Differentiability of the $g$-Drazin inverse

by

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Abstract. If $A(z)$ is a function of a real or complex variable with values in the space $B(X)$ of all bounded linear operators on a Banach space $X$ with each $A(z)$ $g$-Drazin invertible, we study conditions under which the $g$-Drazin inverse $A^D(z)$ is differentiable. From our results we recover a theorem due to Campbell on the differentiability of the Drazin inverse of a matrix valued function and a result on differentiation of the Moore–Penrose inverse in Hilbert spaces.

1 Introduction and preliminaries

The Drazin inverse defined originally for semigroups in [4] in 1958 is an important theoretical and practical tool in algebra and analysis. When $A$ is an algebra and $a \in A$, then $b \in A$ is the Drazin inverse of $a$ if

$$ab = ba, \quad bab = b, \quad aba = a + u \quad \text{where } u \text{ is nilpotent.}$$

It was observed by Harte [7, 8] and by first author in [11] that in Banach algebras it is more natural to replace the nilpotent element $u$ in (1.1) by a quasinilpotent element. If $u$ in (1.1) is allowed to be quasinilpotent, we call $b$ the $g$-Drazin inverse of $a$.

The $g$-Drazin inverse introduced in [11] is a useful construct that finds its applications in a number of areas. In the present paper we concentrate on the $g$-Drazin inverse in the Banach algebra $B(X)$ of bounded linear operators, and continue the investigation of the continuity of the $g$-Drazin

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inverse [14] by studying its differentiability. For matrices, this was studied by Campbell [1] and Hartwig and Shoaf [9]. Drazin [5] considered differentiation of the conventional Drazin inverse in associative rings, using a general derivation in the ring.

We can briefly describe the contents of this paper as follows: If $A(z)$ is a function of a real or complex variable with values in the space of all bounded linear operators on a Banach space with each $A(z)$ $g$-Drazin invertible, we study the conditions under which the $g$-Drazin inverse $A^D(z)$ is differentiable. From our results we recover a theorem due to Campbell on the differentiability of the Drazin inverse of a matrix valued function and a result on differentiation of the Moore–Penrose inverse in Hilbert spaces.

By $B(X)$ we denote the Banach algebra of all bounded linear operators acting on the complex Banach space $X$ with the usual operator norm. By $\rho(T)$, $\sigma(T)$ and $r(T)$ we denote the resolvent set, the spectrum and the spectral radius of $T \in B(X)$, respectively. We also write $\sigma_0(T)$ for $\sigma(T) \setminus \{0\}$. The sets of all isolated and accumulation spectral points of $T$ are denoted by $\text{iso}\sigma(T)$ and $\text{acc}\sigma(T)$. If $\lambda \in \rho(T)$, then $R(\lambda; T) = (\lambda I - T)^{-1}$ is the resolvent of $T$. We recall [12] that $0 \in \text{iso}\sigma(T)$ if and only if there exists a nonzero projection $P \in B(X)$ such that

$$AP = PA$$

is quasinilpotent and $A + P$ is invertible;

$P$ is the spectral projection of $T$ at 0, and is denoted by $A^\pi$ [12, Theorem 1.2].

**Definition 1.1.** (Koliha [11, Definition 2.3]) An operator $A \in B(X)$ is $g$-Drazin invertible if there exists $B \in B(X)$ such that

$$AB = BA, \quad BAB = B, \quad ABA = A + U,$$

where $r(U) = 0$. (1.2)

The operator $B$ is called the $g$-Drazin inverse of $A$, denoted by $A^D$. The Drazin index $i(A)$ of $A$ is 0 if $A$ is invertible, $k$ if $A$ is not invertible and $U$ is nilpotent of index $k$, and $\infty$ otherwise. Definition 1.1 with $i(A)$ finite coincides with the definition of the conventional Drazin inverse (see [3, 4, 10]). An operator $A$ has a conventional Drazin inverse if and only if 0 is at most a pole of the resolvent of $A$; $A$ has the $g$-Drazin inverse if and only if $0 \notin \text{acc}\sigma(A)$ ([11, Theorem 4.2], [12, Theorem 1.2]).
We need a representation of $A^D$ in terms of the holomorphic calculus for $A$. A cycle is a formal linear combination $\Gamma$ of loops with integral coefficients; $\Gamma$ is a Cauchy cycle relative to the pair $(\Omega, K)$, where $K$ is a compact subset of a nonempty open set $\Omega \subset \mathbb{C}$, if $\Gamma \subset \Omega \setminus K$, ind $(\Gamma, \lambda) = 0$ for all $\lambda \notin \Omega$ and ind $(\Gamma, \mu) = 1$ for all $\mu \in K$. The existence of a Cauchy cycle relative to any such pair $(\Omega, K)$ is proved in [16, Theorem 13.5]. By [11, Theorem 4.4],
\[
A^D = \frac{1}{2\pi i} \int_\Gamma \lambda^{-1} R(\lambda; A) \, d\lambda,
\]
where $\Gamma$ is a Cauchy cycle relative to $(\mathbb{C}\setminus\{0\}, \sigma_0(A))$. (In the case that $A$ is quasinilpotent, the formula is interpreted in the following way: As $\sigma_0(A) = \emptyset$, $\Gamma$ can be any cycle in $\mathbb{C}\setminus\{0\}$ with ind $(\Gamma, 0) = 0$. The integral in (1.3) is zero, which agrees with $A^D = 0$.)

In the sequel we use the following perturbation result involving operator resolvents which follows from [6, Lemma VII.6.3].

**Lemma 1.2.** Let $A, A(z) \in B(X)$ for all $z$ in some neighborhood $U$ of $z_0$, and let $\|A(z) - A\| \to 0$ as $z \to z_0$. If $K$ is a compact subset of the complex plane contained in the resolvent sets of $A$ and $A(z)$ for all $z \in U$, then
\[
\lim_{z \to z_0} R(\lambda; A(z)) = R(\lambda; A) \text{ uniformly for } \lambda \in K.
\]

## 2 Differentiability properties of the $g$-Drazin inverse

In this section, $U$ denotes an open interval in $\mathbb{R}$ or an open subset of $\mathbb{C}$, $z_0$ a point in $U$, and $A : U \to B(X)$ an operator valued function. By $A'(z)$ we denote the derivative of $A(z)$ at $z$, and by $A^D(z)$ the $g$-Drazin inverse $A(z)^D$. Our main result on the differentiability of the $g$-Drazin inverse is given in the following theorem:

**Theorem 2.1.** Let $A$ be a $B(X)$-valued function defined on $U$ such that $A(z)$ is $g$-Drazin invertible for all $z \in U$, and differentiable at $z_0 \in U$. Then
\[ A^D(z) \text{ is differentiable at } z_0 \text{ if and only if } A^D(z) \text{ is continuous at } z_0. \] In this case the derivative \( (A^D)'(z_0) \) is given by

\[ (A^D)'(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} R(\lambda; A(z_0)) A'(z_0) R(\lambda; A(z_0)) d\lambda, \]

where \( \Gamma \) is a Cauchy cycle relative to \((\mathbb{C}\setminus\{0\}, \sigma_0(A(z_0)))\).

**Proof.** Assume that \( A^D(z) \) is continuous at \( z_0 \). From [14, Theorem 4.1] (see Equation (2.5) below) it follows that there exists \( r > 0 \) such that

\[ 0 < |\lambda| < r \implies \lambda \in \rho(A(z)) \text{ for all } z \in U. \]

Let \( \Omega = \{ \lambda : |\lambda| > r \} \), and let \( \Omega_1 \) be a bounded open set with \( \sigma_0(A(z_0)) \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega \). From the upper semicontinuity of the spectrum it follows that there exists \( \delta > 0 \) such that the sets \( \sigma_0(A(z)) \) are contained in \( \Omega_1 \) whenever \( |z - z_0| < \delta \). (The cases \( \sigma_0(A(z)) = \emptyset \) or \( \sigma_0(A(z_0)) = \emptyset \) are not excluded.) There exists a Cauchy cycle \( \Gamma \) relative to \((\Omega, \overline{\Omega}_1)\), and

\[ A^D(z) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} R(\lambda; A(z)) d\lambda, \quad |z - z_0| < \delta, \]

by (1.3). Consider the existence of the limit

\[ \lim_{z \to z_0} \frac{A^D(z) - A^D(z_0)}{z - z_0}. \]

Using the second resolvent equation, we get

\[
\frac{A^D(z) - A^D(z_0)}{z - z_0} = \frac{1}{z - z_0} \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} [R(\lambda; A(z)) - R(\lambda; A(z_0))] d\lambda \\
= \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} R(\lambda; A(z)) \frac{A(z) - A(z_0)}{z - z_0} R(\lambda; A(z_0)) d\lambda.
\]

In view of Lemma 1.2,

\[ \lim_{z \to z_0} R(\lambda; A(z)) \frac{A(z) - A(z_0)}{z - z_0} = R(\lambda; A(z_0)) A'(z_0) \]

uniformly for \( \lambda \in \Gamma \). Hence (2.1) follows.

The converse is clear. \( \square \)
Differentiability of the g-Drazin inverse

We note that Theorem 4.1 of [14] holds when sequences are replaced by functions of \( z \); that theorem gives twelve conditions equivalent to the continuity of \( A^D(z) \) at \( z_0 \). For the sake of completeness we restate four of these conditions relevant to the present investigation. Under the hypotheses of Theorem 2.1, \( A^D(z) \to A^D(z_0) \) as \( z \to z_0 \) if and only if any of the following conditions holds:

(2.4) \[ \sup \{ \|A^D(z)\| : |z - z_0| < \delta \} < \infty \text{ for some } \delta > 0, \]

(2.5) \[ \sup \{ r(A^D(z)) : |z - z_0| < \delta \} < \infty \text{ for some } \delta > 0, \]

(2.6) \[ A^D(z)A(z) \to A^D(z_0)A(z_0) \text{ as } z \to z_0, \]

(2.7) \[ A^\pi(z) \to A^\pi(z_0) \text{ as } z \to z_0. \]

We take this opportunity to correct a mistake in [14, Theorem 4.1]: Conditions (4.14) and (4.15) of that theorem should be

\[ C_n \to C \text{ and } \gamma(C_n) \to \gamma(C) \text{ and } \]
\[ C_n \to C \text{ and } \inf_n \gamma(C_n) > 0, \]

respectively, where \( \gamma(A) \) denotes the reduced minimum modulus of an operator \( A \in B(X) \).

Note 2.2. The preceding argument works with appropriate interpretation in the case that \( r(A(t_0)) = 0 \).

Note 2.3. Hartwig and Shoaf [9, (3.10)] used holomorphic calculus to give a formula for the derivative of the Drazin inverse of a complex matrix in terms of the spectral components of \( A(t) \).

In the case that the operators \( A(t) \) have the conventional Drazin inverse and the indices of \( A(t) \) are uniformly bounded, we are able to obtain a stronger result.

Theorem 2.4. Let \( A \) be a \( B(X) \)-valued function defined on \( U \) such that \( A(z) \) is g-Drazin invertible for all \( z \in U \) and differentiable at \( z_0 \in U \). If the indices \( i(A(z)) \) are uniformly bounded and the spectral projections \( A^\pi(z) \) are of finite rank, then \( A^D(z) \) is differentiable at \( z_0 \) if and only if there exists \( \delta > 0 \) such that

\[ \text{rank } A^\pi(z) = \text{rank } A^\pi(z_0) \text{ whenever } |z - z_0| < \delta. \]
Proof. Follows from Theorem 2.1 and [14, Theorem 5.1].

From the preceding theorem we recover the main result of [1] on the differentiability of the matrix Drazin inverse. The core part \( C(z) \) of \( A(z) \) is defined by \( C(z) = A(z)(I - A^\pi(z)) \); the core rank of \( A(z) \) is the rank of \( C(z) \).

**Corollary 2.5.** (Campbell [1, Theorem 4]) Let \( A \) be a \( p \times p \) matrix valued function defined on \( U \) and differentiable at \( z_0 \in U \). Then \( A^D(z) \) is differentiable at \( z_0 \) if and only if the core rank of \( A(z) \) is constant in some neighborhood of \( z_0 \).

Proof. Follows from Theorem 2.4 and the result for the core rank of \( A(z) \), which states that rank \( C(z) = p - \text{rank} A^\pi(z) \).

Let us remark that our approach differs from the one adopted by Campbell in [1], who derived his theorem from the known differentiation result for the Moore–Penrose inverse and from the relation between the Drazin inverse \( A^D \) of a \( p \times p \) matrix \( A \) and the Moore–Penrose inverse \( A^\dagger \) of \( A \):

\[
\]

## 3 Series expansion for \( (A^D)' \)

Let \( U \) be an open interval in \( \mathbb{R} \) or an open set in \( \mathbb{C} \), and \( A(z) \) an operator valued function on \( U \) satisfying the hypotheses of Theorem 2.1 such that \( A^D(z) \) is continuous at \( z_0 \). To simplify notation, we write \( A, A^D, A', A^\pi \) for \( A(z_0), A^D(z_0), A'(z_0), A^\pi(z_0) \). Then (2.2) holds, and we pick \( R > \max \{r, r(A)\} \). In formula (2.1) we choose \( \Gamma = \omega_R - \omega, \) where \( \omega_p(s) = \rho \exp(is) \) for any \( \rho > 0, s \in [0, 2\pi] \). It can be verified that \( \Gamma \) is a Cauchy cycle relative to the pair \((\mathbb{C}\setminus\{0\}, \sigma_0(A))\).

According to (2.1),

\[
(A^D)' = \frac{1}{2\pi i} \int_{\omega_R} \lambda^{-1} R(\lambda; A)A'R(\lambda; A) d\lambda - \frac{1}{2\pi i} \int_{\omega} \lambda^{-1} R(\lambda; A)A'R(\lambda; A) d\lambda.
\]
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Since $R(\lambda; A) = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$, $\| \int_{\omega_R} \lambda^{-1} R(\lambda; A) A' R(\lambda; A) d\lambda \| = O(R^{-2})$ as $R \to \infty$. This shows that

$$\frac{1}{2\pi i} \int_{\omega_R} \lambda^{-1} R(\lambda; A) A' R(\lambda; A) d\lambda = 0.$$ 

By assumption, $0 \not\in \text{acc } \sigma(A)$; in view of [11, Theorem 5.1] there exists $r_0 > 0$ such that

$$R(\lambda; A) = \sum_{n=0}^{\infty} \lambda^{n-1} A^n A^\pi - \sum_{n=0}^{\infty} \lambda^n (A^D)^{n+1} =: U_\lambda - V_\lambda$$

for $0 < |\lambda| < r_0$. If $0 < \rho < \min(r, r_0)$, then

$$\frac{1}{2\pi i} \int_{\omega_r} \lambda^{-1} R(\lambda; A) A' R(\lambda; A) d\lambda = \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-1} (U_\lambda - V_\lambda) A'(U_\lambda - V_\lambda) d\lambda$$

$$= \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-1} U_\lambda A' U_\lambda d\lambda + \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-1} V_\lambda A' V_\lambda d\lambda$$

$$- \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-1} U_\lambda A' V_\lambda d\lambda - \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-1} V_\lambda A' U_\lambda d\lambda$$

$$= \sum_{m,n=0}^{\infty} A^\pi A^m A' A^n A^\pi \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-m-n-3} d\lambda$$

$$+ \sum_{m,n=0}^{\infty} (A^D)^{m+1} A'(A^D)^{n+1} \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{m+n-1} d\lambda$$

$$- \sum_{m,n=0}^{\infty} A^\pi A^m A'(A^D)^{n+1} \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-m-n-2} d\lambda$$

$$- \sum_{m,n=0}^{\infty} (A^D)^{n+1} A'A^m A^\pi \frac{1}{2\pi i} \int_{\omega_\rho} \lambda^{-m-n-2} d\lambda$$

$$= A^D A' A^D - \sum_{n=0}^{\infty} A^\pi A^n A'(A^D)^{n+2} - \sum_{n=0}^{\infty} (A^D)^{n+2} A' A^n A^\pi$$

as $\int_{\omega_\rho} \lambda^k d\lambda$ is equal to $2\pi i$ if $k = -1$ and to $0$ otherwise. Substituting this into (3.1) we get the following result.
Theorem 3.1. Let $A$ be a $B(X)$-valued function defined on $U$ such that $A(z)$ is $g$-Drazin invertible for all $z \in U$ and differentiable at $z_0 \in U$. If $A^D$ is continuous at $z_0$, then

$$
(A^D)' = -A^D A' A^D + \sum_{n=0}^{\infty} A^n A'(A^D)^{n+2} + \sum_{n=0}^{\infty} (A^D)^{n+2} A' A^n A^\pi,
$$

where $A$, $A^D$, $A'$, $A^\pi$ stand for $A(z_0)$, $A^D(z_0)$, $A'(z_0)$, $A^\pi(z_0)$, respectively.

In the case that the Drazin indices $i(A(z))$ are finite and uniformly bounded, the preceding theorem subsumes the differentiation formula of Campbell [1, Theorem 2]; the summation then becomes finite. Let us observe that Campbell’s proof is based on the differentiation of the defining equations in the case that $A$ has the Drazin index 1, that is, on the differentiation of the equations

$$AA^D A = A, \quad A^D AA^D = A^D, \quad AA^D = A^D A.$$  

Hartwig and Shoaf obtained Campbell’s formula from a difference relation [9, (4.16)]. Under the assumption of finite and uniformly bounded indices, formula (3.2) formally agrees with Drazin’s result [5, Theorem 2], which is derived for the conventional Drazin inverse in associative rings.

We note that if $i(A) \leq 1$, formula (3.2) reduces to

$$
(A^D)' = -A^D A' A^D + A^\pi A'(A^D)^2 + (A^D)^2 A' A^\pi.
$$

For matrices this yields [1, Theorem 1].

If $A$ satisfies the hypotheses of Theorem 2.1 and $A^D$ is continuous at $z_0$, Equation (3.3) can be used to describe $(A^D)'$ in terms of the derivative $C'$ of the core part of $A$ bearing in mind that $C$ has Drazin index not exceeding one:

$$
(A^D)' = (C^D)' = -C^D C' C^D + C^\pi C'(C^D)^2 + (C^D)^2 C' C^\pi \\
= -A^D C' A^D + A^\pi C'(A^D)^2 + (A^D)^2 C' A^\pi;
$$

it is known that $A^D = C^D$ and $A^\pi = C^\pi$. 

4 The Moore–Penrose inverse of Hilbert space operators

For $H$ a complex Hilbert space and $A \in B(H)$ it is well known that

$A$ has closed range $\iff A^*A$ has closed range $\iff AA^*$ has closed range $\iff 0 \notin \text{acc } \sigma(A^*A) \iff 0 \notin \text{acc } \sigma(AA^*)$.

For a closed range operator $A \in B(H)$ we can give a definition of the Moore–Penrose inverse $A^\dagger$ of $A$ in terms of the Drazin inverse (see [13, Theorem 2.5]):

\begin{equation}
(A^\dagger)^D = (A^*A)^D A^* = A^*(AA^*)^D.
\end{equation}

This equation enables us to obtain results on the continuity and differentiability of the Moore–Penrose inverse using our results on the $g$-Drazin inverse. (For the continuity of the Moore–Penrose inverse see, for instance, [15].)

**Theorem 4.1.** Let $A$ be a $B(X)$-valued function defined on a real interval $J$ differentiable at $t_0 \in J$ with $A(t)$ closed range operators for all $t \in J$. Write $B(t) = A^*(t)A(t)$ and $E(t) = A(t)A^*(t)$ for all $t \in J$. Then the following conditions are equivalent.

(i) $B^D(t)$ is continuous at $t_0$.
(ii) $E^D(t)$ is continuous at $t_0$.
(iii) $B^D(t)$ is differentiable at $t_0$.
(iv) $E^D(t)$ is differentiable at $t_0$.
(v) $A^\dagger(t)$ is differentiable at $t_0$.
(vi) $A^\dagger(t)$ is continuous at $t_0$.
(vii) $A^\dagger(t)A(t)$ is continuous at $t_0$.
(viii) $A(t)A^\dagger(t)$ is continuous at $t_0$.
(ix) $\|A^\dagger(t)\|$ is bounded in some neighborhood of $t_0$. 

Proof. (i) $\implies$ (iii) $\implies$ (v) $\implies$ (vi) $\implies$ (vii) $\implies$ (i): The first four implications by Theorem 2.1, by the product rule for differentiation applied to $A^\dagger(t) = B^D(t)A^*(t)$, by the relation between differentiability and continuity, and by the continuity of the multiplication in $B(X)$, respectively. The last implication follows when we observe that if (vii) holds, then $A^\dagger(t)A(t) = (A^*(t)A(t))^D(A^*(t)A(t)) = I - A^\pi(t)$ is continuous at $t_0$. Then (i) is true by Theorem 2.1.

(ii) $\implies$ (iv) $\implies$ (v) $\implies$ (vi) $\implies$ (viii) $\implies$ (ii) is proved by a symmetrical argument.

Condition (ix) is equivalent to (vi) when we use the inequality

$$\|A^\dagger(t) - A^\dagger(t_0)\| \leq 3 \max\{\|A^\dagger(t)\|^2, \|A^\dagger(t_0)\|^2\}\|A(t) - A(t_0)\|$$

(see [2, Theorem 10.4.5]).

\[\square\]

**Note 4.2.** We note that in the proof of the implication (iii) $\implies$ (v) the differentiability of $A^*(t)$ follows from the differentiability of $A(t)$ via the identity

$$\frac{dA^*(t)}{dt} = \left(\frac{dA(t)}{dt}\right)^*,$$

which holds only when $t$ is real. The preceding theorem, unlike Theorem 2.1, does not hold for complex differentiation.

**References**


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