Matrices over C^*-Algebras and Applications

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Introduction

C^*-algebra is an important area of research in functional analysis. It is generally believed that C^*-algebras were first considered primarily for their use in quantum mechanics. It was first developed by Gelfand and Naimark in 1943. Gelfand and Naimark showed that among all involutive Banach algebras, C^*-algebras could be characterized by a few simple axioms [8]. In Chapter 3, we will show that a unital C^*-algebra can be considered as a closed *-subalgebra of $\mathcal{B}(H)$, the set of bounded linear operators of a Hilbert space. This is a fundamental result in C^*-algebra, as the theory of C^*-algebra may be easier to handle than more general Banach algebras [2].

This thesis will focus on matrices over C^*-algebras. Consider the set of $n \times n$ matrices over complex numbers, $\mathcal{M}_n(\mathbb{C})$.
Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \mathcal{M}_n(\mathbb{C}).$$

Let $x = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{bmatrix} \in \mathbb{C}^n$. Then,

$$Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{bmatrix} = \begin{bmatrix} a_{11}\zeta_1 + \cdots + a_{1n}\zeta_n \\ \vdots \\ a_{n1}\zeta_1 + \cdots + a_{nn}\zeta_n \end{bmatrix}.$$

Clearly, A is a linear operator from $\mathbb{C}^n$ to $\mathbb{C}^n$. Equip $\mathbb{C}^n$ with Euclidean norm. We will show that A is bounded: Write $A_j = \begin{bmatrix} a_{j1} \\ \vdots a_{jn} \end{bmatrix}$. Then

$$\|Ax\|^2 = \sum_{j=1}^{n} |\langle A_j, x\rangle|^2 \leq \sum_{j=1}^{n} \|A_j\|^2 \|x\|^2$$

where we have used the Cauchy-Schwarz inequality and

$$\|Ax\| \leq \left(\sum_{j=1}^{n} \|A_j\|^2 \right)^{1/2} \|x\| .$$

Now, equip $\mathcal{M}_n(\mathbb{C})$ with the operator norm

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

We will show that this norm satisfies the C^*-identity, that is$\|A^*A\| = \|A\|^2$.
$A \in \mathcal{B}(\mathbb{C}^n)$, so there exists $A^* \in \mathbb{C}^n$ such that for $x, y \in \mathbb{C}^n$,

$$\langle Ax, y \rangle = \langle x, A^*y \rangle.$$

Then, $\|Ax\|^2 = \langle Ax, Ax \rangle = \langle x, A^*Ax \rangle \leq \|x\| \|A^*Ax\|$ by Cauchy-Schwarz inequality. Hence,

$$\frac{\|Ax\|^2}{\|x\|^2} \leq \frac{\|A^*Ax\|}{\|x\|}$$

and

$$\|A\|^2 \leq \|A^*A\| .$$

Furthermore, for any $A, B \in \mathcal{B}(\mathbb{C})$, $\|AB\| \leq \|A\| \|B\|$ and $\|A^*\| = \|A\|$ which imply that $\|A^*A\| \leq \|A^*\| \|A\| = \|A\|^2$. Thus, $|A^*A\| = \|A\|^2$.
Our aim in this thesis is to show that for any C^*-algebra $\mathcal{A}$, $\mathcal{M}_n(\mathcal{A})$, the set of $n \times n$ matrices over $\mathcal{A}$ is a C^*-algebra under an appropriate norm, which will be shown in Chapter 4. In addition, in Chapter 5, we will show that the set of matrices over total system of projections is a C^*-algebra. In Chapter 6, we will present a new proof of Albert's lemmas and the Fuglede-Putnam theorem as applications of the results in Chapter 4 and Chapter 5.

Matrices over C^*-algebra have often been used as a tool to establish some results in functional analysis. However, the objects themselves have not been much investigated. Therefore, this thesis aims to discover some interesting properties of these matrices.

Chapter 1

Banach algebras

The theory of Banach algebras has numerous applications in functional analysis and various mathematical disciplines. In the next chapter, we will be interested in an involutive Banach algebra, that is, a C^*-algebra. The aim of this chapter is to cover some basic theory of Banach algebra, which will include the spectral theory and Gelfand representation. The following discussion on Banach algebra will be fundamental to the study of C^*-algebra. We will follow the materials presented in [1].

1.1 Definition of Banach algebra

An *algebra* is a vector space $\mathcal{A}$, (over complex numbers $\mathbb{C}$), which has a multiplication of elements satisfying

$$(ab)c = a(bc)$$

$$a(b+c) = ab + ac$$

$$(a+b)c = ac + bc$$

$$\lambda(ab) = (\lambda a)b = a(\lambda b).$$

The pair $(\mathcal{A}, \|.\|)$ where $\|.\|$ is submultiplicative is called a *normed algebra.* A norm $\|.\|$ is said to be *submultiplicative* if for $a, b \in \mathcal{A}$,

$$\|ab\| \leq \|a\| \|b\| .$$

If $\mathcal{A}$ admits a unit 1, that is $a1 = 1a = a$, and $\|1\| = 1$, then $\mathcal{A}$ is unital. A complete normed algebra is called a *Banach algebra.*

Some examples of Banach algebras are in [2]:

- The set of bounded linear operators, $\mathcal{B}(X)$, under the operator norm

- $\mathcal{M}_n$, the set of $n \times n$ matrices with entries in $\mathbb{C}$, which can be identified with $\mathcal{B}(\mathbb{C}^n)$
- Let S be a set, $l^\infty(S)$, the set of all bounded complex valued functions, under the supremum norm

The following result will be used in the discussion that follows [1].

Lemma 1.1.1. *Let $\mathcal{A}$ be a unital Banach algebra. If $\|a\| < 1$ for $a \in \mathcal{A}$, then $1 - a \in \mathcal{A}$. Consequently, $\mathcal{A}^{\mathbf{inv}}$ is an open subset of $\mathcal{A}$.*

Proof. Since $\|a^n\| \leq \|a\|^n$, the numerical series $\sum_{n=0}^{\infty} \|a^n\|$ converges. Also, $\mathcal{A}$ is a complete normed space, so $\sum_{n=0}^{\infty} a^n$ converges to some element x in $\mathcal{A}$. Thus, $(1 - a)\sum_{n=0}^{N} a^n = \sum_{n=0}^{N} a^n(1-a) = 1 - a^N \to 1$ as $N \to \infty$. Thus,

$$(1-a)x = 1 = x(1-a)$$

which proves that $x = (1-a)^{-1}$. Further,

$$\left\|(1-a)^{-1}\right\| = \left\|\sum_{n=0}^{\infty} a^n\right\| \leq \sum_{n=0}^{\infty} \|a\|^n = (1 - \|a\|)^{-1}.$$

Now, let $a \in \mathcal{A}$ and $b \in \mathcal{A}^{\mathbf{inv}}$ such that $\|a - b\| < \left\|a^{-1}\right\|^{-1}$. Then,

$$b = a - (a-b) = a(1 - a^{-1}(a-b)).$$

Since$\left\|a^{-1}(a-b)\right\| \leq \left\|a^{-1}\right\| \|a-b\| < 1$, the element $c = 1 - a^{-1}(a-b)$ is invertible in $\mathcal{A}$ and $b = ac$. Then, b is invertible with $b^{-1} = c^{-1}b^{-1}$. Thus, the ball $B(a, \left\|a^{-1}\right\|^{-1})$ lies in $\mathcal{A}^{\mathbf{inv}}$ which shows that $\mathcal{A}^{\mathbf{inv}}$ is open. □

1.2 The spectrum and spectral radius

The concept of spectrum in Banach algebra is a generalization of the concept of eigenvalues of matrices. The spectral theorem gives a condition under which an operator can be 'diagonalized'. Here we will cover some basic results of spectral theory in Banach algebras, in particular the non-emptiness of the spectrum and the spectral radius formula.

Definition 1.2.1. The *resolvent set* of $a \in \mathcal{A}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \in \mathcal{A}^{\mathbf{inv}}\}.$$

Note that we will write $\lambda 1 - a$ as $\lambda - a$.

Lemma 1.2.2. *The resolvent set of a is open.*

Proof. If $\lambda \in \rho a$ and $|\mu - \lambda| < \left\|(\lambda - a)^{-1}\right\|^{-1}$, then Lemma 1.1.1 implies that $\mu \in \rho(a)$. □

Note that the *spectrum* of a is the complement of the resolvent set, $\rho(a)$. We will give the definition below:

Definition 1.2.3. The *spectrum* of $a \in \mathcal{A}$ is defined by

$$\mathbf{Sp}(a) = \left\{\lambda \in \mathbb{C} : \lambda - a \notin \mathcal{A}^{\mathbf{inv}}\right\}.$$

The *spectral radius* of a is $r(a) = \sup_{\lambda \in \mathbf{Sp}(a)} |\lambda|$.

Define the *resolvent* of a as the function

$$R(\lambda; a) = (\lambda - a)^{-1}, \quad \lambda \in \rho(a).$$

Let $a \in \mathcal{A} \setminus \{0\}$. Then, by Lemma 1.1.1,

$$R(\lambda; a) = (\lambda - a)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n, \quad |\lambda\| > \|a\|.$$

Hence, if $|\lambda| > a$, then $\lambda \in \rho(a)$. Thus, if $\lambda \in \mathbf{Sp}(a)$, then $|\lambda\| \leq \|a\|$. So, we get

$$r(a) \leq \|a\|.$$

Now, we will show that $\mathbf{Sp}(a)$ is non-empty and compact. The fact that $\mathbf{Sp}(a)$ is non-empty is nontrivial [1].

Theorem 1.2.4. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then the spectrum of a is nonempty and compact.*

Proof. The spectrum of a is the complement of the resolvent set $\rho(a)$ in $\mathbb{C}$. Since $\rho(a)$ is open, $\mathbf{Sp}(a)$ is closed, and it is bounded by the argument above. hence, Heine- Borel theorem implies that $\mathbf{Sp}(a)$ is compact. First, we will show that if ω is the positively oriented circular loop $|\lambda| = r$ with $r > |a\|$. Then

$$\frac{1}{2\pi i} \int_{\omega} R(\lambda; a) d\lambda = 1.$$

Observe that $\int_{\omega} \lambda^k = 0$ for any $k < -1$. We then have

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\omega} R(\lambda; a) d\lambda &= \frac{1}{2\pi i} \int_{\omega} \left(\sum_{n=0}^{\infty} \lambda^{-n-1} a^n\right) d\lambda \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\omega} \lambda^{-n-1}\right) a^n d\lambda = a^0 = 1
\end{aligned}$$

. Now assume that $\mathbf{Sp}(a)$ is empty. Then $R(\lambda; a)$ is holomorphic and for any loop λ, $\int_{\lambda} R(\lambda; a) d\lambda = 0$ by Cauchy's theorem. But in a unital Banach algebra, $1 \neq 0$. □

The following result due to Beurling will give a calculation of $r(a)$:

Theorem 1.2.5 (The Spectral Radius Formula)**.** *For each $a \in \mathcal{A}$,*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. The function $R(\lambda; a)$ is holomorphic on the annulus $|\lambda| > r(a)$. Hence, it has a unique Laurent expansion in positive and negative powers of λ. Since for $|\lambda| > \|a\|$ we have

$$R(\lambda; a) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n.$$

The same expansion is valid for $|\lambda| > r(a)$. Consequently, for any $r > r(a)$ the terms of the series $\sum_{n=1}^{\infty} r^{-n-1} a^n$ converge to zero, which implies

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a)$$

In view of the compactness of $\mathbf{Sp}(a)$, there exists $\mu \in \mathbf{Sp}(a)$ with $|\mu| = r(a)$. Then $\mu^n \in \mathbf{Sp}(a^n)$ and

$$r(a) = |\mu^n|^{1/n} \le \|a^n\|^{1/n}$$

for all $n \ge 1$. Thus

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a) \le \liminf_{n\to\infty} \|a^n\|^{1/n}$$

which proves that $\lim_{n\to\infty} \|a^n\|^{1/n}$ exists and equals $r(a)$. □

An interesting fact is that the only complex Banach algebra which is a division algebra is the set of complex numbers $\mathbb{C}$, which will be shown here [2]:

Theorem 1.2.6 (Gelfand-Mazur Theorem)**.** *If every element of $\mathcal{A} \setminus \{0\}$ is invertible, then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

Proof. From Theorem 1.2.4, we know that $\mathbf{Sp}(a)$ is non-empty. Let $\lambda \in \mathbf{Sp}(a)$, then $\lambda - a$ is not invertible, so $\lambda 1 - a = 0$. Hence, the map $a = \lambda 1 \mapsto \lambda$ is an isomorphism of $\mathcal{A}$ onto $\mathbb{C}$. □

1.3 Gelfand representation

The idea here is to represent a commutative unital Banach algebra as an algebra of continuous functions on a compact Hausdorff space. This will give a useful way of looking at this algebra.

First, we will define a character of a Banach algebra.

Definition 1.3.1. Let $\mathcal{A}$ be a commutative unital Banach algebra. A nonzero algebra homomorphism $\tau : \mathcal{A} \mapsto (C)$ is called a character. Hence, τ is linear, multiplicative and preserves unit. Denote the set of all characters of $\mathcal{A}$ by $\Omega(\mathcal{A})$.

The following theorem is a celebrated result of I.Gelfand:

Theorem 1.3.2. *Let τ be a character of a commutative unital Banach algebra $\mathcal{A}$. Then, τ is bounded with $\|\tau\| = 1$. The mapping*

$$\tau \mapsto N(\tau) := \{x \in \mathcal{A} : \tau(x) = 0\}$$

is a bijection of $\Omega(\mathcal{A})$ onto the set of all maximal ideals in $\mathcal{A}$.

Proof. Let τ be a character of $\mathcal{A}$. Then, $x \in \mathcal{A}^{\mathbf{inv}}$ if and only if $\tau(x) \neq 0$. Let $A \in \mathcal{A}$ and $\lambda = \tau(a)$. So $\tau(\lambda - a) = 0$ and $\lambda \in \mathbf{Sp}(a)$. Hence,

$$|\tau(a)| \leq r(a) \leq \|a\|$$

and we get $\|\tau\| \leq 1$ Since $\tau(1) = 1$, we get $\|\tau\| = 1$.
Hence, $N(\tau)$ is a closed ideal of $\mathcal{A}$. Note that $a = a - \tau(a)1 + \tau(a)1$, so $\mathcal{A} = N(\tau) \oplus \mathbb{C}1$. Hence, $N(\tau)$ is a maximal ideal.
Suppose that $\tau_1, \tau_2 \in \Omega(\mathcal{A})$ and $N(\tau_1) = N(\tau_2)$. Then, for any $a \in \mathcal{A}$ we have $\tau_1(a - \tau_2(a)) = 0$, so $\tau_1(a) = \tau_2(a)$, that is $\tau_1 = \tau_2$.
Let $\mathcal{M}$ be a maximal ideal of $\mathcal{A}$. Then $\mathcal{M}$ is closed, and $\mathcal{A} \backslash \mathcal{M}$ is unital Banach algebra in which every non-zero element is invertible. By the Gelfand-Mazur Theorem, $\mathcal{A} \backslash \mathcal{M} = \mathbb{C}$ and $\mathcal{A} = \mathcal{M} \oplus \mathbb{C}1$. Define $\tau : \mathcal{A} \mapsto \mathbb{C}$ by $\tau(m + \lambda) = \lambda$ where $a = m + \lambda$, $m \in \mathcal{M}, \lambda \in \mathbb{C}$. Then τ is character with $N(\tau) = \mathcal{M}$. □

The following is a characterisation of the spectrum of $a \in \mathcal{A}$ by the characters of a.

Lemma 1.3.3. *Let $\mathcal{A}$ be a commutative unital Banach algebra and $a \in \mathcal{A}$. Then,*

$$\mathbf{Sp}(a) = \{\tau(a) : \tau \in \Omega(\mathcal{A})\}$$

Proof. Let $\lambda \in \mathbf{Sp}(a)$. Then the ideal $\mathcal{L} = (a - \lambda)\mathcal{A}$ is proper, and therefore contained in a maximal ideal $\mathcal{M} = N(\tau)$ for some $\tau \in \Omega(\mathcal{A})$. Hence, $\tau(a) = \lambda$. The reverse inclusion is established in the proof of Theorem 1.3.2. □

Now, we are ready to prove the main theorem:

Theorem 1.3.4 (Gelfand Representation)**.** *For each $a \in \mathcal{A}$, define the Gelfand transform of a, $\hat{a} : \Omega(\mathcal{A}) \mapsto (C)$ by*

$$\hat{a}(\tau) = \tau(a).$$

Then, the map $\psi : \mathcal{A} \mapsto C(\Omega(\mathcal{A}))$ defined by $\psi(a) = \hat{a}$ is an algebra homomorphism and $\|\hat{a}\|_\infty = r(a) \leq \|a\|$. This map ψ is called the Gelfand representation of $\mathcal{A}$. Further, $\mathbf{Sp}(a) = \hat{a}(\Omega(\mathcal{A}))$.

Proof. By Lemma 1.3.3, $\mathbf{Sp}(a) = \hat{a}(\Omega(\mathcal{A}))$. Hence, $r(a) = \|\hat{a}\|_\infty$. □

Finally, we will consider the topology of $\Omega(\mathcal{A})$.

Theorem 1.3.5. *The space $\Omega(\mathcal{A})$ equipped with the weak*-topology is a compact Hausdorff space.*

Proof. Recall the James embedding $J : \mathcal{A} \mapsto \mathcal{A}^{**}$ defined by $(Ja)(x^*) = x^*(a)$ for $a \in \mathcal{A}$ and all $x^* \in \mathcal{A}^*$. The weak*-topology of $\mathcal{A}^*$ is the weakest topology on A' such in which Ja is continuous for every $a \in \mathcal{A}$ and it is a Hausdorff toppology. Recall that Banach-Alaoglu Theorem states that the closed unit ball of $\mathcal{A}^*$ is weak*-compact. The set $\Omega(\mathcal{A})$ is weak*-closed in the closed unit ball of $\mathcal{A}^*$ and therefore compact. □

Chapter 2

C^*-algebras

This chapter contains some introduction to C^*-algebras. The characterisation of C^*-algebra was given by Gelfand and Naimark in 1943. A C^*-algebra is an involutive Banach algebra, and this has a richer structure than Banach algebra.

First, we will start with a definition of C^*-algebra [1].

2.1 Definition of C^*-algebra

A *C^*-algebra* is a Banach algebra with an involution $x \mapsto x^*$ satisfying

$$(x+y)^* = x^* + y^*$$

$$(xy)^* = y^* x^*$$

$$(\lambda x)^* = \overline{\lambda} x^*$$

$$(x^*)^* = x$$

and the C^*-identity

$$\|x^* x\| = \|x\|^2 .$$

Remark. The space $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H is a C^*-algebra. Moreover, every C^*-algebra can be regarded as a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H, which will be shown in the next chapter.

The following is a main result in C^*-algebra which asserts that all commutative unital C^*-algebras, up to isomorphism, are of the form $C(\Omega(\mathcal{A}))$ [2].
Needed in the proof of this theorem is the following:

Theorem 2.1.1 (Stone-Weirstrass Theorem)**.** *Let K be a compact metric space and $L \subset C(K)$ where we equip $C(K)$ with the supremum norm. In addition, L separates points of K and contains* 1*. Further, if $f \in L$ then $\bar{f} \in L$. Then the algebra generated by L is dense in $C(K)$.*

Theorem 2.1.2. *If $\mathcal{A}$ is a commutative unital C^*-algebra, then the Gelfand representation $\psi : \mathcal{A} \mapsto C(\Omega(\mathcal{A}))$ where $\psi(a) = \hat{a}$ is an isometric *-isomorphism.*

Proof. By Theorem 1.3.4, $\|\psi(a)\| = r(a)$. If $\tau \in \Omega(\mathcal{A})$, then $\psi(a^*)(\tau) = \tau(a^*) = \overline{\tau(a)} = \psi(a)^*(\tau)$, that is ψ is a *-homomorphism. Further, ψ is an isometry since $\|\psi(a)\|^2 = \|\psi(a)^*\psi(a)\| = \|\psi(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2$. Hence, $\psi(\mathcal{A}$ is a closed *-subalgebra of $C(\Omega(\mathcal{A}))$, separating the points of $\Omega(\mathcal{A})$ and for any $\tau \in \Omega(\mathcal{A})$, there is an element $a \in \mathcal{A}$ such that $\psi(a)(\tau) \neq 0$. Thus, the Stone-Weirstrass theorem implies that $\psi(\mathcal{A}) = C(\Omega(\mathcal{A}))$. □

Definition 2.1.3. We say $a \in \mathcal{A}$ is *normal* if $a^*a = aa^*$. Moreover a is *unitary* if $a^*a = aa^* = 1$.

Lemma 2.1.4. *If $u \in \mathcal{A}$ is unitary, then $\mathbf{Sp}(u) \subset \mathbb{T}$ where $\mathbb{T}$ is the unit circle in $\mathbb{C}$.*

Proof. $\|u\| = 1$ since $\|u\|^2 = \|u^*u\| = \|1\| = 1$. Hence, $\mathbf{Sp}(u) \subseteq \mathbb{T}$. Further, if $\lambda \in \mathbf{Sp}(u)$, then $\lambda^{-1} \in \mathbf{Sp}(u^{-1}) = \mathbf{Sp}(u^*)$, so $|\lambda|$ and $\left|\lambda^{-1}\right| \leq 1$, that is $|\lambda| = 1$. □

2.2 Functional calculus

Functional calculus allows one to apply functions to operators. Given that a is a normal element of a C^*-algebra $\mathcal{A}$ and $f : \mathbf{Sp}(a) \mapsto \mathbb{C}$ is a continuous mapping, we will define a function $f(a)$. To do so, we will use the Gelfand representation for commutative algebras [1].

Theorem 2.2.1 (Gelfand-Naimark Calculus)**.** *Let a be a normal element of a unital C^*-algebra $\mathcal{A}$. Then there exists a unique unital *-homomorphism $\varphi_a : C(\mathbf{Sp}(a)) \mapsto \mathcal{A}$ such that $\varphi_a(f) = a$ when $f(\lambda) = \lambda$ for all $\lambda \in \mathbf{Sp}(a)$. Moreover, φ_a is isometric and the range of φ_a is the C^*-subalgebra of $\mathcal{A}$ generated by 1 and a.*

Proof. Let $\mathcal{B}$ be the C^*-algebra generated by 1 and a. Then $\mathcal{B}$ is commutative since a is normal. Also, let $\psi : \mathcal{B} \mapsto C(\Omega(\mathcal{B}))$ be the Gelfand representation. By Theorem 2.1.2, ψ is a *-isomorphism. Moreover, by Theorem 1.3.4, $\hat{a}(\Omega(\mathcal{B})) = \mathbf{Sp}(a)$. Hence, for each $f \in C(\mathbf{Sp}(a))$, the map $f \circ \hat{a} \in C(\Omega(\mathcal{B}))$. Then, we can define

$$\varphi_a(f) := \psi^{-1}(f \circ \hat{a}).$$

Note that $\varphi_a(f) = \psi^{-1}(f(\hat{a})) = \psi^{-1}(\hat{a}) = a$ and φ_a is unital.
From the Stone-Weierstrass Theorem, we know that $C(\mathbf{Sp}(a))$ is generated by 1 and f and therefore φ_a is the unique unital *-homomorphism from $C(\mathbf{Sp}(a))$ to $\mathcal{A}$ such that $\varphi_a(f) = 1$. Also, φ_a is clearly isometric and $\varphi_a(C(\mathbf{Sp}(a)) = \mathcal{B}$. □

This unique unital *-homomorphism φ_a is called the *functional calculus* at a. We usually write

$$f(a) := \varphi_a(f)$$

for $f \in C(\mathbf{Sp}(a))$.

Theorem 2.2.2 (Spectral Mapping Theorem)**.** *Let a be a normal element of a C^*-algebra $\mathcal{A}$, and $f \in C(\mathbf{Sp}(a))$. Then*

$$\mathbf{Sp}(f(a)) = f(\mathbf{Sp}(a)).$$

Proof. Let $\mathcal{B}$ be the C^*-subalgebra generated by 1 and a. Note that, if $\tau \in \Omega(\mathcal{B})$, then $f(\tau(a)) = \tau(f(a))$, since the maps $f \mapsto f(\tau(a))$ and $f \mapsto \tau(f(a))$ from $C(\mathbf{Sp}(a))$ to $\mathbb{C}$ are *-homomorphisms agreeing on the generators of $C(\mathbf{Sp}(a))$, 1 and i where i is the inclusion map of $\mathbf{Sp}(a)$ in $\mathbb{C}$. Then,

$$\begin{aligned}\mathbf{Sp}(f(a)) &= \{\tau(f(a)) : \tau \in \Omega(\mathcal{B})\} \text{ by Lemma 1.3.3} \\ &= \{f(\tau(a)) : \tau \in \Omega(\mathcal{B})\} \\ &= f(\mathbf{Sp}(a))\end{aligned}$$

□

2.3 Positive elements of C^*-algebras

We will introduce a partial ordering on the self-adjoint elements of a C^*-algebra $\mathcal{A}$, then aim to show that all elements of $\mathcal{A}$ of the form a^*a are positive. In addition, we will show that every positive element has a unique positive square root. The proofs of these results will involve functional calculus.

Definition 2.3.1. An element $a \in \mathcal{A}$ is *positive* if a is self-adjoint and $\mathbf{Sp}(a) \subset [0, \infty)$. We write $a \geq 0$ to mean that a is positive. Denote the set of all positive elements of $\mathcal{A}$ by $\mathcal{A}^+$.

Gelfand conjectured in the original paper that a^*a for any $a \in \mathcal{A}$ is positive. However, this was proved about a decade later by Fukamiya and Kaplansky. We will present the proof here. To do so, we will need the following series of results [1]:

Lemma 2.3.2. *Let $a \in \mathcal{A}^{\mathbf{sa}}$. Then,*

(i) $\mathbf{Sp}(a) \subset \mathbb{R}$.

(ii) $\|a\| = r(a)$.

Proof. (1). Let $f(\lambda) = \exp(i\lambda)$ and $u = f(a) = \exp(ia)$. Then u is unitary and by Lemma 2.1.4, $\mathbf{Sp}(u) \subset \mathbb{T}$ where $\mathbb{T}$ is the unit circle. By Spectral Mapping Theorem,

$f(\mathbf{Sp}(u)) = \mathbf{Sp}(f(u)) \subset \mathbb{T}$. Thus for all $\lambda \in \mathbf{Sp}(u)$, $\exp(i\lambda) \in \mathbb{T}$, that is $\lambda \in \mathbb{R}$.
(2). $\left\|a^2\right\| = \|a^*a\| = \|a\|^2$, and therefore by induction $\left\|a^{2^n}\right\| = \|a\|^{2^n}$. By Spectral Radius Formula, $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \left\|a^{2^n}\right\|^{1/2^n} = \|a\|$. □

Lemma 2.3.3. *Let $\mathcal{A}$ be a unital C^*-algebra, $a \in \mathcal{A}^{\mathbf{sa}}$ and $t \in \mathbb{R}$. If $\|t - a\| \leq t$ for $t \geq \|a\|$, then $a \in \mathcal{A}^+$. Conversely, if $\|a\| \leq t$ and $a \in \mathcal{A}^+$, then $\|t - a\| \leq t$.*

Proof. Let $t \geq \|a\|$ and let g be a functional calculus such that $g(a) = t - a$. Note that $r(a) = \|t - a\|$. Now, we will show that $\mathbf{Sp}(a) \subset [0, \infty)$ if and only if $r(t - a) \leq t$. By the Spectral Mapping Theorem, $\mathbf{Sp}(g(a)) = g(\mathbf{Sp}(a))$. Let $\lambda \in \mathbf{Sp}(a)$. If $\lambda \geq 0$, then $|g(\lambda)| = |t - \lambda| \leq t$. Hence, $r(g(a)) = r(t - a) \leq t$. Conversely, if $r(t - a) \leq t$, then $|g(\lambda)| = |t - \lambda| \leq t$. Hence, $\lambda \geq 0$. □

Further, $\mathcal{A}^+$ is closed under addition and multiplication by positive numbers:

Lemma 2.3.4. *$\mathcal{A}^+$ is a cone in $\mathcal{A}$, that is $\mathcal{A}^+ + \mathcal{A}^+ \subset \mathcal{A}^+$, $\mathbb{R}^+\mathcal{A}^+ \subset \mathcal{A}^+$.*

Proof. Let $a, b \in \mathcal{A}^+$. Let $t = \|a\|$ and $s = \|b\|$. By Lemma 2.3.3,

$$\|a - t\| \leq t, \ \ \|b - s\| \leq s.$$

Thus,

$$\|a + b - (t + s)\| \leq \|a - t\| + \|b - s\| \leq t + s$$

Hence, $a + b \in \mathcal{A}^+$ by Lemma 2.3.3. The second part is obvious. □

Lemma 2.3.5. *If $-a^*a \in \mathcal{A}^+$, then $a = 0$.*

Proof. $\mathbf{Sp}(-a^*a) \cup 0 = \mathbf{Sp}(-aa^*) \cup 0$, so $-a^*a \in \mathcal{A}^+$. Write $a = b + ic$ with $b, c \in \mathcal{A}^{\mathbf{sa}}$. Then $a^*a + aa^* = 2b^2 + 2c^2 \in \mathcal{A}^+$ and

$$a^*a = 2b^2 + 2c^2 + (-a^*a) \in \mathcal{A}^+$$

by Lemma 2.3.4. Hence, $\mathbf{Sp}(a^*a) \subset \mathbb{R}^+ \cap (-\mathbb{R}^+) = \{0\}$ and therefore $\|a\|^2 = \|a^*a\| = r(a^*a) = 0$, which implies $a = 0$. □

Now, we are ready to prove this theorem [1]:

Theorem 2.3.6 (Fukamiya-Kaplansky)**.** *If $a \in \mathcal{A}$, then $a^*a \in \mathcal{A}^+$.*

Proof. Let $b = a^*a$ and $c = ab^-$. Then $b \in \mathcal{A}^{\mathbf{sa}}$, so $b = b^+ - b^-$. We get

$$-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \in \mathcal{A}^+$$

which implies $c = 0$ by Lemma 2.3.5. Further,

$$(b^-)^2 = (b^- - b^+)b^- = -bb^- = -a^*ab^- = -a^*c = 0$$

so $b^- = 0$. Hence, $a^*a = b = b^+ \in \mathcal{A}^+$. □

A useful result is that every positive element of a C^*-algebra has a unique square root [1]:

Lemma 2.3.7. *If $a \in \mathcal{A}^+$, then there exists a unique $b \in \mathcal{A}^+$ such that $b^2 = a$. b is called the square root of a and is denoted by $b = a^{1/2}$. Further, $a = xx^*$ for some $x \in \mathcal{A}$.*

Proof. Since $\mathbf{Sp}(a) \subset [0, \|a\|]$, we can define a continuous function $f(t) = \sqrt{(}t)$ on $\mathbf{Sp}(a)$ and set $b = f(a)$. Then, $b^2 = a$, $b \in \mathcal{A}^{\mathbf{sa}}$, and $\mathbf{Sp}(b) = \mathbf{Sp}(f(a)) = f(\mathbf{Sp}(a)) \subset [0, \|a\|^{1/2}]$, so that $b \in \mathcal{A}^+$.
To prove uniqueness, let c be another positive element such that $c^2 = a$. By the functional calculus, $c = f(c^2) = f(a) = b$. We can let $x = a^{1/2}$. Thus, $xx^* = a$. □

As a consequence, we get the following lemma which will be useful later:

Lemma 2.3.8. *Let $a \in \mathcal{A}^+$ and $b \in \mathcal{A}$. Then $b^*ab \in \mathcal{A}^+$.*

Proof. By Lemma 2.3.7, there exists a positive element c such that $c^2 = a$. Consider $(cb)^*(cb) = b^*c^*cb = b^*c^2b = bab$. By Theorem 2.3.6, this is positive. □

Chapter 3

Representations

The aim of this chapter is to show that every C^*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H, which is stated in The Gelfand-Naimark Theorem. This is one of the fundamental results in the theory of C^*-algebra. A key step in its proof is the Gelfand-Naimark-Segal (GNS) construction which sets up a correspondence between the positive linear functionals and the representations of the algebra.

3.1 Positive functionals

In this section, we will study some properties of positive linear functionals [1] which will be essential in the construction of the representations.

Definition 3.1.1. A linear functional $\varphi : \mathcal{A} \mapsto \mathbb{R}$ is positive if $\varphi(a) \geq 0$ for each $a \in \mathcal{A}^+$. If $\varphi(1) = 1$, then φ is called a *state*. The set of all states of $\mathcal{A}$ will be denoted by $\mathcal{S}(\mathcal{A})$.

We observe that $\varphi(a^*) = \overline{\varphi(a)}$ for $a \in \mathcal{A}$. Further, we define the inner product

$$\langle a, b\rangle_\varphi := \varphi(b^* a)$$

where $a, b \in \mathcal{A}$. It can easily be checked that this definition satisfies the properties of an inner product. Further, this function is a sesquilinear form on $\mathcal{A}$. In addition, we have the Cauchy-Schwarz inequality

$$|\varphi(b^* a)|^2 \leq \varphi(a^* a)\varphi(b^* b)$$

and the norm is defined by

$$\|a\|_\varphi = (\varphi(a^* a))^{1/2} = (\langle a, a\rangle_\varphi)^{1/2}$$

Lemma 3.1.2. *Let φ be a positive linear functional on $\mathcal{A}$. Then [2],*

(i) *$\varphi(a^*a) = 0$ if and only if $\varphi(ba) = 0$ for all $b \in \mathcal{A}$.*

(ii) *For all $a, b \in \mathcal{A}$,*

$$\varphi(b^*a^*ab) \leq \|a\|^2 \varphi(b^*b).$$

Proof. (1). It follows from Cauchy- Schwarz inequality.
(2). For $x \in \mathcal{A}$, we have $x^*x \leq \|x\|^2$. Hence, $b^*a^*ab \leq \|a\|^2 b^*b$, and by the monotonicity of φ, we get $\varphi(b^*a^*ab) \leq \|a\|^2 \varphi(b^*b)$. □

Lemma 3.1.3. *Let φ be a nonzero linear functional on $\mathcal{A}$ [1].*

(i) *If φ is positive, then φ is bounded and $\varphi(1) = \|\varphi\|$.*

(ii) *If φ is a bounded linear functional on $\mathcal{A}$ such that $\varphi(1) = \|\varphi\|$, then φ is positive.*

Proof. (1) Let $a \in \mathcal{A}$ and $\|a\| \leq 1$. Then, $\|a^*a\| = \|a\|^2 \leq 1$, so $0 \leq a^*a \leq 1$. By the monotonicity of φ, $0 \leq \varphi(a^*a) \leq \varphi(1)$. By the Cauchy-Schwarz inequality,

$$|\varphi(a)|^2 \leq \varphi(1^*1)\varphi(a^*a) \leq \varphi(1)^2$$

Thus, φ is bounded with $\varphi(1) \leq \varphi(1)$. Conversely, $0 < \varphi(1) \leq \|\varphi\| \|1\| = \|\varphi\|$.
(2) First, we show that φ takes real values on self-adjoint elements. We may assume that $\|\varphi\| = 1$. Let $a \in \mathcal{A}^{\int\dashv}$ and $\|a\| \leq 1$. Also, let $\varphi(a) = \alpha + i\beta$. Note that $\varphi(a - in) = \varphi(a) - in\varphi(1) = \varphi(a) - in$, so we get

$$|\varphi(a - in)|^2 = |\varphi(a) - in|^2 = |\alpha + i\beta - in|^2 = \alpha^2 + (\beta - n)^2$$

and

$$|\varphi(a - in)|^2 \leq \|a - in\|^2 = \|(a - in)^*(a - in)\| = \left\|a^2 + n^2\right\| \leq 1 + n^2.$$

Thus, $\alpha^2 + (\beta - n)^2 \leq 1 + n^2$ and if $\beta \leq 0$, we get $\beta = 0$. If $\beta > 0$, apply the preceding argument $-a$ in place of a to get $\beta = 0$. Let $a \in \mathcal{A}^+$ and $\|a\| \leq 1$. Then $1 - a \in \mathcal{A}^{\int\dashv}$ and $\|1 - a\| \leq 1$. Hence, $1 - \varphi(a) = \varphi(1 - a) \leq 1$. Thus, $\varphi(a) \geq 0$ if $a \in \mathcal{A}^+$. □

Lemma 3.1.4. *Let a be a normal element of a non-zero C^*-algebra $\mathcal{A}$. Then, there exists a state φ such that $\|a\| = |\varphi(a)|$.*

Proof. Assume that $a \neq 0$. Let $\mathcal{B}$ be the C^*-algebra generated by 1 and a. Since $\mathcal{B}$ is commutative and $\hat{a}$ is continuous on the compact space $\Omega(\mathcal{B})$, there is a character φ_2 on $\mathcal{B}$ such that $\|a\| = \|\hat{a}\|_\infty = |\varphi_2(a)|$. By the Hahn-Banach theorem, there is a bounded linear functional φ on $\mathcal{A}$ extending φ_2 and preserving the norm, so $\|\varphi\| = 1$. By Lemma 3.1.3, since $\varphi(1) = \varphi_2(1) = 1$, φ is positive. Hence, φ is a state. □

3.2 Representations

Here, we will define a representation of a C^*-algebra followed by some discussion which will be useful in the proof of Gelfand-Naimark theorem.

Definition 3.2.1. A *representation* of a C^*-algebra $\mathcal{A}$ is a pair (H, π), where H is a Hilbert space and $\pi : \mathcal{A} \mapsto \mathcal{B}(H)$ is a *-homomorphism. A representation (H, π) is *faithful* if π is injective.

Let $\mathcal{S}(\mathcal{A})$ be the set of all states of $\mathcal{A}$ and (H_φ, π_φ) where $\varphi \in \mathcal{S}(\mathcal{A})$ be a family of representations of $\mathcal{A}$. Their direct sum is (H, π) where

$$H = \bigoplus_\varphi H_\varphi$$

with norm $\|x\|^2 = \sum_\varphi \|x_\varphi\|_\varphi^2$ and we define $\pi : \mathcal{A} \mapsto \mathcal{B}(H)$ by

$$\pi(a)x = \bigoplus_\varphi \pi_\varphi(a)x_\varphi$$

Note that π is a representation since each π_φ is a representation. Further, we will prove that H is a Hilbert space.
An element of H is a function $x : \mathcal{S}(\mathcal{A}) \mapsto \bigcup_\varphi H_\varphi$ with $x(\varphi) \in H_\varphi$ and $\sum_\varphi \|x(\varphi)\|_\varphi^2 < \infty$. And for $x, y \in H$, the inner product

$$\langle x, y\rangle := \sum_\varphi \langle x_\varphi, y_\varphi\rangle_\varphi$$

The series $\sum_\varphi \langle x_\varphi, y_\varphi\rangle$ converges absolutely since

$$|\langle x, y\rangle| \leq \|x_\varphi\|_\varphi \|y_\varphi\|_\varphi \leq \tfrac{1}{2}(\|x_\varphi\|_\varphi^2 + \|y_\varphi\|_\varphi^2)$$

Let (x_n) be a Cauchy sequence in H. For each $\varphi \in \mathcal{S}(\mathcal{A})$, and $m, n \geq N(\epsilon)$,

$$\|x_n(\varphi) - x_m(\varphi)\| \leq \|x_n - x_m\| < \epsilon$$

Hence, $(x_n(\varphi))$ is Cauchy in H_φ.
Let E be a finite subset of $\mathcal{S}(\mathcal{A})$. Then,

$$\sum_{\varphi\in E} \|x_n(\varphi) - x_m(\varphi)\|^2 \leq \sum_{\varphi\in D} \|x_n(\varphi) - x_m(\varphi)\|^2 = \|x_n - x_m\|^2 < \epsilon^2$$

Let $n \geq N(\epsilon)$. Then

$$\sum_{\varphi\in E} \|x_n(\varphi) - x(\varphi)\|^2 = \lim_{m\to\infty} \sum_{\varphi\in E} \|x_n(\varphi) - x_m(\varphi)\|^2 \leq \epsilon^2$$

Hence, $x - x_n \in H$ and $x = (x - x_n) + x_n \in H$.
For each $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} \|x_n(\varphi_i) - x(\varphi_i)\|^2 \leq \epsilon^2.$$

Taking the limit as $k \to \infty$, we get $\|x_n - x\| \leq \epsilon$ for $n \geq N(\epsilon)$. Thus, $\|x_n - x\| \to 0$ in H. Hence, (x_n) converges and H is complete.

Definition 3.2.2. Let (H_φ, π_φ) be a family of representations of $\mathcal{A}$ where φ ranges over $\mathcal{S}(\mathcal{A})$. Then, the direct sum of all the representations (H_φ, π_φ) is called a *universal representation.*

3.3 The Gelfand-Naimark Theorem

Now we will proceed to show that with each state, there is associated a representation, which is shown in the GNS construction. The GNS construction will lead to the proof of the Gelfand-Naimark theorem which asserts that every unital C^*-algebra can be regarded as a closed C^*-subalgebra of $\mathcal{B}(H)$.

Definition 3.3.1. A representation (H, π) of a C^*-algebra $\mathcal{A}$ is called a *cyclic representation* if there exists a vector $\xi \in H$ such that $\pi(\mathcal{A})\xi$ is dense in H. Moreover, ξ is called a *cyclic vector* for (H, π).

Theorem 3.3.2 (The Gelfand-Naimark-Segal construction)**.** *Let φ be a state on $\mathcal{A}$. Then there exists a cyclic representation (H, π) and a unit cyclic vector ξinH for (H, π) such that for $a \in \mathcal{A}$,*

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle$$

Proof. Define the set

$$N_\varphi := \{a \in \mathcal{A} : \varphi(a^* a) = 0\}$$

This is a closed left ideal of $\mathcal{A}$ by part (1) of Lemma 3.1.2. Consider the quotient space $\mathcal{A}/\mathcal{N}_\varphi$. An inner product on this space is defined by

$$\langle a + N_\varphi, b + N_\varphi \rangle_\varphi := \varphi(b^* a)$$

Let $N := N_\varphi$. For each $a \in \mathcal{A}$, define $\pi(a) : \mathcal{A}/N \mapsto \mathcal{A}/N$ by

$$\pi(a)(b + N) = ab + N$$

where $b + N \in \mathcal{A}/N$. $\pi(a)$ is a well -defined linear operator :
Clearly, $\pi(a)$ is a linear operator. Moreover, if $b+N = c+N$, then $\pi(a)(b+N) - \pi(a)(c+N) = a(b - c) + N = N$ as N is a left ideal. Hence, $\pi(a)(b + N) = \pi(a)(c + N)$ and therefore π is well-defined.
By part (2) of Lemma 3.1.2, we get

$$\|\pi(a)(b + N)\|^2 = \langle ab + N, ab + N \rangle = \varphi(b^* a^* ab) \leq \|a\|^2 \varphi(b^* b) = \|a\|^2 \|b + N\|^2$$

Thus, $\pi(a)$ is a bounded linear operator.
Let H be a completion of $\mathcal{A}/N$. Then, by continuity, $\pi(a)$ extends to an operator on H with the same norm. Hence, π is a mapping $\pi : \mathcal{A} \mapsto \mathcal{B}(H)$ defined by $a \mapsto \pi(a)$. It can easily be checked that this map is a *-homomorphism. Thus, (H, π) is a representation of $\mathcal{A}$.Let $\xi = 1 + N \in \mathcal{A}/N$ and $a \in \mathcal{A}$. Then

$$\pi(a)\xi = \pi(a)(1 + N) = a + N$$

Hence, $\pi(\mathcal{A})\xi = \mathcal{A}/N$ which is dense in H. Further,

$$\langle \pi(a)\xi, \xi \rangle = \langle a + N, 1 + N \rangle = \varphi(a)$$

Setting $a = 1$, we get $\|\xi\|^2 = \varphi(1) = 1$. □

The representation produced by the GNS construction is not necessarily faithful, but the Gelfand-Naimark theorem further produces an isometric representation.

Theorem 3.3.3 (The Gelfand-Naimark Theorem)**.** *Let $\mathcal{A}$ be a unital C^*-algebra. Then, there exists a faithful representation (H, π) of $\mathcal{A}$, that is, $\mathcal{A}$ is isometrically *-isomorphic to a closed C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

Proof. Let (H, π) be universal representation of $\mathcal{A}$ and a be an element such that $\pi(a) = 0$. By Lemma 3.1.4, there is a state φ on $\mathcal{A}$ such that $\|a^*a\| = \varphi(a^*a)$. Hence, if $b = (a^*a)^{1/4}$, then $\|a\|^2 = \varphi(a^*a) = \varphi(b^4) = \|\pi_\varphi(b)(b + N_\varphi)\|^2 = 0$ since $\pi_\varphi(b^4) = \pi_\varphi(a^*a) = 0$ and therefore $\pi_\varphi(b) = 0$. Hence, $a = 0$ and π is injective. □

Chapter 4

Matrices over C^*-algebra

Matrices over C^*-algebra are often employed in the proofs of some results in functional analysis. However, there has not been much systematic discussion on matrices over C^*-algebras in the literatures. This thesis will discuss some interesting properties of these matrices. In particular, in this section, we will show that a norm can be found so that $\mathcal{M}_n(\mathcal{A})$, a set of $n \times n$ matrices over a C^*-algebra (A), is a C^*-algebra. In our construction of the C^*-norm, we will use the representation of $\mathcal{A}$ and bounded linear operators..

4.1 Operations and the norm on matrices

Let $\mathcal{A}$ be a C^*-algebra. A typical element of $\mathcal{M}_n(\mathcal{A})$ is given by

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

where $a_{ij} \in \mathcal{A}$.
Note that for a matrix $A = [a_{ij}]$, we write $a_{ij} = [A]_{ij}$. Operations in $\mathcal{M}_n(\mathcal{A})$ are analogous to those in the set of numerical matrices. These include the associative, commutative and distributive laws with identity

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}.$$

In addition, the conjugate is defined as $A^* = \begin{bmatrix} a_{11}^* & \cdots & a_{n1}^* \\ \vdots & \vdots & \vdots \\ a_{1n}^* & \cdots & a_{nn}^* \end{bmatrix}$. Also, the reversal law $(AB)^* = B^* A^*$.

We will show that $\mathcal{M}_n(\mathcal{A})$ is a C^*-algebra. To do so, we will use the Gelfand-Naimark representation (H,π) of $\mathcal{A}$
Define $\pi_n \ : \ \mathcal{M}_n(\mathcal{A}) \ \mapsto \ \mathcal{M}_n(B(H))$ by

$$\pi_n(A) = \begin{bmatrix} \pi(a_{11}) & \cdots & \pi(a_{1n}) \\ \vdots & \dots & \vdots \\ \pi(a_{n1}) & \cdots & \pi(a_{nn}) \end{bmatrix} \tag{4.1}$$

Now, π_n embeds $\mathcal{M}_n(\mathcal{A})$ into $\mathcal{M}_n(B(H))$,that is π_n is an injective *-homomorphism.
First, if $\pi_n(A) = 0$ then $\pi(a_{ij}) = 0$ for all i,j and $a_{ij} = 0$ since π is injective
Thus, $A = 0$ and π_n is injective.
Next,

$$\begin{aligned} [\pi_n(A+B)]_{ij} &= \pi(a_{ij} + b_{ij}) = \pi(a_{ij}) + \pi(b_{ij}) \\ &= [\pi_n(A) + \pi_n(B)]_{ij} \end{aligned}$$

And,

$$\begin{aligned} [\pi_n(AB)]_{ij} &= \pi(\sum_{k=1}^{n} a_{ik} b_{kj}) = \sum_{k=1}^{n} \pi(a_{ik})\pi(b_{kj}) \\ &= [\pi_n(A)\pi_n(B)]_{ij} \end{aligned}$$

Also,

$$\begin{aligned} [\pi_n(A^*)] &= \pi(a_{ij}^*) = (\pi(a_{ij}))^* \\ &= [\pi_n(A)]_{ij}^* \end{aligned}$$

Thus, π_n is a *-homomorphism.

Further, recall that the C^* norm on $\mathcal{M}_n(B(H))$ is defined as follows.
Let

$$M = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \dots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

where $A_{ij} \in B(H)$.
Let $x = (x_1, x_2, ...x_n) \in H^{(n)}$, so

$$Mx = \begin{bmatrix} \sum_{j=1}^{n} A_{ij} x_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj} x_j \end{bmatrix}.$$

Clearly, M is a linear operator from $H^{(n)}$ to $H^{(n)}$.
Given that $x = (x_1, ...x_n), y = (y_1, ...y_n) \in H^{(n)}$, the inner product

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$$

so the norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{\frac{1}{2}}$$

The operator norm is defined as

$$\|M\| := \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|}$$

We will show that M is a bounded linear operator. Let $M_j = \begin{bmatrix} m_{j1} \\ \vdots \\ m_{jn} \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.
Then, $\|Mx\|_2^2 = \sum_{j=1}^{n} \left\|{M_j}^T x\right\|^2$. Further, ${M_j}^T x = m_{j1}x_1 + ... + m_{jn}x_n$, and

$$\left\|{M_j}^T x\right\|^2 = \|m_{j1}x_1 + ... + m_{jn}x_n\|^2 \leq (\|m_{j1}\| \|x_1\| + ... + \|m_{jn}\| \|x_n\|)^2$$

by triangle inequality and submultiplicativity of the norm. Then, by the numerical Cauchy Schwarz inequality, we get, $\left\|{M_j}^T x\right\|^2 \leq \left\|{M_j}^T\right\|^2 \|x\|_2^2$. Hence, M is bounded. In addition, since $M \in B(H^{(n)})$ and $H^{(n)}$ is a Hilbert space, this norm satisfies the C^*-identity.

Then, we can define the norm on $\mathcal{M}_n(\mathcal{A})$ as

$$\|A\| := \|\pi_n(A)\| .$$

We will check that the properties of C^*-norm are satisfied.
First, by the subadditivity of operator norm,

$$\begin{aligned} \|A + B\| &= \|\pi_n(A + B)\| = \|\pi_n(A) + \pi_n(B)\| \\ &\leq \|\pi_n(A)\| + \|\pi_n(B)\| = \|A\| + \|B\| . \end{aligned}$$

Next,

$$\begin{aligned} \|\lambda A\| &= \|\pi_n(\lambda A)\| = \|\lambda \pi_n(A)\| \\ &= |\lambda| \|\pi_n(A)\| = |\lambda| \|A\| . \end{aligned}$$

Also, if $\|A\| = 0$,then $\|\pi_n(A)\| = 0$. Hence, $\pi_n(A) = 0$ which follows from the property of operator norm and since π is injective, $a_{ij} = 0$ for all i,j.
Further, $\|A\| = \|\pi_n(A)\|$ is non-negative.
In addition,

$$\begin{aligned} \|AB\| &= \|\pi_n(AB)\| = \|\pi_n(A)\pi_n(B)\| \\ &\leq \|\pi_n(A)\| \|\pi_n(B)\| = \|A\| \|B\| . \end{aligned}$$

Finally, this norm satisfies the C^*-identity:

$$\begin{aligned}\|A^*A\| &= \|\pi_n(A^*A)\| = \|\pi_n(A^*)\pi_n(A)\| \\ &= \|\pi_n(A)^*\pi_n(A)\| = \|\pi_n(A)\|^2 \\ &= \|A\|^2 .\end{aligned}$$

Hence, we have proved the following theorem:

Theorem 4.1.1. *There exists a norm on $\mathcal{M}_n(\mathcal{A})$ which turns it into a C^*-algebra.*

Chapter 5

Matrices over total system of projections

Let $P_1 \in \mathcal{B}(H)$ be an orthogonal projection. Then $P_2 := I - P$ is also an orthogonal projection, $P_1 P_2 = 0 = P_2 P_1$ and $P_1 + P_2 = I$. We note that $H = R(P_1) \oplus R(P_2)$. This can be generalized to n projections, which then form a so called *total system of projections* in $\mathcal{B}(H)$.

5.1 Total projection systems in a C^*-algebra

Let $\mathcal{A}$ be a C^*-algebra. Let $\mathcal{P} = (p_1, p_2, ..., p_n)$ be a total system of projections in $\mathcal{A}$, that is $p_i^2 = p_i = p_i^*$, $p_i p_j = 0$ for $i \neq j$, and $\sum_{i=1}^n p_i = 1$.

The object of our discussion in this section will be a set of $n \times n$ matrices with entries $p_i a p_j$ where $a \in \mathcal{A}$ and p_i, p_j are elements of a total system of projections. Adapting [4] which shows the corresponding result in a Banach algebra, we will show that this set, equipped with an appropriate norm, is a C^*-algebra.

Consider the set $\mathcal{M}_n(\mathcal{A}, \mathcal{P}) \subset \mathcal{M}_n(\mathcal{A})$ where $A \in \mathcal{M}_n(\mathcal{A}, \mathcal{P})$ defined as $A := [a_{ij}]$ with $a_{ij} \in p_i \mathcal{A} p_j$.

We define a norm on $\mathcal{M}_n(\mathcal{A}, \mathcal{P})$ as

$$\|A\| := \left\| \sum_{i,j=1}^n a_{ij} \right\|$$

To check that the properties of norm are satisfied, first,

$$\begin{aligned}\|A+B\| &= \left\|\sum_{i,j=1}^{n}(a_{ij}+b_{ij})\right\| \leq \left\|\sum_{i,j=1}^{n}a_{ij}\right\| + \left\|\sum_{i,j=1}^{n}b_{ij}\right\| \\ &= \|A\|+\|B\|\end{aligned}$$

Next,

$$\begin{aligned}\|\lambda A\| &= \left\|\sum_{i,j=1}^{n}\lambda a_{ij}\right\| = |\lambda|\left\|\sum_{i,j=1}^{n}a_{ij}\right\| \\ &= |\lambda|\,\|A\|\end{aligned}$$

Further, $\|A\| = \left\|\sum_{i,j=1}^{n}a_{ij}\right\| \geq 0$.
Lastly, we will show that $\|A\| = 0$ implies that $A = 0$. $\|A\| = \left\|\sum_{i,j=1}^{n}a_{ij}\right\| = 0$ implies that $\sum_{i,j=1}^{n}a_{ij} = 0$. Note that $a_{ij} = p_i a_{ij} pj$. For any $s, t \in 1, .., n$,

$$0 = p_s(\sum_{i,j=1}^{n}a_{ij})p_t = \sum_{i,j=1}^{n}p_s a_{ij} p_t = 0 = \sum_{i,j=1}^{n}p_s p_i a_{ij} p_j p_t = p_s a_{st} p_t = a_{st} = 0.$$

Hence, $A = 0$.

Our aim is to show that $\mathcal{M}_n(\mathcal{A},\mathcal{P})$ equipped with this norm is a C^*-algebra. This is done by showing that $\mathcal{M}_n(\mathcal{A},\mathcal{P})$ is isometrically *-isomorphic to $\mathcal{A}$ itself. As a consequence of this, by the uniqueness of C^*-norm, the C^*-norm in $\mathcal{M}_n(\mathcal{A},\mathcal{P})$ is given by

$$\|A\| = \left\|\sum_{i,j=i}^{n}a_{ij}\right\| = \|\pi_n(A)\|$$

.

Consider $\varphi : \mathcal{A} \mapsto \mathcal{M}_n(\mathcal{A},\mathcal{P})$ defined by

$$\varphi(x) = \begin{bmatrix} p_1 x p_1 & \cdots & p_1 x p_n \\ \vdots & \ldots & \vdots \\ p_n x p_1 & \cdots & p_n x p_n \end{bmatrix} \tag{5.1}$$

We will show that φ is an isometric *-isomorphism from $\mathcal{A}$ to $\mathcal{M}_n(\mathcal{A},\mathcal{P})$, which implies that $\mathcal{M}_n(\mathcal{A},\mathcal{P})$ is a C^*-algebra
First,

$$\begin{aligned}[\varphi(x+y)]_{ij} &= p_i(x+y)p_j = p_i x p_j + p_i y p_j \\ &= [\varphi(x)]_{ij} + [\varphi(y)]_{ij}\end{aligned}$$

So, $\varphi(x+y) = \varphi(x) + \varphi(y)$.
And,

$$[\varphi(xy)]_{ij} = p_i(xy)p_j = p_i x(\sum_{k=1}^{n} p_k)yp_j$$
$$= \sum_{k=1}^{n}(p_i x p_k)(p_k y p_j) = [\varphi(x)\varphi(y)]_{ij}$$

That is, $\varphi(xy) = \varphi(x)\varphi(y)$. Thus, φ is a homomorphism.
Next, let $x = \sum_{i,j=1}^{n} a_{ij} \in \mathcal{M}_n(\mathcal{A}, \mathcal{P})$. Then,

$$p_s x p_t = p_s(\sum_{i,j=1}^{n} a_{ij})p_t = p_s a_{st} p_t = a_{st}.$$

Hence, φ is surjective.
Note that for each $x \in \mathcal{A}$, $x = \sum_{i,j=1}^{n} p_i x p_j$, since

$$x = \left(\sum_{i=1}^{n} p_i\right) x \left(\sum_{j=1}^{n} p_j\right) = \sum_{i,j=1}^{n} P_i x p_j.$$

Hence, $\|\varphi(x)\| = \|x\|$, that is φ preserves norm.
Also, if $\varphi(x) = 0$, then $\|\varphi(x)\| = \|x\| = 0$, so $x = 0$ and φ is injective.
In addition,

$$[\varphi(x^*)]_{ij} = p_i x^* p_j = (p_j x p_i)^* = [\varphi(x)]_{ij}^*$$

which implies that $\varphi(x^*) = \varphi(x)^*$ and φ is a *-homomorphism. Thus, φ is an isometric *-isomorphism. The identity in $\mathcal{M}_n(\mathcal{A}, \mathcal{P})$ is given by

$$I = \varphi(1_\alpha) = \begin{bmatrix} p_1 & 0 & 0 & ... & 0 \\ 0 & p_2 & 0 & ... & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & ... & ... & p_n \end{bmatrix}$$

Thus, we have proved the following

Theorem 5.1.1. *There exists a norm on $\mathcal{M}_n(\mathcal{A}, \mathcal{P})$ which turns it into a C^*-algebra.*

Chapter 6

Applications

In this chapter, we shall see some examples of the use of matrices over C^*-algebras. By making use of matrices over C^*-algebra, one can construct a more elegant proof.

6.1 Albert's Lemmas

In this section, we will present a new proof of Albert's Lemmas [6]. We will make use of the results in section 4 and 5. The method used here to establish these results is new. First, we will begin with some definitions [5].

Definition 6.1.1. $a \in \mathcal{A}$ is *regular* if there exists $b \in \mathcal{A}$ such that $aba = a$, that is $a \in a\mathcal{A}a$. Further, b is called the *inner inverse* of a, and we write $b := a^-$.

Definition 6.1.2. The *Moore-Penrose* inverse of an element $a \in \mathcal{A}$ is the unique element $a^\dagger \in \mathcal{A}$ satisfying the equations

$$aa^\dagger a = a,\ a^\dagger aa^\dagger = a^\dagger,\ (aa^\dagger)^* = aa^\dagger,\ (a^\dagger a)^* = a^\dagger a$$

Definition 6.1.3. Let

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in \mathcal{M}_2(\mathcal{A}).$$

Then the *Schur complement* $s(X) := x_{22} - x_{21}x_{11}{}^-x_{12}$.

To proceed, we will state a well-known theorem [7] which will be needed in our proof.

Theorem 6.1.4 (Harte-Mbekhta)**.** *Let $a \in \mathcal{A}$. Then, the followings are equivalent:*

(i) *a is regular.*

(ii) *a has Moore-Penrose inverse.*

(iii) $\overline{a\mathcal{A}} = a\mathcal{A}$, *that is* $a\mathcal{A}$ *is closed.*

In addition, we will need the following lemma in the proof:

Lemma 6.1.5. *Let* $a \in \mathcal{A}$. *If* a^*a *is regular, then* a *is regular.*

Proof. Let $b = (a^*a)^- a^*$.Then,

$$\begin{aligned} a^*aba &= a^*a(a^*a)^- a^*a \\ &= a^*a \end{aligned}$$

The law of *-cancellation states that if $a^*x = a^*y$, then $x = y$. This follows from the C^*-identity. Thus, we get $aba = a$. □

Lemma 6.1.6. *Consider* $X = \begin{bmatrix} x_{11} & 0 \\ 0 & x_{22} \end{bmatrix}$ *with* x_{11} *and* x_{22} *self-adjoint.* X *is positive is and only if* x_{11} *and* x_{22} *are positive.*

Proof. Suppose that x_{11} and x_{22} are positive. Consider $\pi_2(X)$, where the map π_2 is as defined in 4.1.

$$\pi_2(X) = \begin{bmatrix} \pi(x_{11}) & 0 \\ 0 & \pi(x_{22}) \end{bmatrix}$$

and π is the Gelfand representation. For any $a = [a_1\ a_2]$,

$$\begin{aligned} \langle \pi_2(X)a, a\rangle &= \langle \pi(x_{11})a_1, a_1\rangle + \langle \pi(x_{22})a_2, a_2\rangle \\ &\geq 0 \end{aligned}$$

Hence, $\pi_2(X) \geq 0$, and therefore X is positive. Next, suppose that X is positive. For any $a = [a_1\ a_2]$, $\langle \pi_2(X)a, a\rangle \geq 0$. Take $a = [0\ y]$. Then,

$$\begin{aligned} \langle \pi_2(X)a, a\rangle &= \langle \pi(x_{11})0, 0\rangle + \langle \pi(x_{22})y, y\rangle \\ &= 0 + \langle \pi(x_{22})y, y\rangle \\ &= \langle \pi(x_{22})y, y\rangle \geq 0 \end{aligned}$$

for any $y \in \mathcal{A}$. Hence, x_{22} is positive. Similarly, take $a = [y\ 0]$ and we will get x_{11} is positive. □

Now we are ready to state the main result. This result consists of two Lemmas. The first Lemma will characterise the positiveness of matrices over C^*-algebras. The second Lemma will characterise the positiveness of elements of C^*-algebras. The proof of the second Lemma will make use of the result of the first Lemma.

Lemma 6.1.7 (Albert's Lemma 1). *Let*

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

be a matrix over a C^*-algebra $\mathcal{A}$ such that X is self-adjoint and x_{11} is regular. Then $X \geq 0$ if and only if

(i) $x_{11} \geq 0$.

(ii) $x_{11}{x_{11}}^{-}x_{12} = x_{12}$.

(iii) $s(X) = x_{22} - x_{21}{x_{11}}^{-}x_{12} \geq 0$.

Proof. Since X is self-adjoint, x_{11} and x_{22} are self-adjoint and ${x_{12}}^* = x_{21}$.

Assume that conditions (1)-(3) hold. Define

$$S := \begin{bmatrix} 1 & -{x_{11}}^{-}x_{12} \\ 0 & 1 \end{bmatrix}.$$

S is invertible with an inverse

$$S^{-1} = \begin{bmatrix} 1 & {x_{11}}^{-1}x_{12} \\ 0 & 1 \end{bmatrix}$$

Consider the matrix $A = S^*XS$. We will show that A is positive, which will imply that $(S^{-1})^*(S^*XS)S^{-1} = (S^*)^{-1}(S^*XS)S^{-1} = X$ is positive by Lemma 2.3.8. Now,

$$A = S^*XS = \begin{bmatrix} x_{11} & 0 \\ 0 & s(X) \end{bmatrix}.$$

We apply the function π_2 as defined in 4.1.

$$\pi_2(A) = \begin{bmatrix} \pi(x_{11}) & 0 \\ 0 & \pi(s(X)) \end{bmatrix}$$

where π is the Gelfand representation of $\mathcal{A}$. Then, for $u = (x, y) \in H^2$,

$$\begin{aligned} \langle \pi_2(A)(u), u \rangle &= \langle \begin{bmatrix} \pi(x_{11}) & 0 \\ 0 & \pi(s(X)) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle \\ &= \langle \begin{bmatrix} \pi(x_{11}) & x \\ \pi(s(X)) & y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \rangle \\ &= \langle \pi(x_{11})x, x \rangle + \langle \pi(s(X))y, y \rangle \\ &\geq 0 \end{aligned}$$

since π is an injective *-homomorphism, and therefore preserves the positivity. Thus, $\pi_2(A)$ is positive, and since π_2 is an injective *-homomorphism, this implies that A is positive. Hence, we have proved that X is positive.

Next, let $X \geq 0$. By Lemma 2.3.7, $X = Y^*Y$ for some $Y \in \mathcal{M}_2(\mathcal{A})$. Write

$$Y_1 = \begin{bmatrix} y_{11} & 0 \\ y_{21} & 0 \end{bmatrix}, \ Y_2 = \begin{bmatrix} 0 & y_{12} \\ 0 & y_{22} \end{bmatrix}$$

Then, $Y = Y_1 + Y_2$, and so $X = Y^*Y = (Y_1+Y_2)^*(Y_1+Y_2) = Y_1^*Y_1 + Y_1^*Y_2 + Y_2^*Y_1 + Y_2^*Y_2$. Also,

$$Y_1^*Y_1 = \begin{bmatrix} x_{11} & 0 \\ 0 & 0 \end{bmatrix}, \ Y_1^*Y_2 = \begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix}$$

$$Y_2^*Y_1 = \begin{bmatrix} 0 & 0 \\ x_{21} & 0 \end{bmatrix}, \ Y_2^*Y_2 = \begin{bmatrix} 0 & 0 \\ 0 & x_{22} \end{bmatrix}$$

Lemma 6.1.6 implies that x_{11} is positive since $Y_1^*Y_1$ is positive by Theorem 2.3.6.

Let $Z = Y_1^*Y_1$ and

$$W = \begin{bmatrix} x_{11}{}^- & 0 \\ 0 & 0 \end{bmatrix}$$

Then,

$$\begin{aligned} ZWZ &= \begin{bmatrix} x_{11}x_{11}{}^-x_{11} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & 0 \\ 0 & 0 \end{bmatrix} \\ &= Z \end{aligned}$$

which implies that Z is regular.
By Lemma 6.1.5, Y_1^* is also regular, and $(Y_1^*)^- = Y_1(Y_1^*Y_1)^-$.
We have

$$\begin{aligned} x_{11}x_{11}{}^-x_{12} &= Y_1^*Y_1(Y_1^*Y_1)^-Y_1^*Y_2 \\ &= Y_1^*(Y_1^*)^-Y_1^*Y_2 \\ &= Y_1^*Y_2 \\ &= x_{12} \end{aligned}$$

Next, by Lemma 2.3.8, $A = S^*XS = \begin{bmatrix} x_{11} & 0 \\ 0 & s(X) \end{bmatrix}$ is positive since X is positive. Hence, $s(X)$ is positive by Lemma 6.1.6. □

In [6], Albert proved this Lemma for finite block matrices. The method presented above also applies to finite block matrices.

Lemma 6.1.8 (Albert's Lemma 2)**.** *Let $\mathcal{P} = (p1, p2)$ be a total system of projections and self-adjoint $X \in \mathcal{M}_2(\mathcal{A})$*

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

with $x_{ij} = p_i x p_j$ for some $x \in \mathcal{A}$ and x_{11} is regular. Then, x is positive if and only if

(i) $x_{11} \geq 0$.

(ii) $x_{11}x_{11}{}^{-}x_{12} = x_{12}$.

(iii) $s(X) = x_{22} - x_{21}x_{11}{}^{-}x_{12} \geq 0$.

Proof. Apply the function $\varphi : \mathcal{A} \mapsto \mathcal{M}_n(\mathcal{A}, \mathcal{P})$ as defined in 5.1. φ is an isometric *-isomorphism. Thus, x is positive if and only if X is positive. By Albert's Lemma 1, x is positive if and only if (1)-(3) hold. □

6.2 2 × 2 matrices over C^*-algebras

Applying 2×2 matrices over C^*-algebra, one can immediately solve some problems which can not be treated easily in the C^*-algebra itself. We will give some examples of these in this section. In particular, we will prove the Fuglede-Putnam theorem and various applications due to Moslehian [11].

6.2.1 Fuglede-Putnam Theorem

We will present here a result concerning normal elements in a C^*-algebra by using matrices over C^*-algebras. First, we will need the following result given in [3]:

Theorem 6.2.1 (Fuglede's Theorem)**.** *Let a be a normal element of $\mathcal{A}$. Then a^* double commutes with a.*

Proof. Let $b \in \mathcal{A}$ such that b commutes with a. Then, b commutes with all powers of a, so b commutes with $p(a)$ where p is a polynomial. Further, b commutes with $f(a)$ where f is a uniform limit of polynomials on $\mathbf{Sp}(a)$. In particular, b commutes with $\exp(\mu a)$ for any complex μ. Thus, for any $\lambda \in \mathbb{C}$,

$$\exp(\lambda a^*)\ b\ \exp(-\lambda a^*) = \exp(\lambda a^*)\ \exp(-\overline{\lambda} a)\ b\ \exp(\overline{\lambda} a)\ \exp(-\lambda a^*)$$

As $aa^* = a^*a$, we get

$$\exp(\lambda a^*)\ b\ \exp(-\lambda a^*) = \exp(\lambda a^* - \overline{\lambda} a)\ b\ \exp(\overline{\lambda} a - \lambda a^*)$$

Further, for any $x \in \mathcal{A}$, the element $-i(x - x^*)$ is self-adjoint. Thus,

$$g(\lambda) = -i(\lambda a^* - \overline{\lambda} a) = -i(\lambda a^* - (\lambda a^*)^*)$$

is self-adjoint. Hence

$$\|\exp(ig(\lambda))\| = \|\exp(-ig(\lambda))\| = 1$$

and so

$$\|\exp(\lambda a^*)\ b\ \exp(-\lambda a^*)\| = \|\exp(ig(\lambda))\ b\ \exp(-ig(\lambda))\| \leq \|b\|$$

The map $F(\lambda) = \exp(\lambda a^*)\ b\ \exp(-\lambda a^*)$ is therefore a bounded entire function. By Liouville's theorem, F is a constant function. Hence,

$$b = \exp(\lambda a^*)\ b\ \exp(-\lambda a^*)$$

and therefore,

$$\exp(\lambda a^*)\ b = b\ \exp(\lambda a^*)$$

for all $\lambda \in \mathbb{C}$. Equating coefficients of λ on both sides, we get $a^*b = ba^*$. □

Now, we will employ matrices over C^*-algebras, together with the theorem above to prove the following theorem:

Theorem 6.2.2 (Fuglede-Putnam Theorem)**.** *Let a, b be normal elements of $\mathcal{A}$. If $c \in \mathcal{A}$ such that $ac = cb$, then $a^*c = cb^*$.*

Proof. Let

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$$

Then, $A, B \in \mathcal{M}_2(\mathcal{A})$. A is normal, since

$$\begin{aligned} A^*A &= \begin{bmatrix} a^* & 0 \\ 0 & b^* \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^*a & 0 \\ 0 & b^*b \end{bmatrix} \\ &= \begin{bmatrix} aa^* & 0 \\ 0 & bb^* \end{bmatrix} = AA^* \end{aligned}$$

Also,

$$\begin{aligned} AB &= \begin{bmatrix} 0 & ac \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & cb \\ 0 & 0 \end{bmatrix} \\ &= BA \end{aligned}$$

That is, A commutes with B.
By Fuglede's theorem, $A^*B = BA^*$,

$$\begin{bmatrix} 0 & a^*c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & cb^* \\ 0 & 0 \end{bmatrix}$$

Hence, we get $a^*c = cb^*$. □

6.2.2 Moslehian's results

In his paper [11], Moslehian surveys the applications of 2×2 matrices over C^*-algebras to simplify various problems in functional analysis. Here we will present some of them.

The following two theorems will concern linear maps between C^*-algebras. Our aim is to show how the proofs of these results are simplified by utilizing 2×2 matrices over C^*-algebras. Before we state the first theorem, note the following definition:

Definition 6.2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. A linear map $\phi : \mathcal{A} \mapsto \mathcal{B}$ is said to be *completely positive* if for all n, $\phi_n : \mathcal{M}_n(\mathcal{A}) \mapsto \mathcal{M}_n(\mathcal{B})$ given by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$ is positive.

Theorem 6.2.4. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\phi : \mathcal{A} \mapsto \mathcal{B}$ is a completely positive linear map with $\phi(1) = 1$. Then, for each $a \in \mathcal{A}$, $\phi(a^*a) \geq \phi(a)^*\phi(a)$.*

Proof. Note that $\begin{bmatrix} 1 & a \\ a^* & a^*a \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \geq 0$. By the complete positivity of ϕ, we get $\begin{bmatrix} \phi(1) & \phi(a) \\ \phi(a^*) & \phi(a^*a) \end{bmatrix} = \begin{bmatrix} 1 & \phi(a) \\ \phi(a^*) & \phi(a^*a) \end{bmatrix} \geq 0$. Thus, $\phi(a^*a) \geq \phi(a^*)\phi(a)$. □

To prove the next result, we need the followings:

Lemma 6.2.5 (Kadison's inequality)**.** *If $\Phi : \mathcal{A} \mapsto \mathcal{B}$ is a positive unital linear map between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, then for every normal element $a \in \mathcal{A}$*

$$\Phi(a^*a) \leq \Phi(a)^*\Phi(a)$$

Definition 6.2.6. $\Phi : \mathcal{A} \mapsto \mathcal{B}$ is a contraction if and only if $\|\Phi\| \leq 1$.

An interesting fact is that a positive unital linear map is a contraction:

Theorem 6.2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\phi : \mathcal{A} \mapsto \mathcal{B}$ be a unital positive linear map. Then, ϕ is a contraction.*

Proof. Let $u = \begin{bmatrix} a & (1-aa^*)^{1/2} \\ (1-aa^*)^{1/2} & -a^* \end{bmatrix} \in \mathcal{M}_2(\mathcal{A})$ where $a \in \mathcal{A}$ and $\|a\| \leq 1$. Note that u is unitary. Define a map $\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \phi(a)$. ψ is a positive unital linear map. Applying Kadison's inequality, $\|\phi(a)\| = \|\psi(u)\| = \|\psi(u)^*\psi(u)\|^{1/2} \leq \|\psi(u^*u)\|^{1/2} = \|\psi(1)\| = 1$. Hence, ϕ is a contraction. □

We know that for every $x \in \mathcal{A}^+$, there exists $y \in \mathcal{B}$ such that $x = y^*y$. Now, the next result states that for every $x \in \mathcal{A}$, there exists $y \in \mathcal{A}$ such that $x = yy^*y$, and we will use matrices to prove this theorem.

Theorem 6.2.8. *For each element $x \in \mathcal{A}$, there exists a unique $y \in \mathcal{A}$ such that $x = yy^*y$.*

Proof. Consider $\begin{bmatrix} 0 & x^* \\ x & 0 \end{bmatrix}$. This matrix is self-adjoint. Further

$$\begin{bmatrix} 0 & x^* \\ x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence, by functional calculus,

$$\begin{bmatrix} 0 & x^* \\ x & 0 \end{bmatrix}^{1/3} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & x^* \\ x & 0 \end{bmatrix}^{1/3}$$

Let $\begin{bmatrix} 0 & x^* \\ x & 0 \end{bmatrix}^{1/3} = \begin{bmatrix} u & y^* \\ y & v \end{bmatrix}$ We get $u = v = 0$ and $\begin{bmatrix} 0 & x^* \\ x & 0 \end{bmatrix} = \begin{bmatrix} 0 & y^* \\ y & 0 \end{bmatrix}^3$ implies that $x = yy^*y$. □

6.3 Common self-adjoint solutions

Here, we will use matrices over C^*-algebras to prove the existence of self-adjoint solutions to system of linear equations in C^*-algebras.

First, note the following result given in [9].

Theorem 6.3.1. *Let $a, c \in \mathcal{A}$ and a be regular. Then, $ax = c$ has self-adjoint solutions if and only if $aa^-c = c$ and ca^* is self-adjoint*

Recall that a^- is an inner inverse to a regular a, that is $aa^-a = a$.

Define the matrices

$$A = \begin{bmatrix} a & 0 \\ b^* & 0 \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 \\ d^* & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that matrix X is self-adjoint if and only if x is self-adjoint.
Consider the equations

$$\begin{aligned} ax &= c \\ xb &= d. \end{aligned} \tag{6.1}$$

We can write these equations as

$$\begin{aligned} ax &= c \\ b^*x &= d^* \end{aligned}$$

and turn this into matrix equation $AX = C$. Then, equations 6.1 will have a common self-adjoint solution x if and only if the matrix equation $AX = C$ has a self adjoint solution $X = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$. Theorem 6.3.1 shows that $AX = C$ has a self adjoint solution if and only if $AA^-C = C$ and CA^* is self-adjoint. Thus, we get the following theorem:

Theorem 6.3.2. *Let $a, b, c, d \in \mathcal{A}$. Let $m = b^*(a - a^-a)$ and a, b, m be regular. Let matrices A, C, X be defined as above. Then A is regular. The equations $ax = c$ and $xb = d$ have a common self-adjoint solution if and only if*

$$AA^-C = C \quad \textit{and} \quad CA^* \quad \textit{is self adjoint.}$$

It was shown in [10] that an inner inverse of A exists and is given by

$$A^- = \begin{bmatrix} a^- - (1-a^-a)m^-b^*a^- & (1-a^-a)m^- \\ 0 & 0 \end{bmatrix}.$$

Then,

$$AA^-C = \begin{bmatrix} a & 0 \\ b^* & 0 \end{bmatrix} \begin{bmatrix} a^- - (1-a^-a)m^-b^*a^- & (1-a^-a)m^- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ d^* & 0 \end{bmatrix} = \begin{bmatrix} aa^-c & 0 \\ (1-mm^-)b^*a^-c + mm^-d^* & 0 \end{bmatrix}.$$

Hence, $AA^-C = C$ if and only if

$$aa^-c = c \quad \text{and} \quad (1-mm^-)b^*a^-c = (1-mm^-)d^*$$

Next,

$$CA^* = \begin{bmatrix} c & 0 \\ d^* & 0 \end{bmatrix} \begin{bmatrix} a^* & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ca^* & cb \\ d^*a^* & d^*b \end{bmatrix}$$

Therefore, CA^* is self-adjoint if and only if

$$ad = cb \quad \text{and} \quad ac^*, b^*d \quad \text{are self-adjoint.}$$

Hence, we have proved the following theorem:

Theorem 6.3.3. *Let $a, b, c, d \in \mathcal{A}$. Let $m = b^*(1-a^-a)$ and a, b, m be regular. Then the equations $ax = b$ and $xb = d$ have a common self-adjoint solutions if and only if*

$$aa^-c = c \quad \textit{and} \quad (1-mm^-)b^*a^-c = (1-mm^-)d^*$$

and

$$ad = cb \quad \textit{and} \quad ac^*, b^*d \quad \textit{are self-adjoint.}$$

Bibliography

[1] J.J Koliha, *Lecture Notes on Banach and C^*-algebras*, University of Melbourne, 2005.

[2] Gerard J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, Inc., 1990.

[3] J.J Koliha, *Spectral Theory of Operators*, University of Melbourne, 2006.

[4] N. Castro González and J.J Koliha, *New Additive Results for the g-Drazin inverse*, Proceedings of the Royal Society of Edinburgh, **134A** (2004), 1085-1097.

[5] Dragana S. Cvetković-Ilić, Dragan S. Djordević and Vladimir Rakočević, *Schur Complements in C^*-algebras*, Math. Nachr, **278**, No. 7-8, (2005), 808-814.

[6] Arthur Albert, *Conditions for Positive and Nonnegative Definiteness in Terms of Pseudoinverses*, SIAM Journal on Applied Mathematics, **17**, No. 2, (1969), 434-440.

[7] R.E Harte and M. Mbekhta, *On Generalized Inverses in C^*-algebras*, Studia Math, **103**, (1992), 71-77.

[8] Jacques Dixmier, *C^*-algebras*, North-Holland Publishing Company, 1977.

[9] C.G Khatri and S.K Mitra, *Hermitian and Nonnegative Definite Solutions of Linear Matrix Equations*, SIAM Journal of Applied Mathematics, **31**, (1976), 579-585.

[10] P.Patrício and R.Puystjens, *About the von-Neumann Regularity of Triangular Block Matrices*, Linear Algebra Appl, **332-334**, (2001), 485-502.

[11] Mohammad Sal Moslehian, *Trick with 2×2 Matrices over C^*-algebras*, Austral. Math. Soc. Gaz, **30**, (2003), no. 3, 150-157.