \( \mathbb{k} \), it must be a field, and so we see that \( B/\sigma^{-1}(M_x) \) is also a field; therefore \( \sigma^{-1}(M_x) \) is a maximal ideal in \( B \). Thus \( \sigma \) gives rise to a map \( a\sigma : \Max(A) \to \Max(B) \) defined by \( M_x \mapsto \sigma^{-1}(M_x) \).

**Definition 2.22.** A morphism \( \varphi : \Max(A) \to \Max(B) \) is \( k \)-algebra homomorphism \( \varphi : B \to A \), or equivalently, the map \( a\varphi : \Max(A) \to \Max(B) \) described above.

**Remark.** Perhaps an interesting point to notice is that the main technical result needed to sensibly define affinoid varieties was the Noether Normalisation Lemma 2.19. This is in contrast with the classical case for algebraic geometry over \( \mathbb{C} \), where the Hilbert Nullstellensatz is the main tricky technical result.

**Example 2.23.** An important affinoid variety that we will need later on corresponds to the algebra \( T_{n,\rho} \) of convergent power series on the polydisc

\[
P\rho(\mathbb{k}) = P\rho = \{ x \in \mathbb{k}^n : |x_i| \leq \rho_i \text{ for } 1 = 1, \ldots, n \}.
\]

By a similar argument to Lemma 2.1, we see that a formal power series \( f = \sum a_i x^i \) converges on \( P\rho \) if \( \lim |a_i|\rho^i \to 0 \) as \( ||i|| \to \infty \). Hence we define

\[
T_{n,\rho} = \left\{ \sum a_i x^i : \lim_{||i|| \to \infty} |a_i|\rho^i = 0 \right\}.
\]

Just as we did for \( T_n \), by taking evaluation homomorphisms \( \eta_a : T_{n,\rho} \to P\rho \), for each \( a \in \mathbb{F}_\rho = P\rho(\mathbb{k}) \), we see that the affinoid variety \( \Max(T_{n,\rho}) \) is equal to \( \mathbb{F}_\rho / \Gamma \), where \( \Gamma \) is induced by the Galois group of \( \overline{\mathbb{k}} \) over \( \mathbb{k} \).

### 2.5 Equivalence of \( T(X) \) and \( BG(X) \)

We would like to prove the following

**Theorem 2.24.** Let \( X = \mathbb{k}[x_1^\pm, \ldots, x_n^\pm] / I \) for some ideal \( I \). Then the sets \( T(X) \) and \( BG(X) \) are equal.

Before turning to the proof, we make a few observations. As we noted in Section 1.2, the inclusion \( T(X) \subset BG(X) \) is immediate. Since \( BG(X) \) is a \( v(\overline{\mathbb{k}}) \)-rational polyhedron, the set \( BG(X) \cap v(\overline{\mathbb{k}}) \) is dense in \( BG(X) \). Therefore to prove that \( BG(X) \subset T(X) \), it is enough to prove that \( \mathbb{B}(X) \cap v(\overline{\mathbb{k}}) \subset T(X) \). So let \( u = (u_1, \ldots, u_n) \in \mathbb{B}(X) \cap v(\overline{\mathbb{k}}) \). Let \( a_i \in \overline{\mathbb{k}} x \) with \( v(a_i) = u_i \). Then under the change of coordinates in \( \mathbb{k}[X] x_i' = x_i/a_i \), the set \( \mathbb{B}(X) \) becomes

\[
\{ w(x') \}_{w \in W(A)} = \{ w(x/a) \}_{w \in W(A)} = \{ w(x) - u \}_{w \in W(A)}.
\]
and so we see that this change of coordinates corresponds to translating $BG(X)$ by the vector $-u$, and similarly for $T(X)$. Hence any $w \in W(A)$ with $\beta(w) = u$ in the original coordinates will now have $\beta(w) = 0$. Therefore to prove Theorem 2.24 it is enough to show that $0 \in T(X)$ implies $0 \in BG(X)$. We proceed by constructing an affinoid algebra corresponding to the point $0 \in T(X) \subset \mathbb{R}^n$.

**Definition 2.25.** The algebra $A_0$ consists of Laurent series $\sum_{i \in \mathbb{Z}^n} a_i x^i$ with $|a_i| \to 0$ as $||n|| \to \infty$.

Note that $A_0$ is the quotient of $T_{2n} = k(z_1, w_1, \ldots, z_n, w_n)$ by the ideal generated by the elements $z_iw_i - 1$. Hence $A_0$ is $k$-affinoid.

**Lemma 2.26.** The variety of $A_0$ corresponds to

$$\text{val}^{-1}(0) = \{ z \in \mathbb{B}^n(k) : \text{val}(z) = 0 \}.$$

**Proof.** Suppose $z = (z_1, w_1, \ldots, z_n, w_n) \in \mathbb{B}^n$ with $z_iw_i = 1$ for each $i$. Then $w_i = z_i^{-1}$ for each $i$, and hence $z$ is determined by specifying $z' = (z_1, \ldots, z_n) \in \mathbb{B}^n(k)$. Also since we have $z_i, w_i \in \mathbb{B}$, $z_i w_i = 1$ implies $|z_i| = 1$ for each $i$, i.e., $\text{val}(z_i) = 0$ for each $i$.

**Theorem 2.27.** If $0 \in BG(X)$ then $0 \in T(X)$.

**Proof.** Suppose $0 \notin T(X)$. Then $\text{val}(z) \neq 0$ for each $z \in V_0(X)$. Let $X$ be defined by the polynomials $f_1, \ldots, f_r$. Put $\tilde{A}_0 = A_0/(f_1, \ldots, f_r)$. Then

$$\text{Max}(\tilde{A}_0) = (X(\overline{k}) \cap \text{val}^{-1}(0))/G = \emptyset.$$ Since every non-zero ring has maximal ideals, we must have that $\overline{A}_0 = 0$. Hence $(f_1, \ldots, f_r) = A$, i.e., there are $g_1, \ldots, g_r \in A_0$ such that $f_1g_1 + \cdots + f_rg_r = 1$.

Now suppose $0 \in BG(X)$. Then there is a valuation on

$$A = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/(f_1, \ldots, f_r) = k[X]$$

such that $w(x_i) = 0$ for each $i$. We can consider $w$ as a ring valuation on $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that equals $\infty$ on the ideal $(f_1, \ldots, f_r)$. Since $w(x_i) = 0$ we can extend $w$ by continuity $\hat{w}$ on $A_0$ via

$$\hat{w} \left( \sum_{i \in \mathbb{Z}^n} a_i x^i \right) = \lim_{N \to \infty} w \left( \sum_{i \in [-N,N]^d} a_i x^i \right).$$

But then we have

$$0 = \hat{w}(1) = \hat{w}(f_1g_1 + \cdots + f_rg_r) \geq \min \hat{w}(f_1g_1) \geq \min(w(f_i) + \hat{w}(g_i)) = \infty,$$

a contradiction. \(\square\)
Chapter 3

A global result

In the previous chapter, we were essentially performing analysis locally. In proving the equivalence of the tropical variety and the Bieri-Groves set, we were able to change coordinates so that we only needed to concern ourselves with what was happening at the origin. We looked at the ring of convergent power series at the origin, and the correspondence between maximal ideals of that ring and points of the unit ball $\mathbb{B}^n(k)$ modulo the Galois group. Then by looking at valuations on the ring itself we obtained the result we needed.

We now wish to prove the connectedness of a general irreducible tropical variety (that is, a tropical variety $T(X)$ of an irreducible scheme $X$). In contrast to the previous chapter, local techniques appear to be insufficient here. Analysis over our non-Archimedean field $k$ only happens in the unit ball, whereas the scheme $X$ lies in the affine space $\mathbb{A}^n$.

It turns out that analytic continuation of affinoid functions is only possible along finite coverings of a special kind of open set. This leads us to define Grothendieck topologies, which keep track of coverings of open sets, as well as keeping track of the open sets of a space. As a bonus, we get a non-trivial notion of connectedness (note that under the ordinary topological definition of connectedness, $\mathbb{A}^n$ is totally disconnected since $k$ is).

Specifically, given a set $S$ with a Grothendieck topology, we will say that a subset $U \subset X$ is disconnected only if it has can be written as a disjoint union if open sets $U = A \cup B$, where $\{A, B\}$ is allowed as a covering under the Grothendieck topology. In particular, we will have that an ball $\mathbb{B}^n(k)$ together with its complement $\mathbb{A}^n \setminus \mathbb{B}^n(k)$, while both closed and open, will not form an admissible covering of $k$. This can be thought of as playing an analogous role to path-connectedness for analytic continuation of complex-analytic functions. We will define a Grothendieck topology on our scheme $X$, then using a result of Conrad that shows that $X$ is connected with respect to this Grothendieck topology, we will see that that this implies that the tropical variety $T(X)$ is also connected.
3.1 Affinoid subdomains

We define affinoid subdomains via the following universal property: Let Max $A$ be an affinoid variety and $U$ a subset of Max $A$. Then a given affinoid map $\varphi : \text{Max } A' \to \text{Max } A$ is said to represent all affinoid maps into $U$ if $\varphi$ maps $\text{Max } A'$ into $U$ and if $\varphi$ satisfies the following universal property:

Given any affinoid map $\psi : \text{Max } B \to \text{Max } A$ such that $\psi(\text{Max } B) \subset U$, there exists a unique affinoid map $\psi' : \text{Max } B \to A'$ such that $\psi = \varphi \circ \psi'$.

**Definition 3.1.** Let Max $A$ be an affinoid variety and let $U$ be a subset of Max $A$. Then $U$ is called an affinoid subdomain of Max $A$ if there exists an affinoid map $\varphi : \text{Max } A' \to \text{Max } A$ representing all affinoid maps into $U$.

Next we give an important class of examples of affinoid subdomains.

**Example 3.2.** Let $X = \text{Max } A$ be an affinoid variety and let $f = (f_1, \ldots, f_m)$ Then we set

$$A(f) = A(z_1, \ldots, z_m)/(z_1 - f_1, \ldots, z_m - f_m)$$

$$X(f) = \{x \in X : |f_i(x)| \leq 1\}.$$  

We say that $X(f)$ is a Weierstrass domain in $X$.

**Proposition 3.3.** The set $X(f)$ is an affinoid subdomain of $X = \text{Max } (A)$, and the affinoid map $\psi : \text{Max } A(f)$ corresponding to the canonical homomorphism $\psi^* : A \to A(f)$ represents all affinoid maps into $X(f)$.

**Proof.** Suppose $\varphi : \text{Max } (B) \to \text{Max } (A)$ is an arbitrary affinoid map satisfying $\text{im}(\varphi) \subset X(f)$. Let $\varphi^* : A \to B$ be the induced map on affinoid algebras, defined by $\varphi^*(f_i(x)) = f(\varphi(x))$. Now for each $f_i$, and for each $y \in \text{Max } (B)$ we must have

$$|f_i(\varphi(y))| \leq 1 = |\varphi^*(f_i)(y)| \leq 1,$$

and therefore the elements $\varphi^*(f_i) \in B$ satisfy $|\varphi^*(f_i)| \leq 1$. Then by Lemma 2.16, we get a unique continuous homomorphism $\Phi^* : A(f) \to B$ satisfying $\Phi \circ \psi = \varphi$. By Proposition 2.22, this corresponds to a map $\Phi : \text{Max } (B) \to X(f)$ with $\varphi \circ \Phi = \psi$, and hence we see that $\psi$ represents all affinoid maps into $X(f)$.

3.2 Grothendieck topologies

Let $X$ be a set. By a covering of a subset $U \subset X$ we mean a collection $\{U_i\}_{i \in I}$ of subsets in $X$ satisfying $\cup_{i \in I} = U$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be coverings of a set $V$. We say that $\mathcal{V}$ is a refinement of $\mathcal{U}$ if each $V_j$ is contained in some $U_i$. 

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Definition 3.4. A Grothendieck topology $\mathcal{S}$ on $X$ (usually abbreviated as $G$-topology) consists of

(a) a system $S$ of subsets in $X$, called admissible open or $\mathcal{S}$-open subsets of $X$, and

(b) a family $\{\text{Cov } U\}_{U \in S}$ of systems of coverings, called admissible or $\mathcal{S}$-coverings, where $\text{Cov } U$ for $U \in S$ contains coverings $\{U_i\}$ of $U$ by sets $U_i \in S$.

The system $S$ and the family $\{\text{Cov } U\}$ are subject to the following conditions:

(i) $U, V \in S \Rightarrow U \cap V \in S$

(ii) $U \in S \Rightarrow \{U\} \in \text{Cov } U$

(iii) If $U \in S$, $\{U_i\}_{i \in I} \in \text{Cov } U$, and $\{V_{ij}\} \in \text{Cov } U_i$ for $i \in I$, then $\{V_{ij}\}_{i \in I, j \in J_i} \in \text{Cov } U$.

(iv) If $U, V \in S$ with $V \subset U$ and if $\{U_i\}_{i \in I} \in \text{Cov } U$, then $\{V \cap U_i\}_{i \in I} \in \text{Cov } V$.

If $\mathcal{S}$ and $\mathcal{S}'$ are $G$-topologies on a set $X$, then we say that $\mathcal{S}'$ is stronger than $\mathcal{S}$ if every $\mathcal{S}$-open subset is also $\mathcal{S}'$-open, and every $\mathcal{S}$-admissible covering is also an $\mathcal{S}'$-admissible covering.

Let $X$ be a $G$-topological space with $G$-topology $\mathcal{S}$ and system of $\mathcal{S}$-open subsets $S$. Then a system $B \subset S$ is called a basis for $\mathcal{S}$ if each $\mathcal{S}$-open subset in $X$ admits a $\mathcal{S}$-covering by sets belonging to $B$. The following conditions will be satisfied by the $G$-topologies we will be interested in:

(G0) The sets $\emptyset$ and $X$ are admissible open subsets in $X$

(G1) Let $U \subset X$ be an admissible open set and let $V \subset U$ be a subset. Assume that there exists an admissible covering $\{U_i\}_{i \in I}$ of $U$ such that $V \cap U_i$ is admissible open in $X$ for all $i \in I$. Then $V$ is admissible open in $X$.

(G2) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of an admissible open set $U \subset X$ such that $U_i$ is admissible open in $X$ for all $i \in I$. Assume further that $\mathcal{U}$ has a refinement which is admissible. Then $\mathcal{U}$ is an admissible covering of $U$.

Definition 3.5. Let $X$ be a set admitting $G$-topologies $\mathcal{S}$ and $\mathcal{S}'$. We say that $\mathcal{S}'$ is slightly finer than $\mathcal{S}$ if the following conditions are satisfied:

(i) $\mathcal{S}'$ is finer than $\mathcal{S}$.

(ii) The $\mathcal{S}$-open subsets of $X$ form a basis for $\mathcal{S}'$. 

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(iii) For each $\mathcal{T}'$-covering $\mathcal{U}$ of a $\mathcal{T}$-open subset $U \subset X$, there exists a $\mathcal{T}$-covering which refines $U$.

Note that condition (ii) requires that $\mathcal{T}'$-open sets generate the same topology as the $\mathcal{T}$-open sets, and so of course there are no non-trivial analogues of slightly finer topologies in the ordinary topological sense.

It is a natural question to ask if there is a unique $G$-topology among those slightly finer than $\mathcal{T}$. It will be convenient to look at this question in a slightly more general setting. Let $\mathfrak{C}$ be a subcategory of the category of sets, and suppose that each $X \in \mathfrak{C}$ carries a $G$-topology $\mathfrak{T}_X$ such that all morphisms of $\mathfrak{C}$ are continuous. Then we say that $\mathfrak{T} = \{\mathfrak{T}_X\}_{X \in \mathfrak{C}}$ is a $G$-topology on $\mathfrak{C}$. We say that for $G$-topologies $\mathfrak{T}$, $\mathfrak{T}'$ on $\mathfrak{C}$, $\mathfrak{T}'$ is finer (or slightly finer) than $\mathfrak{T}$ if $\mathfrak{T}'_X$ is finer (or slightly finer) than $\mathfrak{T}_X$ for each $X \in \mathfrak{C}$.

**Proposition 3.6.** Let $\mathfrak{T}$ be a $G$-topology on a category of sets $\mathfrak{C}$ as above. Then there exists a unique finest $G$-topology $\mathfrak{T}'$ on $\mathfrak{C}$ among all the $G$-topologies slightly finer than $\mathfrak{T}$. The $G$-topology $\mathfrak{T}'$ satisfies conditions $(G_1)$ and $(G_2)$. If condition $(G_3)$ is satisfied by $\mathfrak{T}$ then it is also satisfied by $\mathfrak{T}'$.

Before proving the Proposition, we describe the $G$-topology $\mathfrak{T}'$ explicitly. Let $X \in \mathfrak{C}$ and $U \subset X$. An arbitrary covering $\{U_i\}_{i \in I}$ of $U$ is called compatible with $\mathfrak{T}$ if for any $\mathfrak{C}$-morphism $\varphi : Y \to X$ and any $\mathfrak{T}_Y$-open subset $Y' \subset Y$ satisfying $\varphi(Y') \subset U$, the covering $\{\varphi^{-1}(U_i) \cap Y'\}_{i \in I}$ of $Y$ has a $\mathfrak{T}_Y$-covering which refines it. The idea here is that adding in a compatible covering gives a slightly finer topology on $\mathfrak{C}$. Explicitly, if $U \subset X \in \mathfrak{C}$, and $\{U_i\}_{i \in I}$ is a compatible covering, then define $\mathfrak{T}'$ on $\mathfrak{C}$ by taking $\mathfrak{T}$ and adding in all the sets and coverings needed to make sure all the morphisms $\varphi : Y \to X$ are still continuous; that is for every continuous $\varphi : Y \to X$, add in the sets $\varphi^{-1}(U)$ as $U$ runs through $\mathfrak{T}'$-open sets, and the coverings $\mathcal{U} = \{U_i\}_{i \in I}$ as $\mathcal{U}$ runs over $\mathfrak{T}'$-admissible coverings. To prove that this is indeed the $G$-topology that we want, we will need the following technical result.

**Lemma 3.7.** (i) Let $U, V$ be subsets of some $X \in \mathfrak{C}$. If $\{U_i\}_{i \in I}$ is a covering of $U$ compatible with $\mathfrak{T}$, then also the covering $\{U_i \cap V\}_{i \in I}$ is compatible with $\mathfrak{T}$.

(ii) Let $U$ be a subset of some $X \in \mathfrak{C}$. If the coverings $\{U_i\}_{i \in I}$ of $U$ and $\{U_{ij}\}_{j \in J_i}$ of $U_i$, $i \in I$ are compatible with $\mathfrak{T}$, then also $\{U_{ij}\}_{i \in I, j \in J_i}$ is a covering of $U$ compatible with $\mathfrak{T}$.

**Proof.** Let $\varphi : Y \to X$ be a $\mathfrak{C}$-morphism and $Y' \subset Y$ with $\varphi(Y') \subset U$. Since $\{U_i\}_{i \in I}$ is compatible, we have that $\{\varphi^{-1}(U_i) \cap Y'\}_{i \in I}$ has an admissible covering $\{Y_i\}_{i \in I}$ in $Y$ which refines it. Then we have:

(i) Since $\varphi$ is continuous, $\varphi^{-1}(V)$ is $\mathfrak{T}_Y$-open and so by $G$-topology axiom (iv), $\{Y_i \cap \varphi^{-1}(V)\}_{i \in I}$ is admissible. Since it is a refinement of $\{\varphi(U_i) \cap Y_i \cap \varphi^{-1}(V)\}_{i \in I}$, it is also admissible.
Lemma 3.8. \( \varphi(V) \}_{i \in I} = \{ \varphi(U_i \cap \varphi V) \}_{i \in I} \), we have that \( \{ U_i \cap V \}_{i \in I} \) is compatible with \( \mathfrak{T} \).

(ii) Since \( \{ Y_i \}_{i \in I} \) is a refinement, we have that each each \( Y_i \) is contained in some \( \varphi^{-1}(U) \), so let \( \tau : L \to I \) be such that \( Y_i \subset \varphi^{-1}(U_{\tau(i)}) \). Note that this is equivalent to \( \varphi(Y_i) \subset U_{\tau(i)} \). Since for each \( i \), the coverings \( \{ U_{ij} \}_{j \in J} \) are compatible with \( \mathfrak{T} \), then in particular, for every \( l \in L \), the covering \( \{ \varphi^{-1}(U_{\tau(lj)}) \}_{j \in J_{\tau(l)}} \) has a refinement \( \{ Z_{lk} \}_{k \in K} \) which is admissible. The covering \( \{ Y_{lk} \}_{k \in K} = \{ Z_{lk} \cap Y_i \}_{k \in K} \) is an admissible refinement of the covering \( \{ \varphi^{-1}(U_{\tau(lj)}) \cap Y_i \}_{j \in J_{\tau(l)}} \). Then \( \{ Y_{lk} \}_{l \in L, k \in K} \) is a \( \Sigma_Y \)-covering of \( Y' \), and by construction it is a refinement of \( \{ \varphi^{-1}(U_{ij} \cap Y') \}_{i \in I, j \in J} \). Thus we have that \( \{ U_{ij} \}_{i \in I, j \in J} \) is compatible with \( \mathfrak{T} \).

Then we can now prove the following

**Lemma 3.8.** (i) A \( G \)-topology \( \mathfrak{T}' \) on \( \mathfrak{C} \) can be defined by as follows: For \( X \in \mathfrak{C} \), a subset \( U \subset X \) is called \( \mathfrak{T}'_X \)-open if it admits a covering \( \{ U_i \}_{i \in I} \) by \( \mathfrak{T}_X \)-open sets \( U_i \), which is compatible with \( \mathfrak{T} \). Similarly, a covering \( \{ V_j \}_{j \in J} \) of any \( \mathfrak{T}'_X \)-open subset \( V \subset X \) by \( \mathfrak{T}_X \)-open sets \( V_j \) is called a \( \mathfrak{T}'_X \)-covering if \( \{ V_j \}_{j \in J} \) is compatible with \( \mathfrak{T} \).

**Proof.** Fix an \( X \in \mathfrak{C} \). We show that the \( \mathfrak{T}'_X \)-open sets and the \( \mathfrak{T}'_X \)-compatible coverings for a \( G \)-topology on \( X \), by checking each of the \( G \)-topology axioms.

(i) Let \( U, V \subset X \) be \( \mathfrak{T}'_X \)-open sets. We want to find a compatible covering of \( U \cap V \) by \( \mathfrak{T}_X \)-open sets. Let \( \{ U_{ij} \}_{i \in I}, \{ V_j \}_{j \in J} \) coverings of \( U \) and \( V \) compatible with \( \mathfrak{T} \), consisting of \( \mathfrak{T}_X \)-open sets. By (i) of Lemma 3.7, for every \( i \), the covering \( \{ U_i \cap V \}_{i \in I} \) of \( U \cap V \) is compatible with \( \mathfrak{T} \); also, for every \( i \), the covering \( \{ U_i \cap V_j \}_{j \in J} \) of \( U \cap V \) is compatible with \( \mathfrak{T} \). Thus by (ii) of Lemma 3.7, the covering \( \{ U_i \cap V_j \}_{i \in I, j \in J} \) is compatible with \( \mathfrak{T} \). Since all the sets \( U_i, V_j \) are \( \mathfrak{T}_X \)-open, by \( G \)-topology axiom (i) for \( \mathfrak{T} \), we have that all the sets \( U_i \cap V_j \) are open. Hence by the definition of \( \mathfrak{T}' \), \( U \cap V \) is \( \mathfrak{T}' \)-open.

(ii) This is immediate since for a \( \mathfrak{T}' \)-open \( U \subset X \), we have a covering \( \{ U_i \}_{i \in I} \) of \( U \) which is compatible with \( \mathfrak{T} \); then \( \{ U_i \}_{i \in I} \) is a refinement of the trivial covering \( \{ U \} \) of \( U \), we have that \( \{ U \} \) is admissible.

(iii) Follows from part (ii) of Lemma 3.7.

(iv) Follows from part (i) of Lemma 3.7.

To finish the proof we still have to check that the \( \mathfrak{T}'_X \) are a topology on \( \mathfrak{C} \). That is, we need to check that all \( \mathfrak{C} \)-morphisms are continuous with
respect to the $G$-topologies $\mathcal{T}'_X$. So let $\varphi : Y \to X$ be an arbitrary $\mathcal{C}$-morphism. If $U \subset X$ is $\mathcal{T}'_X$-open, then there is an $\mathcal{T}_X$-admissible covering of $\{U_i\}_{i \in I}$ of $U$ which is compatible with $\mathcal{T}$. Then applying the definition of a compatible covering, we have a $\mathcal{T}_X$-covering $\{Y_j\}_{j \in J}$ of $\varphi^{-1}(U)$ refining $\{\varphi^{-1}(U_i)\}_{i \in I}$. We just need to check that $\{Y_j\}_{j \in J}$ is compatible with $\mathcal{T}$. So suppose $\psi : Z \to Y$ is another $\mathcal{C}$-morphism with $W \subset Z$ such that $\psi(W) \subset \varphi^{-1}(U)$. Then $\varphi \circ \psi : Z \to X$ is a $\mathcal{C}$-morphism with $\varphi \circ \psi(W) \subset U$, and therefore since $\{U_i\}_{i \in I}$ is compatible with $\mathcal{T}$, there is a $\mathcal{T}_Z$-covering $\{Z_k\}_{k \in K}$ refining $\{\varphi \circ \psi(U_i) \cap W\}_{i \in I}$. By the continuity of $\psi$, $\psi^{-1}(Y_j)$ is $\mathcal{T}_Z$-open for each $j \in J$, and hence by axioms (iii) and (iv) for $G$-topologies, we have that $\{Z_k \cap \psi^{-1}(Y_j)\}_{k \in K, j \in J}$ is a $\mathcal{T}_Z$-covering refining $\{\psi^{-1}(Y_j) \cap W\}_{j \in J}$, and hence $\{Y_j\}_{j \in J}$ is compatible with $\mathcal{T}$. The proof that $\mathcal{C}$-morphisms respect $\mathcal{T}'$, admissible coverings is very similar. This completes the proof.

Finally we come to the

Proof of Proposition 3.6. By the construction of $\mathcal{T}'$ we have that $\mathcal{T}'$ is finer than any $G$-topology $\mathcal{T}''$ on $\mathcal{C}$ which is slightly finer than $\mathcal{T}$—and in particular is slightly finer than $\mathcal{T}$ itself—since for each $X \in \mathcal{C}$,

(i) Let $U$ be $\mathcal{T}_X''$ open. Applying 3.5(iii) we have a $\mathcal{T}_X$-covering $\Psi$ of $U$ refining the trivial covering $\mathcal{T}''$-covering $\{U\}$ of $U$. Since any $\mathcal{T}_X$-covering is trivially compatible with $\mathcal{T}$, we see that $U$ is $\mathcal{T}'$-open, and hence $\mathcal{T}'$ is finer than $\mathcal{T}''$.

(ii) By definition $\mathcal{T}'_X$-open sets have a $\mathcal{T}_X$-admissible covering consisting of $\mathcal{T}_X$-open sets. But since $\mathcal{T}''$ is finer than $\mathcal{T}$, then any such covering is also a $\mathcal{T}'_X$-admissible covering consisting of $\mathcal{T}'_X$-open sets, and so the $\mathcal{T}''$-open sets form a basis for $\mathcal{T}'$.

(iii) If $\mathcal{U}$ is a $\mathcal{T}'$-covering of a $\mathcal{T}''$-open $U$, then since $\mathcal{U}$ is, by definition, compatible with $\mathcal{T}$, then applying the identity $\mathcal{C}$-morphism $X \to X$, we see that there exists a $\mathcal{T}_X$-covering refining $\mathcal{U}$. Again, since $\mathcal{T}''$ is finer than $\mathcal{T}$, such a covering is also a $\mathcal{T}'_X$-covering refining $\mathcal{U}$.

So to finish the proof, we just have to verify conditions ($G_0$), ($G_1$) and ($G_2$). If $\mathcal{T}$ satisfies ($G_0$), then since $\mathcal{T}'$ is finer than $\mathcal{T}$, we have that $\mathcal{T}'$ also satisfies ($G_0$). Also condition ($G_2$) is immediate from the definition of $\mathcal{T}'$. For ($G_1$), let $X \in \mathcal{C}$, and let $U \subset X$ be a $\mathcal{T}'$-open subset admitting $\{U_i\}_{i \in I}$ as a $\mathcal{T}'$-covering; also let $V \subset U$ be a subset such that $V \cap U_i$ is $\mathcal{T}'_X$-open in $X$ for all $i \in I$. Then since $\{U_i\}_{i \in I}$ is a $\mathcal{T}'$-covering, it is compatible with $\mathcal{T}$, and hence due to (i) of Lemma 3.7, the covering $\{V \cap U_i\}_{i \in I}$ is also compatible with $\mathcal{T}$. Choosing a $\mathcal{T}_X$-covering $\{V_{ij}\}_{j \in J_i}$ of $V \cap U_i$ consisting of $\mathcal{T}_X$-open sets for all $i \in I$, then by (ii) of Lemma 3.7, $\{V_{ij}\}_{i \in I, j \in J_i}$ is a covering of $V$ compatible with $\mathcal{T}$. Then $V$ is $\mathcal{T}'_X$-open.
Corollary 3.9. Let $X$ be a set and $\mathfrak{T}$ a $G$-topology on $X$. Then there exists a unique finest $G$-topology $\mathfrak{T}'$ of $X$ among all the $G$-topologies slightly finer than $\mathfrak{T}$. The $G$-topology $\mathfrak{T}'$ satisfies conditions $(G_1)$ and $(G_2)$. If condition $(G_3)$ is satisfied by $\mathfrak{T}$ then it is also satisfied by $\mathfrak{T}'$.

Proof. Apply 3.6 to the special situation where the category $\mathfrak{C}$ consists of a single set $X$, and where the identity map $X \to X$ is the only morphism. \(\square\)

3.3 Pasting of $G$-topological spaces

We now look at an important method of forming new $G$-topological spaces. Let $X$ be a set, and let $X = \{X_i\}_{i \in I}$ be a covering of $X$, where each $X_i$ carries a $G$-topology $\mathfrak{T}_i$ such that for all $i, j \in I$, we have the following Pasting Conditions:

(a) $X_i \cap X_j$ is $\mathfrak{T}_i$-open in $X_i$ and $\mathfrak{T}_j$-open in $X_j$ and

(b) $\mathfrak{T}_i$ and $\mathfrak{T}_j$ induce the same $G$-topology on $X_i \cap X_j$.

We would like a $G$-topology on $X$ admitting the sets $X_i$ as admissible open sets and inducing the $G$-topology $\mathfrak{T}_i$ on each $X_i$. Denote by $\mathfrak{T}^w$ the weakest such $G$-topology. We need to have $\mathfrak{T}^w$ include as admissible open sets at least each $\mathfrak{T}_i$-open $U \subset X_i$, and as admissible covers at least all the $\mathfrak{T}_i$-covers. In fact this is enough to give a $G$-topology. Firstly we note that the condition (a) implies that each $X_i$ will be open since letting $i = j$, we have that $X_i = X_i \cap X_i$ is $\mathfrak{T}_i$-open. For the first $G$-topology axiom, suppose $U \subset X_i$ is $\mathfrak{T}_i$-open and $V \subset X_j$ is $\mathfrak{T}_j$-open. By condition (a) we have that $X_i \cap X_j$ is $\mathfrak{T}_j$-open in $X_j$, and hence so is $V \cap (X_i \cap X_j)$. Condition (b) requires that $\mathfrak{T}_i$ and $\mathfrak{T}_j$ agree on $X_i \cap X_j$, and hence $V \cap (X_i \cap X_j) \subset X_i$ must be $\mathfrak{T}_i$-open as well. Then we have

$$U \cap V = U \cap [V \cap (X_i \cap X_j)],$$

which is an intersection of $\mathfrak{T}_i$-open sets, and hence since $\mathfrak{T}$ satisfies $G$-topology axiom (i), is again $\mathfrak{T}_i$-open, and hence open in $\mathfrak{T}^w$. The other axioms are entirely trivial, since the cases that arise are always contained in a single $X_i$. For example, condition (iii) requires a certain cover to be admissible given an open set $U \subset X$, a cover $\{U_i\}_{i \in I}$ of $U$, and covers $\{V_{ij}\}_{j \in J_i}$ of each of the $U_i$. But since $U$ must be contained in some $X_i$, then so are all the $U_i$ and $V_{ij}$, so we can just use the fact that $\mathfrak{T}_i$ is a $G$-topology.

We are also interested in the $G$-topology $\mathfrak{T}$, the finest $G$-topology inducing the $G$-topology $\mathfrak{T}_i$ on each $X_i$. For this to be true, if $U \subset X$ is $\mathfrak{T}$-open, we require that $U \cap X_i$ is $\mathfrak{T}_i$-open for all $i \in I$; if covering $\{U_j\}_{j \in J}$ is a $\mathfrak{T}$-covering we require that $\{U_j \cap X_i\}_{j \in J}$ is a $\mathfrak{T}_i$-covering for all $i \in I$. Then we define $\mathfrak{T}$ as follows: call a subset $U \subset X$ $\mathfrak{T}$-open if and only if $U \cap X_i$
is $\mathcal{I}_i$-open for all $i \in I$; call a covering $\{U_j\}_{j \in J}$ a $\mathcal{I}$-covering if and only if $\{U_j \cap X_i\}_{j \in J}$ is a $\mathcal{I}_i$-covering for all $i \in I$. We have that $\mathcal{I}$ is a $G$-topology since:

(i) If $U, V$ are $\mathcal{I}$-open, then $U \cap X_i$ and $V \cap X_i$ is $\mathcal{I}_i$-open for all $i \in I$. Then

$$(U \cap V) \cap X_i = (U \cap X_i) \cap (V \cap X_i)$$

is $\mathcal{I}_i$-open for each $i \in I$, since it is an intersection of $\mathcal{I}_i$-open sets, and each $\mathcal{I}_i$ satisfies axiom (i).

(ii) If $U$ is $\mathcal{I}$-open, then $U \cap X_i$ is $\mathcal{I}_i$-open for each $i \in I$, so since each $\mathcal{I}_i$ satisfies axiom (ii), $\{U \cap X_i\}$ is a $\mathcal{I}_i$-covering of $U \cap X_i$. Then by the definition of $\mathcal{I}$, $\{U\}$ is a $\mathcal{I}$-covering of $U$.

(iii) Let $U \subset X$, $\{U_j\}_{j \in I}$ be a $\mathcal{I}$-cover of $U$, and $\{V_{jk}\}_{k \in K_j}$ be $\mathcal{I}_j$-covers of each of the $U_j$. By the definition of $\mathcal{I}$, we have for each $i \in I$ that $\{U_j \cap X_i\}_{j \in I}$ is a $\mathcal{I}_i$ covering, and also $\{V_{jk} \cap X_i\}_{k \in K_j}$ is a $\mathcal{I}_i$-covering for every $j \in J$. Then applying axiom (iii) in each $X_i$ we get that $\{V_{jk} \cap X_i\}_{j \in I, k \in K_j}$ to be a $\mathcal{I}_i$-covering for each $i \in I$. Then by the definition of $\mathcal{I}$, $\{V_{jk}\}_{j \in I, k \in K_j}$ is a $\mathcal{I}$-covering.

(iv) Let $U, V$ be $\mathcal{I}$-open with $V \subset U$, and let $\{U_j\}$ be a $\mathcal{I}$-covering of $U$. Then $\{U_j \cap X_i\}_{j \in J}$ is a $\mathcal{I}_i$ covering of $U \cap X_i$ for each $X_i$, and since $V \cap U \cap X_i \subset U \cap X_i$, applying axiom (iv) in $X_i$, we have that $\{V \cap U_j \cap X_i\}_{j \in J}$ is a $\mathcal{I}_i$-covering of $U \cap X_i$ for each $i \in I$. Then by the definition of $\mathcal{I}$, $\{V \cap U_j\}_{j \in I}$ is a $\mathcal{I}$-covering.

**Proposition 3.10.** The $G$-topology $\mathcal{I}$ is slightly finer than $\mathcal{I}^w$ if all $G$-topologies $\mathcal{I}_i$ satisfy condition $(G_2)$.

**Proof.** First we show that the $\mathcal{I}^w$-open sets form a basis for $\mathcal{I}$. Let $U \subset X$ be $\mathcal{I}$-open. We have to find a $\mathcal{I}$-covering of $U$ by $\mathcal{I}^w$ open sets. Since each of the $X_i$ are $\mathcal{I}^w$-open, letting $V = X_i \cap U \subset X_i$, by $G$-topology axiom (iv), we have that $V \cap X_i = U \cap X_i$ is $\mathcal{I}^w$-open for each $X_i$. Then $\{U \cap X_i\}_{i \in I}$ is a covering of $U$ by $\mathcal{I}^w$-open sets. Restricting to an arbitrary $X_i$, let $\mathcal{U}_i$ be the covering $\{U \cap X_i \cap X_j\}_{i \in I}$ of $U \cap X_i$. The trivial covering $\{U \cap X_j\}_{j \in J}$ of $U \cap X_j$ is $\mathcal{I}_j$-admissible; further it is a refinement of $\mathcal{U}_j$ since its only element $U \cap X_j$ is contained in $U \cap X_j \cap X_j \in \{U \cap X_i \cap X_j\}_{i \in I} = \mathcal{U}_j$. Hence by $(G_2)$, $\mathcal{U} + j$ is a $\mathcal{I}_j$-covering. Then by the definition of $\mathcal{I}$, $\{U \cap X_i\}_{i \in I}$ is a $\mathcal{I}$ admissible covering of $U$.

Now suppose $\mathcal{U}$ is a $\mathcal{I}$-covering of a $\mathcal{I}^w$-open subset $U \subset X$. By the definition of $\mathcal{I}$, we have $U \subset X_i$ for some $i \in I$. But inside a single $X_i$, $\mathcal{I}$-admissible coverings and $\mathcal{I}^w$-admissible coverings are the same (since they have to induce the $G$-topologies $\mathcal{I}_i$). Then $\mathcal{U}$ is $\mathcal{I}^w$-admissible, so $\mathcal{U}$ is itself a $\mathcal{I}^w$-covering which refines $\mathcal{U}$. □
Proposition 3.11. Assume that all $G$-topologies $\mathfrak{T}_i$, $i \in I$, satisfy conditions $(G_0)$, $(G_1)$ and $(G_2)$. Then the $G$-topology $\mathfrak{T}$ is uniquely characterised on $X$ by the following properties:

(i) $X_i$ is $\mathfrak{T}$-open in $X$ and $\mathfrak{T}$ induces the $G$-topology $\mathfrak{T}_i$ on each $X_i$.

(ii) $\mathfrak{T}$ satisfies conditions $(G_0)$, $(G_1)$ and $(G_2)$.

(iii) $\{X_i\}_{i \in I}$ is a $\mathfrak{T}$-covering of $X$.

Proof. The $G$-topology $\mathfrak{T}$ satisfies (i) by construction. For (ii), we have:

$(G_0)$ $X \cap X_i = X_i$ and $\emptyset \cap X_i = \emptyset$ are admissible open in each $X_i$ since each $X_i$ satisfies $(G_0)$. Then $X$ and $\emptyset$ are admissible open in $\mathfrak{T}$ and hence in $\mathfrak{T}$.

$(G_1)$ Let $U \subset X$ be $\mathfrak{T}$-open, and let $V \subset U$ be a subset. Suppose we have an admissible covering $\{U_j\}_{j \in J}$ of $U$ such that $V \cap U_j$ is admissible open in $X$ for all $J \in I$. By the definition of $\mathfrak{T}$, this means that $V \cap U_j \cap X_i$ is $\mathfrak{T}_i$-open for each $i$. Since by the definition of $\mathfrak{T}$ we also have that $\{U_j \cap X_i\}_{j \in J}$ is a $T_i$-covering for each $i \in I$, then since each $X_i$ satisfies $(G_2)$, we have that $V \cap X_i$ is $\mathfrak{T}_i$-open for each $i$, and hence $V$ is $\mathfrak{T}$-open.

$(G_2)$ Suppose $\{U_j\}_{j \in J}$ is a covering of an admissible open $U \subset X$ such that $U_i$ is admissible open in $X$ for all $i \in I$, and suppose that $\{U_j\}_{j \in J}$ has an admissible refinement $\{V_k\}_{k \in K}$. Then $\{V_k \cap X_i\}_{k \in K}$ is an admissible refinement of $\{U_j \cap X_i\}_{j \in J}$ in each $X_i$, and therefore, since each $X_i$ satisfies $(G_2)$, $\{U_j \cap X_i\}_{j \in J}$ is an admissible covering in each $X_i$. Thus by the definition of $\mathfrak{T}$, $\{U_j\}_{j \in J}$ is an admissible covering in $X$.

For (iii), we have that $\{X_j\}$ is a refinement of $\{X_j \cap X_i\}_{i \in I}$ for each $j \in I$, which is admissible by $G$-topology axiom (ii). Then since $X_j$ satisfies $(G_2)$, we have that $\{X_j \cap X_i\}_{i \in I}$ is $\mathfrak{T}_i$-admissible. Then by the definition of $\mathfrak{T}$, $\{X_i\}_{i \in I}$ is a $\mathfrak{T}$-admissible covering.

Thus if $\mathfrak{T}'$ is another $G$-topology on $X$ with properties (i), (ii) and (iii), $\mathfrak{T}'$ will be weaker than $\mathfrak{T}$ by the definition of $\mathfrak{T}$. To see that $\mathfrak{T}'$ is in fact equal to $\mathfrak{T}$, let $U \subset X$ be $\mathfrak{T}$-open. Then $U \cap X_i$ is $\mathfrak{T}_i$-open in $X_i$ for all $i \in I$, and hence is also $\mathfrak{T}'$-open since $\mathfrak{T}'$ satisfies (i). Therefore, applying condition $(G_1)$ to the $\mathfrak{T}'$-covering $\{X_i\}_{i \in I}$ and the subset $U \subset X$, we see that $U$ is $\mathfrak{T}'$-open with $\{X_i \cap U\}_{i \in I}$ an admissible covering of $U$. Similarly, if $\{U_j\}_{j \in J}$ is a $\mathfrak{T}$-covering of $U$, its restriction $\{X_i \cap U_j\}_{j \in J}$ to $X_i$ is a $\mathfrak{T}_i$-covering of $X_i \cap U$ for all $i \in I$, and hence is also a $\mathfrak{T}'$-covering by (i). Then by $G$-topology axiom (iii), the covering $\{X_i \cap U_j\}_{i \in I, j \in J}$ is seen to be a $\mathfrak{T}'$-covering of $U$ which is a refinement of the covering $\{U_j\}_{j \in J}$. So $\{U_j\}_{j \in J}$ is a $\mathfrak{T}'$-covering due to condition $(G_2)$. Thus we conclude than $\mathfrak{T}' = \mathfrak{T}$. 

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We are now in a position to define the correct $G$-topology on affinoid variety (although we will need a little sheaf theory, presented in the following section, to see precisely why it is what we need). Let $X$ be an affinoid variety. We define a $G$-topology $\mathcal{T}_X$ on $X$ as follows: call all affinoid subdomains admissible open, and call all emphfinite coverings by affinoid subdomains admissible coverings. This gives a $G$-topology called the weak $G$-topology on $X$. Applying Corollary 3.9, we get a unique finest topology $\mathcal{T}'_X$ on $X$ being slightly finer than $\mathcal{T}_X$. For any affinoid variety $X$, the resulting $G$-topology $\mathcal{T}'_X$ is called the strong $G$-topology on $X$. Although we won’t need it, we note that this definition is motivated by the following fact, which is given without proof.

**Proposition 3.12.** The affinoid subdomains form a basis for the canonical topology on $X$.

Then the ordinary topology generated by the $\mathcal{T}_X$-open sets is the one that we would expect, and so the $G$-topology $\mathcal{T}_X$ can be viewed as adding extra structure on top of the existing topological structure on $X$.

The following fact is obvious from our definitions.

**Proposition 3.13.** The strong $G$-topology on an affinoid variety $X$ satisfies conditions $(G_0)$, $(G_1)$ and $(G_2)$.

### 3.4 Sheaf theory

**Definition 3.14.** A presheaf of rings on a $G$-topological space $X$ is a contravariant functor $\mathcal{F}$ from the category of admissible open subsets of $X$, with inclusions as morphisms, into the category of rings.

Explicitly, a presheaf $\mathcal{F}$ of rings on $X$, maps each admissible opens subsets $U \subset X$ to a ring $\mathcal{F}(U)$, and to each inclusion $U \hookrightarrow V$ a restriction homomorphism $\mathcal{F}(V) \to \mathcal{F}(U)$ which we will write as $f \mapsto f|_U$, so that the following properties are fulfilled: the trivial inclusion $U \subset U$ corresponds to the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and whenever $U \subset V \subset W$ are admissible open subsets of $X$, the restriction homomorphism $\mathcal{F}(W) \to \mathcal{F}(V)$ is the composition of the restriction homomorphisms $\mathcal{F}(W) \to \mathcal{F}(V)$ and $\mathcal{F}(V) \to \mathcal{F}(U)$.

We can recover the ordinary definition of presheaf by calling all open sets and all open coverings admissible. Then, for example, we can get the ordinary structure sheaf on a variety $X = \text{Max}(k[x_1^\pm, \ldots, x_n^\pm]/I)$ by setting $\mathcal{F}(U)$ equal to the ring of affine functions on $U$, i.e.,

$$\mathcal{F}(U) = \{f|_U : f \in k[X] = k[x_1^\pm, \ldots, x_n^\pm]/I\}.$$ 

We can now see why the definition requires that $\mathcal{F}$ changes the direction of morphisms. Suppose we have a subset $V \supset U$, open in $X$, inducing the
inclusion $U \hookrightarrow V$. We might expect to be able to construct a homomorphism $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$, preserving the direction of the arrow. But given a function $f \in \mathcal{F}(U)$, there are likely to be many different ways to extend $f$ to a function on $V$, and so it is not clear that such an homomorphism $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ even exists. On the other hand the restriction $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ given by $f \mapsto f_U$ is entirely natural, and is always defined.

**Definition 3.15.** A presheaf $\mathcal{F}$ of rings on $X$ is called a sheaf if, for all admissible open $U \subset X$ and all admissible coverings $\{U_i\}_{i \in I}$ of $U$, the following conditions are satisfied:

(i) If $f, g \in \mathcal{F}(U)$ are elements such that $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.

(ii) If $\{f_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a family of elements such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists an $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$ for all $i \in I$.

Note that by (i), the element $f$ in (iii) is uniquely determined, since suppose we have another $g \in \mathcal{F}(U)$ such that $g|_{U_i} = f_i$ for each $i \in I$. Since we also have $f|_{U_i} = f_i$, we get $f|_{U_i} = g|_{U_i}$, for each $i \in I$, and so by (i), $f = g$.

We define morphisms in the category sheaves so that they respect the restriction homomorphisms: a homomorphism $\varphi : \mathcal{F} \rightarrow \mathcal{F}'$ of presheaves or sheaves on $X$ is a collection of homomorphisms $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{F}'(U)$ for each admissible open $U \subset X$, such that if $U \subset V$ are admissible open subsets of $X$, then for all $f \in \mathcal{F}(V)$

$$\varphi_U(f|_U) = \varphi_V(f)|_U.$$  

Let $\mathcal{T}$ and $\mathcal{T}'$ be $G$-topologies on a space $X$, with $\mathcal{T}'$ finer than $\mathcal{T}$. When dealing with sheaves on $X$, we will say $\mathcal{T}$-sheaf, when we mean a sheaf with respect to $\mathcal{T}$-open sets and -covers; $\mathcal{T}'$-sheaf when we mean a sheaf with respect to $\mathcal{T}'$-open sets and -covers. Since each $\mathcal{T}'$-open set is also $\mathcal{T}$ open, and each $\mathcal{T}'$-admissible cover is also $\mathcal{T}$-admissible, each $\mathcal{T}'$-sheaf $\mathcal{F}'$ restricts to a $\mathcal{T}$-sheaf $\mathcal{F}$ by setting $\mathcal{F}(U) = \mathcal{F}'(U)$. In this situation, we call $\mathcal{F}'$ a $\mathcal{T}'$-extension of $\mathcal{F}$.

The proof of the following Proposition using sheaf cohomology can be found in Section 9.2 of [2].

**Proposition 3.16.** Let $\mathcal{T}'$ be slightly finer than $\mathcal{T}$. Then any $\mathcal{T}$-sheaf $\mathcal{F}$ extends to a $\mathcal{T}'$-sheaf $\mathcal{F}'$ which is uniquely determined up to isomorphism. Furthermore, any homomorphism of $\mathcal{T}$-sheaves $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ extends uniquely to an homomorphism $\mathcal{F}_1' \rightarrow \mathcal{F}_2'$ of the corresponding $\mathcal{T}'$-extensions.
3.5 Analytic varieties

We would like to define an analytic variety as a global space which has some sort of local affinoid structure defined in an openneighbourhood of each point. We proceed in an analogous way to the classical construction of abstract varieties (see, e.g., [8], Part II). We define a \( G \)-ringed-space over \( k \) to be a pair \((X, \mathcal{O}_X)\), where \( X \) is a \( G \)-topological space, and \( \mathcal{O}_X \) is a sheaf of \( k \)-algebras on \( X \) called the structure sheaf of \( X \). If \( X = \text{Max}(A) \) is an affinoid variety, we have a natural presheaf \( \mathcal{O}_X \) which associates each affinoid subdomain \( V = \text{Max}(B) \subset \text{Max}(A) \) its corresponding affinoid algebra \( B \).

We have the following result due to [10] (see [2], Chapter 8 for a proof using Čech cohomology).

**Theorem 3.17** (Tate’s acyclicity theorem). *The presheaf \( \mathcal{O}_X \) is a sheaf with respect to the weak \( G \)-topology on \( X \).*

Now considering the strong \( G \)-topology on \( X \), which is slightly finer than the weak one, by Proposition 3.16, the sheaf \( \mathcal{O}_X \) can be extended uniquely to a sheaf of \( k \)-algebras \( \mathcal{O}'_X \) with respect to the strong \( G \)-topology on \( X \). From now on we will always associate the strong \( G \)-topology which satisfies \((G_0), (G_1), \text{ and } (G_2)\), and is characterised by Proposition ???. Then \((X, \mathcal{O}'_X)\) is a locally \( G \)-ringed space, and is called the \( G \)-ringed space associated with \( X \). We are now in a position to define rigid analytic varieties.

**Remark.** For an algebraic variety over \( C \), the ordinary structure presheaf is a sheaf in its own right ([8] Theorem V.2.3/1). In particular, this means that there exist analytic functions which can be defined locally over arbitrary open sets. For example over the complex plane, the analytic function \( 1/(1 - z) \) can be defined by a Taylor expansion in any open disc which doesn’t contain the point \( 1 \in C \), and sheaf axiom (ii) says that this corresponds to an analytic function on all of \( C \setminus 1 \). On the other hand, for an affinoid variety, we cannot expect this to happen—an analytic function which can be defined locally on a \( G \)-open subset \( U \) is only guaranteed by sheaf axiom (ii) to correspond to an affinoid function on all of \( U \) if \( U \) can be covered by a finite number of affinoidsubdomains . In particular, a locally-defined affinoid function on \( X \) is only guaranteed to define an affinoid function over all of \( X \) if \( X \) has an admissible covering \( \{X_i\} \) by affinoid subdomains.

**Definition 3.18.** A rigid analytic variety over \( k \) (also called a \( k \)-analytic variety) is a locally \( G \)-ringed space \((X, \mathcal{O}_X)\) over \( k \) (also denoted by \( X \)) such that the followings conditions are satisfied:

1. The \( G \)-topology of \( X \) fulfils conditions \((G_0), (G_1) \text{ and } (G_2)\).
2. \( X \) admits an admissible covering \( \{X_i\}_{i \in I} \) with \((X_i, \mathcal{O}_X|_{X_i})\) being a \( k \)-affinoid variety for all \( i \in I \).
We will now present a general recipe for gluing together analytic varieties. The uniqueness of the resulting analytic variety strongly relies on the conditions \((G_0), (G_1)\) and \((G_2)\) for \(G\)-topologies.

**Proposition 3.19.** Consider the following data:

(i) analytic varieties \(X_i\), where \(i\) runs through some index set \(I\),

(ii) open subvarieties \(X_{ij} \subset X_i\) (possibly equal to the empty set) and isomorphisms \(\varphi_{ij} : X_{ij} \to X_{ji}\) for all \(i, j \in I\).

Assume that the following conditions are satisfied:

(a) \(\varphi_{ij} \circ \varphi_{ji} = \text{id}, X_{ii} = X_i, \varphi_{ii} = \text{id}\) for all \(i, j \in I\).

(b) The map \(\varphi_{ij}\) induces isomorphisms \(\varphi_{ijl} X_{ij} \cap X_{il} \to X_{jl} \cap X_{il}\) such that \(\varphi_{ijl} = \varphi_{iji} \circ \varphi_{jil}\), for all \(i, j, l \in I\).

Then it is possible to construct an analytic variety \(X\) by pasting together the varieties \(X_i\), with \(X_{ij}\) and \(X_{ji}\) being identified via \(\varphi_{ij}\) (these will play the part of the intersection “\(X_i \cap X_j\)”) such that \(\{X_i\}_{i \in I}\) is an admissible covering of \(X\).

**Proof.** Let \(\hat{X}\) be the disjoint union of the \(X_i\), viewed as sets. We define a relation \(\sim\) on \(\hat{X}\) by calling \(x \in X_i, y \in X_j\) equivalent if and only if \(x \in X_{ij}, y \in X_{ji}\) and \(\varphi_{ij}(x) = y\). Then \(\sim\) is reflexive and symmetric due to condition (a). For transitivity, let \(x \in X_i, y \in X_l, z \in X_j\) such that \(x \sim y\) and \(y \sim z\). Then we have \(x \in X_{il}, y \in X_{li}\) and \(y = \varphi_{il}(x)\) as well as \(y \in X_{ij}, z \in X_{jl}\) and \(z = \varphi_{ij}(y)\). Hence \(y\) belongs to the intersection \(X_{il} \cap X_{ij}\), which implies \(x \in X_{il} \cap X_{ij}\) by condition (b). We get

\[
  z = \varphi_{ij} (\varphi_{il}(x)) = \varphi_{ij} (\varphi_{il}(x)) = \varphi_{ijl}(x) = \varphi_{ij}(x),
\]

and therefore \(x \sim z\). Thus \(\sim\) is an equivalence relation.

Now define \(X = \hat{X}/\sim\). Then the canonical maps \(X_i \to X\) are injective and, for all \(i \in I\), we identify \(X_i\) with its image in \(X\). We would like to apply the pasting procedure of Section 3.3, to obtain a \(G\)-topology on \(X\). For pasting condition (a), we need that each of the intersections \(X_i \cap X_j = X_{ij} = X_{ji}\) is open in \(X_i\) and in \(X_j\); this is immediate from the assumption (i). For pasting condition (b), since for any \(i, j \in I\), the map \(\varphi_{ij}\) is an isomorphism with respect to the \(G\)-topologies on \(X_{ij}\) and \(X_{ji}\), so in particular it is a homeomorphism with respect to the coverings \(G\)-topologies. Since these are, by definition, the \(G\)-topologies induced by \(X_i\) and \(X_j\), we have that pasting condition (b) is satisfied. Then applying Proposition 3.11 with respect to the covering \(\{X_i\}\) of \(X\), there exists a \(G\)-topology \(\mathcal{T}\) on \(X\) which satisfies \((G_0), (G_1), (G_2)\), admits the \(X_i\) as admissible open subsets, induces on \(X_i\) the given \(G\)-topology, and allows \(\{X_i\}\) as an admissible covering of \(X\)
In order to define the structure sheaf on $X$, we also consider the weak topology $\mathcal{T}^w$ on $X$. Recall that a subset $U \subset X$ is $\mathcal{T}^w$-open if and only if it is admissible open in $X_i$, for some $i \in I$. For a $\mathcal{T}^w$-open subset $U \subset X$, we denote by $I_U$ the set of all indices $i \in I$ such that $U$ is contained in $X_i$. Then $U$ is admissible open in $X_i$ for all $i \in I_U$, and $I_U$ is not empty. Further, if $\mathcal{O}_{X_i}$ denotes the structure sheaf of $X_i$, for each $i, j \in I_U$, the isomorphisms $\varphi_{ij} : X_{ij} \to X_{ji}$ give rise to isomorphisms

$$(\varphi_{ij}^*)_U : \mathcal{O}_{X_j}(U) \to \mathcal{O}_{X_i}(U).$$

Then similarly to the identifications above, for each $\mathcal{T}^w$-open set $U \subset X$, we identify all rings $\mathcal{O}_{X_i}(U)$ via the maps $(\varphi_{ij}^*)_U$, $i, j \in I_U$. Since each $X_i \cap X_j$ is admissible open, then in particular, we can choose $U = X_i \cap X_j$, and the the isomorphism

$$\mathcal{O}_{X_j}(X_i \cap X_j) \cong \mathcal{O}_{X_i}(X_i \cap X_j)$$

given by $(\varphi_{ij}^*)$ becomes an identity. This means that we can define a $\mathcal{T}^w$-sheaf of rings $\mathcal{O}^w$ on $X$, by

$$\mathcal{O}^w(U) = \mathcal{O}_{X_i}(U)$$

for any $X_i \supset U$, which is well-defined since for any other $X_j \supset U$, under the identification, we now have $\mathcal{O}_{X_j}(U) = \mathcal{O}_{X_i}(U)$. By Proposition 3.10, the $G$-topology $\mathcal{T}$ is slightly finer than $\mathcal{T}^w$, and therefore we can apply Proposition 3.16. So $\mathcal{O}^w$ extends to a $\mathcal{T}$-sheaf $\mathcal{O}_X$, which is essentially unique. Then $X$ together with $\mathcal{O}_X$ is the required analytic variety. 

We now come to our main objects of study.

**Example 3.20.** The affine $n$-space $\mathbb{A}^n$—Let $x = (x_1, \ldots, x_n)$ be a system of indeterminants, and choose a constant $c \in k$, with $|c| > 1$. Recall that by Example 2.23, we have an affinoid variety $\text{Max}(T_n, \rho)$ corresponding to the closed polydisc of radius $\rho$, for any $\rho \geq 1$. We will define an affinoid structure on $\mathbb{K}^n$ by taking a set of affinoid varieties of this type as a covering and then pasting.

For each integer $i \geq 0$, we denote by $A_i$ the $k$-algebra $T_{n, |c|^i}$ of all formal power series converging on the ball of radius $|c|^i$, centered at 0, in $\mathbb{K}^n$. Then the $A_i$ occur in the decreasing sequence

$$(*) \quad k(x) = A_0 \supset A_1 \supset A_2 \supset \cdots \supset k[x].$$

In the language of Example 3.2, we have $\text{Max}(A_i)$ is isomorphic to the affinoid subdomain $\text{Max}(A_{i+1})\langle c^{-i}x \rangle$, in $\text{Max}(A_{i+1})$, since $\text{Max}(A_{i+1})\langle c^{-i}x \rangle$ consists precisely of those points $x \in \text{Max}(A_{i+1})$ such that $|c^{-i}x| \leq 1$, i.e., $|x| \leq |c|^i$. Therefore the inclusions $A_{i+1} \hookrightarrow A_i$ induce the sequence of affinoid subdomains

$$\mathbb{K}^n(k) = \text{Max}(A_0) \hookrightarrow \text{Max}(A_1) \hookrightarrow \cdots.$$
Then setting \( X_i = \text{Max}(A_i), \) \( X_{ij} = X_i \cap X_j = X_{\min i,j}, \) id = \( \varphi_{ij} : X_{ij} \to X_{ji}, \)
for all \( i, j, \) we can apply 3.19 to construct an analytic variety admitting \( \{X_i\} \)
as a covering, which we call \( \mathbb{A}^n, \) the analytic affine \( n \)-space.

Let us check that this analytic variety makes sense as an affinoid version of \( \mathbb{A}^n = \text{Max}(\mathbb{k}[x_1, \ldots, x_n]). \) By (\textdagger), we have that the polynomial ring \( \mathbb{k}[x] \) is contained in each of the \( A_i, \) so by sheaf axiom (ii) we have that \( \mathbb{k}[x] \subset \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n). \) In particular each of the functions \( x_1, \ldots, x_n \) lie in \( \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n), \) and we call these a set of coordinates for \( \mathbb{A}^n. \)
The functions \( x_{1|A_i}, \ldots, x_{n|A_i} \in \mathcal{O}_{\mathbb{A}^n}(A_i) \) have a unique zero in \( A_i, \) for each \( i; \) since
\[
x_1|_{A_0}, \ldots, x_n|_{A_0} = x_1|_{A_i}, \ldots, x_n|_{A_i},
\]
we have that all these zeroes are in fact the same point. Hence we have a unique common zero for the \( x_1, \ldots, x_n, \) which we call the origin of \( \mathbb{A}^n. \)

Now by an analogous argument to the one at the start of Section 2.4, we have that each maximal ideal \( x \in \text{Max}(\mathbb{k}[x]) \) occurs as the kernel of a \( \mathbb{k}\)-homomorphism \( \sigma : \mathbb{k}[x] \to \mathbb{k}. \) Let \( i \geq 0 \) be such that \( |c|^i \geq \sigma(x_i) \) for each coordinate \( x_i. \) Then for \( f \in A_i, \) we can extend \( \sigma \) to a continuous homomorphism \( \sigma' : A_i \to \mathbb{k} \)
by
\[
\sum a_kx^k \mapsto \sum a_k\sigma(x)^k,
\]
which is in \( \mathbb{k} \) since \( \sigma(x_i) \in \mathbb{B}_{|c|^i} \) for each \( x_i, \) and by definition, \( f \in A_i \)
converges on \( \mathbb{B}_{|c|^i}. \) Hence \( x \in \text{Max}(A_i), \) and so \( \text{Max}(\mathbb{k}[x]) \subset \bigcup_i \text{Max}(A_i). \)
Since any \( \sigma : A_i \to \mathbb{k} \) restricts to a homomorphism \( \sigma|_{\mathbb{k}[x]} : \mathbb{k}[x] \to \mathbb{k}, \) we have the reverse inclusion \( \text{Max}(\mathbb{k}[x]) \supset \bigcup_i \text{Max}(A_i), \) and so we are justified in identifying \( \mathbb{A}^n \) with the affine algebraic \( n \)-space.

The open subvariety \( \text{Max}(A_i) \subset \mathbb{A}^n \) corresponds to the ball of radius \( |c|^i \)
centered at the origin of \( \mathbb{k}^n. \) Of course this has no counterpart in algebraic geometry. This is unavoidable since, in general, affinoid varieties will contain many subdomains (such the Weierstrass domains \( \text{Max}(A_i)(f) \)) with no counterpart in algebraic geometry.

\textbf{Example 3.21.} Algebraic varieties—We can generalise the above procedure to view an arbitrary algebraic variety as an analytic space. Let \( B = \mathbb{k}[x_1, \ldots, x_n]/I \) with \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) an arbitrary ideal. Using the same notation as in the Example above, we get a sequence of \( \mathbb{k}\)-algebra homomorphisms
\[
A_0/IA_0 \leftarrow A_1/IA_1 \leftarrow \cdots \leftarrow B,
\]
which corresponds to an increasing sequence
\[
\text{Max}(A_0/IA_0) \to \text{Max}(A_1/IA_1) \to \cdots
\]
and just as above, one constructs an analytic variety admitting
\[
\{\text{Max}(A_i/IA_i)\}_{i \geq 0}
\]
as covering. We call this variety \( X^{\text{an}}, \) the analytic variety associated with \( X. \)
3.6 Connectedness of $T(X)$

We will need the following generalisation of 2.26 for arbitrary bounded $v(\mathbb{k})$-rational polyhedron. As we noted in the introduction to Chapter 1, the tropical variety $T(X)$ is a finite union of convex polyhedra.

**Definition 3.22.** Let $\Delta \subset \mathbb{R}^n$ be a convex bounded $v(\mathbb{k})$-rational polyhedron. Then $k\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ consists of formal Laurent series $f(x) = \sum_{i \in \mathbb{Z}^n} a_i x^i$ satisfying the condition

$$\lim_{\|i\| \to \infty} (v(a_i) + i \cdot x) = \infty$$

for all $x \in \Delta$.

Note that in this notation, the algebra $A_0$ in Section 2 corresponds to $k\langle\langle \text{val}^{-1}(0) \rangle\rangle$, since in this case the condition $\lim_{\|i\| \to \infty} (v(a_i) + i \cdot x) = \infty$ reduces to $\lim_{\|i\| \to \infty} v(a_i) = \infty$; this is equivalent to $\lim_{\|i\| \to \infty} |a_i| = 0$, since $|a_n| = e^{-v(a_n)} \to 0$ if $v(a_n) \to \infty$.

We can see directly that $k\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ is a ring as follows. Let $f(x), g(x) \in k\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$, with $f(x) = \sum_{i \in \mathbb{Z}^n} a_i x^i, g(x) = \sum_{i \in \mathbb{Z}^i} b_i x^i$. Then the coefficient $c_i$ of the $x^i = x^{i_1} \cdots x^{i_n}$ term in the expansion of $f + g$ is given by $a_i - b_i$, so for $x \in \Delta$, we have

$$v(c_i) + i \cdot x \geq \min(a_i, b_i) + i \cdot x \to \infty$$
as $\|i\| \to \infty$, and hence $f + g \in k\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$. The coefficient $c_i$ of the $x^i = x^{i_1} \cdots x^{i_n}$ term in the expansion $fg$ is given by $\sum a_{j_1 \cdots j_n} \cdot b_{k_1 \cdots k_n}$ where the sum is take over all $(j_1 \cdots j_n), (k_1 \cdots k_n) \in \mathbb{Z}^n$ such that each $j_m + k_m = i_m$. In particular, each term in the sum has either $(j_1 \cdots j_n)$ or $(k_1 \cdots k_n)$ greater than or equal to $i/2 = (i_1/2, \ldots, i_n/2)$, and so we have

$$v(c_i) + i \cdot x \geq v(a_i/2 + b_i/2) + i \cdot x = v(a_i/2) + v(b_i/2) + i \cdot x \to \infty$$
as $\|i\| \to \infty$, and hence $fg \in k\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$.

We define a norm on $k\langle\langle \text{val}^{-1}(\Delta) \rangle\rangle$ via the valuation

$$v(f) = \inf\{v(a_i) + i \cdot x : i \in \mathbb{Z}^n, x \in \Delta\}.$$
multiples of the inequalities if necessary. Let \( b = (b_1, \ldots, b_d) \in \mathbb{Z}^n \) and choose \( w_i \in k \) with \( c_i = v(w_i) \). We rewrite the condition \( \text{val}(x) \in \Delta \) in terms of the norm \( | \cdot | = -e^{\nu(x)} \) as follows: If \( \text{val}(x) \in \Delta \), we must have \( v(x_1)b_1 + \cdots + v(x_1)b_i \geq c_i \) for each \( i \). Taking exponents and multiplying by \(-1\) yields

\[
-e^{v(x_1)b_1} + \cdots + e^{v(x_1)b_i} = |x_1^{b_1}| \cdots |x_i^{b_i}| = |x_i^{b_i}| \leq |w_i|
\]

Hence we have \( |x_i^{b_i}/w_i| \leq 1 \) for each \( i \). Thus we have a continuous homomorphism

\[
\varphi : T_r = \mathbb{k}\langle \langle y_1, \ldots, y_r \rangle \rangle \to \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle
\]

defined by \( y_i \mapsto x_i^{b_1} \cdots x_i^{b_n}/w_i \). By 2.17 it is enough to prove that \( \varphi \) is finite. Let \( S \subset \mathbb{Z}^n \) be the semigroup with 0 generated by the \( b_i \), and \( C \) its convex hull. Denote by \( \mathbb{k}[S] \) the set of all formal Laurent series \( \sum_{i \in S} a_i x^i \), and by \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle \) the intersection \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle \cap \mathbb{k}[S] \). Then it is enough to prove the following two statements.

(a) The image of \( \varphi \) is \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle S \).

(b) The extension \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle \supset \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle S \) is finite.

For (a), let \( f(x) = \sum_{i \in S} a_i x^i \) lie in \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle S \). For any \( i \in S \), let

\[
i = m_{i_1} b_1 + \cdots + m_{i_r} b_r
\]

with \( m_i = (m_{i_1}, \ldots, m_{i_r}) \in \mathbb{Z}^r_+ \). Then the series \( g(y) = \sum_{i \in S} a_i y^{m_i} \) lies in \( T_r \), and \( \varphi(g) = f \).

For (b), Let \( \mathbb{k}[S] \) and \( \mathbb{k}[C \cap \mathbb{Z}]^n \) be the semigroup algebras of \( S \) and \( C \cap \mathbb{Z}^n \). It is a standard result in discrete geometry that \( \mathbb{k}[C \cap \mathbb{Z}]^n \) is finite over \( \mathbb{k}[S] \) (see, e.g., [3], Chapter 2 for a proof).

Also we have that \( C \cap \mathbb{Z}^n = \mathbb{Z}^n \). This is because supposing not, we must have that all the \( b_i \) are contained in some half space \( k \cdot x \geq 0 \), and so \( b_i \cdot k \geq 0 \). Let \( a \) be an interior point of \( \Delta \). Let \( l \) be a vector such that \( k \cdot l \leq 0 \), for example we could take \( l = -k \). Then since \( \Delta \) is bounded and non-degenerate, the line through \( a \) in the direction \( l \) intersects the boundary of \( \Delta \), and by perturbing \( l \) slightly if necessary, we may assume that the line intersects a face. Let \( b_i \) correspond to this face. Then we have \( l \cdot b_i \geq 0 \), and since also \( k \cdot b_i \geq 0 \), we must have \( k \cdot l \geq 0 \), a contradiction.

Thus we see that a system of module generators of \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle \) over \( \mathbb{k}[S] \) will be a system of module generators of \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle \) over \( \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle S \). This completes the proof of the proposition.

**Proposition 3.24.** Let \( A = \mathbb{k}\langle \langle \text{val}^{-1}(\Delta) \rangle \rangle \) for a bounded \( v(\mathbb{k}) \)-rational polyhedron \( \Delta \), so that \( \text{Max}(A) = \text{val}^{-1}(\Delta)/\Gamma \). For any \( v(\mathbb{k}) \)-rational subpolyhedron \( \Sigma \subset \Delta \), the subset \( \text{val}^{-1}(\Delta)/\Gamma \) is admissible open.
We will rely on the following result of Conrad.

**Theorem 3.25** (Thm. 2.3.1 of [4]). Let $X$ be an irreducible algebraic variety over $\mathbb{k}$. Then the analytic space $X^{an}$ is connected.

We now assume $X$ is a closed subvariety in $\mathbb{k}^n$, possibly reducible, and let $T = T(X) \subset \mathbb{R}^n$ be its tropical variety. Suppose $T$ is disconnected. That is, suppose $T = T_1 \cup T_2$ with $T_1 \cap T_2 = \emptyset$, and $T_1$ and $T_2$ are both open and closed in $T$. Define subsets $X_1, X_2 \subset X^{an} = X(\mathbb{k})/\Gamma$ by

$$X_1 = (X(\mathbb{k}) \cap \text{val}^{-1}(T_1))/\Gamma$$

$$X_2 = (X(\mathbb{k}) \cap \text{val}^{-1}(T_2))/\Gamma.$$  

Since the tropical variety is defined as the closure of $\text{val}(X(\mathbb{k}))$ in $\mathbb{R}^n$, we have that $X^{an}$ is the disjoint union of $X_1$ and $X_2$, since given any $x \in X^{an}$, $\text{val}(x) \subset T$ is contained exactly one of $T_1$ and $T_2$, and hence $x$ is contained in exactly one of $X_1$ and $X_2$. If we can show that this implies that $X^{an}$ is disconnected in the $G$-topological sense, then by Theorem 3.25, we must have that $X$ is reducible. Therefore, it is enough to prove the following.

**Proposition 3.26.** (i) $X_1$ and $X_2$ are open in $X^{an}$, and (ii) The covering of $X^{an}$ by $X_1$ and $X_2$ is admissible.

**Proof.** Let $\mathbb{R}^n = \bigcup_{i \in I} \Delta_i$ be a decomposition of $\mathbb{R}^n$ into parallel cubes of side-length $\epsilon$. Then $\{\text{val}^{-1}(\Delta_i)/\Gamma : i \in I\}$ is an admissible covering of $(\mathbb{k}^n)/\Gamma$ since $\text{val}^{-1}(\Delta_i) = \text{Max}(\mathbb{k}(\text{val}^{-1}(\Delta_i)))$ is an affinoid variety, by Proposition 3.23, and since each of the $A_i$ in the definition of $A_i^{an}$ is covered by a finite number of $\Delta_i$. Hence by axiom (iv) for $G$-topologies, the sets

$$D_i = (X(\mathbb{k}) \cap \text{val}^{-1}(\Delta_i))/\Gamma$$  

form an admissible covering of $X^{an}$. Let $J_1 = \{i \in I : \Delta_i \cap T_1 \neq \emptyset\}$, $J_2 = \{i \in I : \Delta_i \cap T_2 \neq \emptyset\}$, and $J = J_1 \cup J_2$. Since $T_1$ and $T_2$ are finite unions of polyhedra, $T_1 \cap T_2 = \emptyset$ implies the distance $d(T_1, T_2)$ between $T_1$ and $T_2$ must be equal to some $\delta > 0$: Suppose we have $d(T_1, T_2) = 0$. Then let $(t_i)_{i=1}^{\infty} \in T_1$, $(s_i)_{i=1}^{\infty} \in T_2$ with $|t_i - s_i| \to 0$ as $i \to \infty$. We can take subsequences $(t'_i)_{i=1}^{\infty}, (s'_i)_{i=1}^{\infty}$ such that all the $t'_i$ lie on a single polyhedral component $P$ of $T_1$, and all the $s'_i$ lie on a single polyhedral component $P'$ of $T_2$. Then $|t'_i - s'_i| \to 0$ implies that $P$ and $P'$ will in fact meet, contradicting $T_1 \cap T_2 = \emptyset$. Therefore by taking $\epsilon < \delta$ small enough, we can assume $J_1 \cap J_2 = \emptyset$.

Since $D_i = \emptyset$ for $i \notin J$, we have that the subcover $\{D_i\}_{i \in J} \subset \{D_i\}_{i \in I}$ is in fact an admissible subcover of $X^{an}$. We want to use $(G_1)$ to show that $X_1$ and $X_2$ are admissible open in $X$. Let the analytic space $Z$ in the statement of $(G_1)$ be equal to $X^{an}$, and take $V \subset U$ equal to $X_1 \subset X^{an}$. Then since we have the admissible covering $\{D_i\}_{i \in J}$ of $X^{an}$, and since $D_i \cap X^{an}$ is either
equal to \( D_i \subset X_1 \) if \( i \in J_1 \)—and so is trivially admissible open in \( X_1 \); or is equal to \( \emptyset \) if \( i \in J_2 \)—which is admissible open in \( X_1 \) by \((G_0)\). Then we can conclude that \( X_1 \) is admissible open in \( X^{\text{an}} \). A similar argument shows that \( X_2 \) is admissible open in \( X^{\text{an}} \). This completes part (i) of the proposition.

For part (ii), the covering \( X^{\text{an}} = X_1 \cup X_2 \) has a refinement

\[
X = \left( \bigcup_{i \in J_1} D_i \right) \cup \left( \bigcup_{i \in J_2} D_i \right) = \bigcup_{i \in J} D_i
\]

which is admissible, so by \((G_2)\) we have that \( \{X_1, X_2\} \) is an admissible covering. \(\square\)
Bibliography


