Studies of Barrier Options and their Sensitivities

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Abstract

Barrier options are cheaper than the respective standard European options, because a zero payoff may occur before expiry time $T$. Lower premiums are usually offered for more exotic barrier options, which make them particularly attractive to hedgers in the financial market. Under the Black-Scholes framework, we explicitly derive and present pricing formulae for a range of different European barrier options depending the options barrier variety, direction, activation time and whether it will be a call or put. A new pricing formulae is also presented, which to the best of our knowledge has not yet appeared in the literature. We compare numerical results of analytical formulae for option prices with Monte Carlo simulation where efficiency is improved via the variance reduction technique of antithetic variables. We also present numerical results for sensitivity estimation. We used finite differences to estimate the values of two Greeks, the Delta and the Eta, that characterise the changes in the specified options prices in response to small changes in the initial asset price $S_0$ and barrier height $H$. 
Politics is for the present, but an equation is for eternity...

Albert Einstein
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Chapter 1

Introduction

Barrier options are financial derivative securities that are cheaper than the respective standard vanilla options. They differ from ordinary options in that the underlying’s price must either touch or not touch a specified barrier $H$ before or on the expiry $T$. This also depends on whether the barrier is hit from above or from below and the period during which the underlying price is monitored for barrier hits. There are two broad types of barrier options: a kick-out option, which results in a zero payoff if the barrier is hit, and a kick-in option, which results in zero payoff if the barrier is not hit. This indeed does make the option price cheaper compared to vanilla options, since it may expire worthless if the barrier is touched (or not touched) in the situation in which the vanilla option would have paid off.

Under the Black-Scholes model, such options can be priced under the assumption that the price follows the Geometric Brownian Motion. Under this assumption, we can price barrier options in terms of boundary hitting probabilities for the Brownian Motion. We price barrier options by evaluating the integral of its expected payoff under a risk-neutral probability measure. We first condition on the values of the price process at the starting and ending points of the barrier. We then multiply the integral by the boundary hitting probability for the Brownian Motion, which can be found using the Brownian Bridge property. This eventually leads to dealing with linear combinations of exponentials. For standard barrier options, where the barrier starts at 0 and ends at maturity time $T$, pricing is not very difficult and closed form solutions have been available for some time. Pricing becomes less elementary when the barrier time-period is shorter or when combinations of kick-in and kick-out barriers are considered during the option’s lifetime. In most cases one can use Monte Carlo simulation, which has proven to be an effective and simple tool for pricing more complex structures of options. Estimating payoff prices can be improved by introducing variance reduction techniques. When dealing with financial securities, most practitioners also seek information on option sensitivities, which are used for hedging purposes and risk management. Sensitivities are also used for investigating small changes in pricing formula with respect to some underlying parameter. In terms of pricing formulae, we take the partial derivative of the formula with respect to the parameter of interest. In addition to this, sensitivities can also be estimated by using simulation, where one often then requires the use of numerical differentiation.

In this present thesis, we concentrate on deriving closed-form formulae for the expected
payoff price for a number of different barrier options and verify their accuracy using Monte Carlo simulation. A variety of different types of barrier options will be considered. Since more complex barrier options are built on concepts from simpler ones, we begin by deriving and presenting the pricing formula for a standard barrier option. We will present pricing formulae for partial-time barrier options, where different barrier monitoring times are considered. Using techniques for partial-time barrier options, we then derive and present the pricing formula for a window barrier option. Finally, we present pricing formulae for two barrier options, which to our knowledge have not yet been presented in the literature. We will also look at their respective sensitivities, Delta and Eta, and compare our numerical results with simulation.

Chapter 3 covers the fundamentals of financial theory and the mathematical concepts that will be used throughout the thesis. This will include a discussion of the Black-Scholes model, the Brownian bridge properties, option sensitivities and approximation for bivariate and trivariate normal distributions. In Chapter 4, we discuss different barrier options types and their properties that will be considered for pricing purposes. Chapter 5 covers the techniques that will be used for simulation, where the basics of Monte Carlo simulation and numerical differentiation are discussed. In Chapter 7 we derive and present closed form formulae for a selected number of barrier options. Chapter 8 compares numerical results obtained from formulae and their respective sensitivities with the results of corresponding Monte Carlo simulations. Conclusions and proposals for further work are presented in chapter 9. Simulation coding was done in R, and we also used Mathematica for differentiation of the very complex formulae of barrier options’ prices. Codes used for simulation, Mathematica output and extensions to proofs can be found in the Appendix.
Chapter 2

Literature Review

The birth of theoretical options pricing came from the groundbreaking paper by Black and Scholes (1973) with joint works of Merton (1973). Together, they showed that under certain conditions, one could perfectly hedge the profits or losses of a European vanilla option, by following a self-financing replicating portfolio strategy. This gave rise to the first successful option pricing formula for a European call option, aptly named the Black-Scholes formula. Since then, a large interest in theory and simulation for derivatives securities was developed. Merton (1973) extended on the Black-Scholes model by pricing a standard barrier call option. This was then further extended by the works of Reiner and Rubinstein (1991), where formulae are presented for every type of standard barrier option. In the early nineties more complicated structures for barriers were studied; Kunitomo and Ikeda (1992) price barrier options with curved boundaries, Heynen and Kat (1994) derive pricing formulae for partial-time barrier options, where barriers are active for only a period of the option lifetime, Geman and Yor (1996) look at double barrier options, where the underlying price is sandwiched between a barrier from above and barrier from below, and more recently Armstrong (2001) derives the pricing formulae for window barrier options, where the barrier is only active for some period of time, between 0 and the maturity time $T$.

Monte Carlo simulation was first introduced for derivative securities by Boyle (1977), where the payoff was simulated for vanilla options, and several variance reduction techniques were used. This was further extended by Boyle, Brodie and Glasserman (1997), where more variance reduction techniques are discussed and Monte Carlo simulation was used for Asian options, barrier options and American options. Conditional Monte Carlo and Quasi Monte Carlo was also introduced and some numerical differential techniques for estimating price sensitivities are discussed and presented. Brodie and Glasserman (1996) present three approaches for estimating security price derivatives, illustrated by numerical results.
Chapter 3

Preliminaries

3.1 Options

Options are financial instruments which are bought and sold in a market place. They are known as derivatives securities, a.k.a contingent claims, whose characteristics and value depend on the characteristics and value of the underlying asset.

**Definition 3.1.** A derivatives security with maturity date $T$ is a function

$$X = X(\omega) = g(S_T(\omega)) \geq 0$$

of the underlying asset price $S_T$ at time $T$, $\forall \omega$, where $\omega \in$ the sample space $\Omega$.

Contracts that pay owners of the claims the amount $X$ at the time of maturity $T$ are referred to as a claims. In financial terms, options are contracts that give buyers the right, but not the obligation, to perform a specified transaction at some specified strike price $K$ on or before a specified time.

**Definition 3.2.** A call option is a contract which gives the holder (or owner) of the option the right to buy.

**Definition 3.3.** A put option is a contract which gives the holder (or owner) of the option the right to sell.

Options can be on stock, currency exchange, fuel etc. We use options to reduce risk. The two common types of options are European and American. A European option has fixed maturity time which we normally denote by $T$ and can only be exercised at this maturity time, otherwise the option simply expires. An American option is more flexible: one can exercise it at any time before or at the maturity time $T$. American options tend to be harder to understand than European options, therefore we only focus on the latter throughout the thesis.

3.1.1 Payoffs

The price which is paid for the asset when our European option is exercised is called the Strike Price and will be denoted as $K$. Our profit on the maturity date is known as the
payoff and can be summarized as follows:

For a European call option with the strike $K$ on one share with price at maturity $S_T$ and expiry $T$;

$$\text{Payoff} = \max[0, (S_T - K)] = (S_T - K)^+. $$

This is a contingent claim with $X = g(S_T) = (S_T - K)^+ \geq 0$. Figure 3.1 shows its value at exercise as a function of the price of the underlying.

![Figure 3.1: Payoff for a Call Option with $K = 100$.](image)

Similarly, for a European put option with the strike price $K$ on one share $S_T$ and expiry $T$;

$$\text{Payoff} = \min[0, (S_T - X)] = (S_T - X)^-. $$

This is also a contingent claim with $X = g(S_T) = (S_T - K)^- \geq 0$, we illustrate this in Figure 3.2.

![Figure 3.2: Payoff for a Put Option with $K = 100$.](image)

For most of the options considered in this thesis we will deal with a more general form, with a payoff $g(S_t, t \in [0, T])$.
3.1.2 Arbitrage and Portfolios

An arbitrage opportunity can be defined as a “free lunch”, that is we make profit by buying and selling something without taking any risk. Opportunities for arbitrage are very short-lived as many traders will seek to advantage from such opportunities and gradually move the market where these opportunities can no longer exist. A common principle in economics says that “there is no such thing as a free lunch”, which states that there exists no arbitrage or the market is arbitrage-free. When working in real-life financial markets, it becomes reasonable to assume that all prices are such that no arbitrage is possible. The arbitrage pricing theory with can be found in most introductory financial text books, so we will only briefly discuss this topic. We will see later on in this section the role it plays when using the Black-Scholes model, but first we must give some definitions on which the Black-Scholes model was built on. Proofs for the following definitions and theorems can be found in Klebaner (2005) pp.289-304.

The main principal for pricing financial securities with no arbitrage is to construct a portfolio of stock and bond to replicate the payoff of the security at maturity time \( T \). The portfolio’s value must equal the price of the derivative security at all times, such that it excludes any arbitrage opportunities. We construct the portfolio as follows. Suppose we have a portfolio consisting of two instruments; \( \pi(t) \) shares of a risky with a price \( S_t \) at time \( t \) and \( b(t) \) units of bond held at time \( t \) with price \( B_t \) each. Therefore the value of the portfolio at time \( t \) is given by

\[
V(t) = \pi(t)S_t + b(t)B_t.
\]

Note that we will be working with continuous time models. A portfolio consisting of \( (\pi(t), b(t)) \) is called self-financing, if the \( V(t) \) obeys the following:

\[
V(t) = V(0) + \int_0^t \pi(u)dS_u + \int_0^t b(u)dB_u,
\]

the change in \( V(t) \) only comes from changes in the prices of assets that constructed it. That is, no more money is added in or withdrawn from the portfolio during the time period \( (0, T) \). We denote by \( \mathbb{P} \) the “real-world” probability measure. For contingent claims we need the following definition:

**Definition 3.4.1.** A claim with a payoff \( X \) is said to be replicable if there exists a self-financing portfolio \( V(T) \geq 0 \) that can replicate this claim so that we have for any state of the world:

\[
X = V(T).
\]

The following result is central to the theory:

**Theorem 3.1.** Suppose there is a probability measure \( \mathbb{Q} \), such that the discounted stock process \( Z_t = S_t/B_t \) is a \( \mathbb{Q} - \) martingale. Then for any replicating trading strategy, the discounted value process \( V(t)/B_t \) is also a \( \mathbb{Q} - \) martingale.

The probability measure \( \mathbb{Q} \) can also be referred to as an equivalent martingale measure (EMM) or the risk-neutral probability measure.
Definition 3.4.2. Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on a common measurable space $(\Omega, \mathcal{F})$ are called equivalent if they have the same null sets, that is, for any set $A$ with $\mathbb{P}(A) = 0$ one also has, $\mathbb{Q}(A) = 0$ and vice versa.

Theorem 3.2. (First Fundamental Theorem ) A market model does not have arbitrage opportunities if and only if there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the discounted stock process $Z_t = S_t/B_t$ is $\mathbb{Q}$-martingale.

A market model is complete if any integrable claim is replicable. If a market model is complete, then any claim can be priced by no arbitrage. Furthermore the market model is complete if and only if the EMM $\mathbb{Q}$ is unique. We also need the following theorem for pricing claims

Theorem 3.3. (Fundamental asset pricing formula) The time $t = 0$ price of a claim with payoff $X$ is given by
$$ \frac{1}{B_T} \mathbb{E}_X, $$
where the expectation is taken under the EMM.

Theorem 3.3 will be become central to pricing options, this is also referred to as discounted price. Note that we will not be using dividend yields or rebates for the entire thesis.

3.2 Brownian Motion

The Brownian motion process (a.k.a. the Weiner process) becomes of great importance to modelling in financial mathematics.

Definition 3.5. A stochastic process $\{W_t\}_{t \geq 0}$ with $W_0 = 0$ is called a standard Brownian Motion Process if the following properties hold:
(i) stationary Gaussian(normal) increments:
$$ W_t - W_s \sim N(0, t-s), \ 0 \leq s \leq t; $$
(ii) independent increments: $W_t - W_s$ is independent of the $\sigma-$algebra
$$ \mathcal{F}_s = \sigma(W_u : u \leq s), \ 0 \leq s \leq t; $$
(iii) continuous trajectories: $W_t(\omega)$ is continuous in $t$ for almost all $\omega$'s.

Brownian motion is clearly a Markov process. Two popular related process are the arithmetic Brownian Motion and the geometric Brownian motion.

Definition 3.6. Arithmetic Brownian motion process is given by
$$ X_t = X_0 + \mu t + \sigma W_t, \quad (3.1) $$
where $\mu$ and $\sigma$ are some constants and $\{W_t\}_{t \geq 0}$ is a standard Brownian Motion.

Definition 3.7. Geometric Brownian motion process is given by
$$ Z_t = \exp(X_t) = Z_0 \exp(\mu t + \sigma W_t), $$
where $\mu$ and $\sigma$ are some constants and $\{W_t\}_{t \geq 0}$ is a standard Brownian Motion.

The geometric Brownian motion will become important when using the Black-Scholes model.
3.2.1 Brownian Bridge

Perhaps the most important feature that will be used for pricing barrier options throughout the thesis, is the Brownian Bridge property. Since we will be dealing with a random process that will be restricted by some boundary, it will become important to know the probability that the process crosses this boundary. To find the probability of such an event, we implement the idea of the Brownian Bridge.

**Definition 3.8.** A Brownian Bridge (a.k.a. the tied down Brownian motion), is a conditional Brownian Motion process specified as

\[ X_t \sim (W_t | W_1 = 0), \ t \in [0, 1], \]

where \( \{W_t\} \) is the standard Brownian Motion.

Alternatively one can write the above definition as follows:

**Definition 3.9.** A Brownian Bridge is a process given by

\[ X_t = (1 - t)W_{\frac{t_1}{1-t}}, \ t \in [0, 1]. \]

**Remark 3.1.** Instead of conditioning on \( W_1 = 0 \), we can condition on \( W_1 = z \). Then the conditional process will be:

\[ (W_t | W_1 = z) \sim X_t + zt, \ t \in [0, 1], \]

where \( \{X_t\} \) is the Brownian Bridge from definition 3.8. Similarly, we can condition on two points \( W_{t_1} = z_1 \) and \( W_{t_2} = z_2 \). Then the trajectory of \( W_t \) between these two points can be distributed as a scaled Brownian Bridge:

\[ (W_t | W_{t_1} = z_1, W_{t_2} = z_2) \sim z_1 \frac{t_2 - t}{t_2 - t_1} + \sqrt{t_2 - t_1}X_{\frac{t - t_1}{t_2 - t_1}} + z_2 \frac{t - t_1}{t_2 - t_1}, \ t \in [t_1, t_2], \quad (3.2) \]

where \( \{X_t\} \) is the Brownian Bridge process.

Consider the following example. Suppose \( \{X_t\} \) is a Brownian Bridge process and \( H \) is a linear boundary connecting the two points \((0, b_1)\) and \((1, b_2), b_i > 0\). This is illustrated in Figure 3.3, where the process starts at zero and is tied down back to zero at \( t = 1 \), with a boundary \( H \) above the process.

The probability that \( \{X_t\} \) crosses the boundary \( H \) can be written as

\[ P(X_t > b_1 + (b_2 - b_1)t, \ for \ some \ t \in [0, 1]) \]

(3.3)

Using Definition 3.9, we have:

\[ X_t > b_1 + (b_2 - b_1)t \iff (1 - t)W_{\frac{t_1}{1-t}} > b_1 + (b_2 - b_1)t. \]

Using the change of variables, let \( u := \frac{t}{1-t} \iff t = \frac{u}{1+u}, \) we see that the probability (3.3) is equivalent to
Figure 3.3: Trajectory of a Brownian Bridge $X_t$ and a boundary connecting the points $(0, b_1)$ and $(1, b_2)$.

\[
P \left( (1 + u)W_u > b_1 + \frac{(b_2 - b_1)u}{1 + u}; \text{ for some } u \geq 0 \right)
= P \left( W_u > b_1 + b_2u; \text{ for some } u \geq 0 \right),
= P \left( W_u - b_2u > b_1; \text{ for some } u \geq 0 \right).
\]

But we know that for an arithmetic Brownian Motion process (3.1) with $\mu < 0$, one has

\[
P \left( \max_{t \geq 0} (\sigma W_t + \mu t) > y \right) = \exp \left( -\frac{2|\mu|y}{\sigma^2} \right), \quad y > 0.
\]

Substituting $\mu = -b_2$, $y = b_1$ and $\sigma^2 = 1$, we obtain:

\[
P \left( W_u - b_2u > b_1 \right) = \exp \left( -2b_1b_2 \right). \tag{3.4}
\]

This result can be found e.g in Karatzas and Shreve (1988) pp.265. Suppose now we wish to find the probability that a Brownian motion process \{\text{\textit{W}}_t\} stays below a boundary starting at $t_1 > 0$ and finishing at $t_2 > t_1$. Again we can illustrate this on Figure 3.4, where the boundary starts at level $h_1 > 0$ and finishes at a level $h_2 > 0$.

We can write this in a linear boundary form as:

\[
\left\{ W_t < h_1 \frac{t_2 - t}{t_2 - t_1} + h_2 \frac{t - t_1}{t_2 - t_1}, \quad t \in [t_1, t_2] \right\}.
\]

To find the conditional probability of this event given $W_{t_1} = y_1$ and $W_{t_2} = y_2$,

\[
P \left( W_t < h_1 \frac{t_2 - t}{t_2 - t_1} + h_2 \frac{t - t_1}{t_2 - t_1}, \quad t \in [t_1, t_2] \mid W_{t_1} = y_1, W_{t_2} = y_2 \right), \tag{3.5}
\]

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we can use (3.2), to re-express it as

\[
P \left( y_1 \frac{t_2 - t}{t_2 - t_1} + \sqrt{t_2 - t_1} X \frac{t - t_1}{t_2 - t_1} + y_2 \frac{t - t_1}{t_2 - t_1} < h_1 \frac{t_2 - t}{t_2 - t_1} + h_2 \frac{t - t_1}{t_2 - t_1}, t \in [t_1, t_2] \right),
\]

where \( X_t \) is the Brownian Bridge, using the change of variables \( u = \frac{t - t_1}{t_2 - t_1} \), this becomes

\[
P \left( X_u < \frac{h_1 - y_1}{\sqrt{t_2 - t_1}} (1 - u) + \frac{h_2 - y_2}{\sqrt{t_2 - t_1}} u, 0 \leq u \leq 1 \right).
\]

Let \( b_1 = \frac{h_1 - y_1}{\sqrt{t_2 - t_1}} \) and \( b_2 = \frac{h_2 - y_2}{\sqrt{t_2 - t_1}} \), this is equivalent to,

\[
1 - P \left( X_u > b_1 + (b_2 - b_1)u \text{ for some } u \in [0, 1] \right).
\]

The following result will be central to this thesis; by using (3.4) we obtain that the conditional probability (3.5) is given by

\[
1 - \exp \left( -2b_1b_2 \right) = 1 - \exp \left( -\frac{2}{t_2 - t_1} (h_1 - y_1)(h_2 - y_2) \right). \tag{3.6}
\]

So for any boundary starting at some time \( t_1 \) and finishing at \( t_2 \), provided we can specify \( h_1 \) and \( h_2 \), the probability that the Brownian motion process \( W_t \) does not touch this boundary can be obtained using the above result and the total probability formula. Note that we can also find the probability that the Brownian Motion will touch such a boundary. This is simply given by:

\[
\exp \left( -\frac{2}{t_2 - t_1} (h_1 - y_1)(h_2 - y_2) \right). \tag{3.7}
\]

### 3.3 Black-Scholes-Merton Model

As we have said in the previous section, we will be working under the assumption of risk neutrality, where no arbitrage opportunities exist. The most commonly used model
for pricing security derivatives is the famous Black-Scholes model, which was constructed through the works of Black and Scholes (1973) and Merton (1973). In this thesis we will be working under the Black-Scholes model framework, where in most cases we will refer to it as the BSM. The reader may wish to refer to Klebaner (2005) pp.289-304 for a more detailed explanation of the BSM derivation.

First we assume that the stock price is given by a diffusion model in continuous time $t \in [0, T]$:

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t,$$  \hspace{1cm} (3.8)

where $\mu$ is the drift, $\sigma$ is the volatility, $W_t$ is the standard Brownian Motion under the probability measure $\mathbb{P}$. The bond price $B_t$ is assumed to be deterministic and continuous:

$$B_t = \exp\left(\int_0^t r(u)\,du\right).$$

The BSM stipulates the following key assumptions:

- The riskless interest rate $r$ and the initial asset price $S_0$ are constant.
- $\sigma(S_t) = \sigma S_t$, where the volatility $\sigma$ is constant.

Since we are working in an arbitrage free market and the BSM is a complete market model, there will exist a unique EMM $\mathbb{Q}$. The unique EMM $\mathbb{Q}$ makes $S_t \exp(-rt)$ a martingale. Using Itô’s formula\textsuperscript{1} and martingale theory, one can show that under these BSM assumption and changes, instead of (3.8) we have under the probability $\mathbb{Q}$ the following stochastic differential equation for $S_t$.

$$dS_t = rS_t dt + \sigma S_t dW_t.$$  \hspace{1cm} (3.9)

where $W_t$ is also a standard Brownian motion under the measure probability $\mathbb{Q}$.

The solution to (3.9) is given by:

$$S_t = S_0 \exp\left((r - \frac{1}{2} \sigma^2)t + \sigma W_t\right).$$  \hspace{1cm} (3.10)

for $t \geq 0$, where (3.10) is the geometric Brownian motion as seen in earlier in the section.

\textsuperscript{1}Itô’s formula can be found in most elementary Stochastic Calculus hand books. See also Klebaner (2005) pp.303 for a more detailed proof
3.4 Vanilla Options

Options with no special features are known as vanilla options, they are also sometimes referred to as time-independent options, since there are no conditions to how the underlying arrives on maturity. Due to their simple behaviour, they act as a great building block to constructing other complex options.

3.4.1 Pricing Vanilla options

Following the derivation of the BSM, Black and Scholes (1973) went on to deriving the famous Black-Scholes formula for European call option.

**Theorem 3.4. (Black-Scholes Formula) The time $t = 0$ price of a European call option with initial price $S_0$, strike price $K$, interest rate $r$, volatility $\sigma$ and maturity time $T$ is**

$$S_0 N(h) - e^{-rt} K N(h - \sigma T),$$

where

$$h = \left( \ln\left( \frac{S_0}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) \right) / \sigma \sqrt{T} \quad \text{and} \quad N(x) \quad \text{is the standard normal CDF}.$$

**Proof.** The price of the European call option can be written as

$$C_0 = e^{-rT} e^{-rt} \mathbb{E} \left( (S_T - K^+) \right) = \mathbb{E} \left( (S_T - K) 1_{\{S_T > K\}} \right) = e^{-rT} \mathbb{E}(S_T; S_T > K) - e^{-rT} \mathbb{E}(K; S_T > K).$$

Using (3.10) we see that the event $\{S_T > K\}$ is equivalent to Since $\{S_T > K\}$, then this is equivalent to

$$W_T > \left( \ln(\frac{S_0}{K}) - \left( r - \frac{1}{2} \sigma^2 \right) \right) / \sigma \sqrt{T} =: w$$

$$\Leftrightarrow Z > w / \sqrt{T},$$

where $Z := W_T / \sqrt{T} \sim N(0, 1)$.

So now we have

$$C_0 = S_0 e^{-\frac{1}{2} \sigma^2 T} \mathbb{E} \left( e^{\sigma W_T}; Z > \frac{w}{\sqrt{T}} \right) - e^{-rT} K \mathbb{P} \left( Z > \frac{w}{\sqrt{T}} \right),$$

where

$$\mathbb{P} \left( Z > \frac{w}{\sqrt{T}} \right) = N \left( -w / \sqrt{T} \right) = N \left( h - \sigma \sqrt{T} \right)$$

(3.11)

and

$$\mathbb{E} \left( e^{\sigma W_T}; Z > \frac{w}{\sqrt{T}} \right) = \int_{w / \sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( \sigma \sqrt{T} x - \frac{x^2}{2} \right) dx.$$
Note that

\[-\frac{1}{2} \left( x^2 - 2\sigma \sqrt{T} x \right) = -\frac{1}{2} \left( x - \sigma \sqrt{T} \right)^2 + \frac{\sigma^2 T}{2},\]

and so (3.12) is equal to

\[e^{\sigma^2 T/2} \int_{\frac{x}{\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left\{ \left( x - \sigma \sqrt{T} \right)^2 \right\} dx.\]

Changing the variables \( x' = x - \sigma \sqrt{T} \), we obtain

\[e^{\sigma^2 T/2} \int_{\frac{w}{\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(x'^2)dx' = e^{\sigma^2 T/2} P(Z > -h) = e^{\frac{x^2}{2}} N(h),\]

which together with (3.11), completes the proof.

The above formula is very important when pricing more complex options as well. In some cases we will need to refer back to this formula under different circumstances.

### 3.4.2 Put-Call parity

A convenient extension of the Black-Scholes formula is the put-call parity. From this formula we can find the price of a call or put option, assuming you know the price for one or the other, for the same stock, strike price \( K \), interest rate \( r \), volatility \( \sigma \) and maturity time \( T \).

**Theorem 3.5. (Put-call parity)**

\[S_t + P_t - C_t = K e^{-r(T-t)}, \ \forall t \in [0, T],\]

where \( C_t \) is the call price and \( P_t \) is the put price for the option.

The proof of this result is based on a simple no-arbitrage argument and can be found in most introductory finance text books.

### 3.5 Option Sensitivities

Sensitivities (a.k.a. the Greeks\(^2\)) are a very important tool used for mathematical finance and financial risk management. They are of great practical and theoretical importance for hedging purposes or to know the impact of the underlying parameters. They are simply the partial derivatives of security prices with respect to the parameter of interest that participate in the pricing formula. Sensitivity pricing formulae are very easy to compute under the BSM, which adds to popularity of the model. A great coverage for many different Greeks for European vanilla call and put options can be found in Haug (2007) pp. 21-95, where formulae, diagrams and some numerical calculations are presented. In practice, the Greeks are very important for reducing the risk of a portfolio of securities, when closing the

\(^2\)Due to the use of Greek letters to denote them.
position is not practical or desirable. One such example, where the derivative is taken with respect to the initial price (i.e. the Delta), indicates the number of units of the security to hold in the hedge portfolio. We also need sensitivities to see how pricing formulae behave under small changes to the formulae parameters. We will not concentrate too much on hedging securities since this is beyond the scope of this thesis. The reader is encouraged to refer to Hull (2006) for material on hedging securities.

As we have already mentioned, partial derivatives need to be taken to obtain closed-form expressions for the sensitivities. We will see in Chapter 7 that, the pricing formulae for some barrier options tend to be quite complicated and depend on the length, direction and variety of the barrier. One could envisage the formulae would become even longer and more complex when derivatives are taken. This is the case indeed. Therefore we had to use the Mathematica software package to compute them. An explicit formula for Delta and Eta Greeks is given in the Appendix section.

Here we will present only a few of the more common Greeks, where we denote by $C$ the price of a barrier call option.

**Definition 3.10. Delta** is the sensitivity with respect to the initial price $S_0$

$$\Delta = \frac{\partial C}{\partial S_0}.$$  

**Definition 3.11. Gamma** is the sensitivity with respect to small changes in initial price $S_0$

$$\Gamma = \frac{\partial^2 C}{\partial S_0^2}.$$  

**Definition 3.12. Vega** is the sensitivity with respect to the volatility $\sigma$

$$\nu = \frac{\partial C}{\partial \sigma}.$$  

**Definition 3.13. Rho** is the sensitivity with respect to the interest rate $r$

$$\rho = \frac{\partial C}{\partial r}.$$  

**Definition 3.14. Eta** is the sensitivity with respect to the barrier $H$

$$\eta = \frac{\partial C}{\partial H}.$$  

To the best of our knowledge, sensitivities for barrier options have not been considered in past literature. A study of Delta and Vega is presented in Dolgova (2005), but only for standard barrier options. For some barrier options considered for pricing, we will only compute the Delta and the Eta. Numerical results are presented in Chapter 8.
3.6 Bivariate and Trivariate Normal Distribution Functions

We have already seen the use of the standard normal distribution when pricing European vanilla call option under the Black-Scholes. When pricing more exotic derivative securities, one may need to use multivariate normal distributions, such as the bivariate and trivariate normal distribution functions. In particular, the bivariate normal distribution will become of great importance to partial-time barrier derivative securities.

In this section we will discuss some general properties of the bivariate normal distribution, and lightly touch on the trivariate normal distribution. In the last part of this section we will present approximations for the bivariate and trivariate normal cumulative distribution (CDF) as no closed-form solution exists for any of these.

3.6.1 Bivariate Normal Distribution Functions

For a random vector \( X = (X_1, ..., X_N)^T \), the probability density function of a non-degenerate multi-normal distribution is given by:

\[
f_X(x) = \frac{1}{(2\pi)^{N/2} \sqrt{|\Sigma|}} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right), \quad x \in \mathbb{R} \tag{3.13}
\]

where \( \Sigma \) is the covariance matrix (positive-definite, real, \( N \times N \)), \( |\Sigma| \) its determinant, \( \mu = (\mu_1, ..., \mu_N)^T \) is the mean vector and \( N \) is the dimensionality.

The normal bivariate density is of order \( N = 2 \), so that from (3.13) we have:

\[
f_{X_1,X_2}(x_1, x_2) = \frac{1}{(2\pi)^{1/2} \sqrt{|\Sigma|}} \exp \left( -\frac{1}{2} (x_1 - \mu_1, x_2 - \mu_2)^T \Sigma^{-1} (x_1 - \mu_1, x_2 - \mu_2)^T \right), \tag{3.14}
\]

where

\[
\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}.
\]

\( \rho \) is the correlation between \( X_1 \) and \( X_2 \), \( \sigma_1^2 \) is the variance of \( X_1 \) and \( \sigma_2^2 \) is the variance of \( X_2 \).

The cumulative distribution function for the standard Normal bivariate distribution with correlation \( \rho \) is given by

\[
N_2[a, b; \rho] = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{2\pi \sqrt{\Sigma}} \exp \left( -\frac{1}{2} (x_1, x_2)^T \Sigma^{-1} (x_1, x_2)^T \right) dx_1 dx_2.
\]

Substituting for \( \Sigma \) with \( \sigma_1 = \sigma_2 = 1 \), we can express this as
In most cases, we will be dealing with a multiple of the normal probability and \( \exp(cx_i), i = 1, 2 \) and \( c \) is some constant. This still remains a bivariate normal distribution, only we “shift” the mean by some constant \( c \). We can still express this in a nice closed form expression of the standard normal CDF, however we will need to reconstruct the density.

When pricing vanilla options, we needed to “complete the square” in the exponential part of the density. By making a change of variables, we were then able to express the integral in a closed form expression using the CDF of the standard normal distribution. The same technique can be applied in the shifted bivariate normal distribution case, where we will be dealing with quadratic expression of the form

\[
-\frac{1}{2} \left( \gamma_{11}(x_1 - \mu_1)^2 + 2\gamma_{12}(x_2 - \mu_2)(x_1 - \mu_1) + \gamma_{22}(x_2 - \mu_2)^2 \right),
\]

for some constants \( \mu_1, \mu_2, \gamma_{11}, \gamma_{22} \) and \( \gamma_{12} \). This is clearly the exponent in the bivariate normal density (3.14), with

\[
\Sigma^{-1} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}.
\]

By equating this to the exponent in the shifted bivariate normal density function, we can then find the constants \( \mu_1, \mu_2, \gamma_{11}, \gamma_{22} \) and \( \gamma_{12} \). Furthermore, we can then construct the covariance matrix \( \Sigma \) to find \( \rho \).

### 3.6.2 Trivariate Normal Distribution Functions

For the standard trivariate normal distribution, we will be using the CDF, given by:

\[
N_3[a, b, c; \rho_{12}, \rho_{13}, \rho_{23}] = \int_a^b \int_c^c \int_{-\infty}^{-\infty} \frac{1}{\sqrt{(2\pi)^3|\Sigma|}} \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right) dx,
\]

where \( x = (x_1, x_2, x_3)^T \), is a vector and

\[
\Sigma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix},
\]

\( \rho_{ij} = \text{corr}(X_i, X_j) \)
### 3.6.3 Approximations for Bivariate and Trivariate CDF’s

We use algorithm approximations for the closed-form representation of the bivariate and trivariate CDF’s. The statistical package R has an inbuilt algorithm for both cases. In most cases we will need to take derivatives of the CDF’s, which is not possible for R since it is not a computer algebra system. We can however, use the Mathematica package and manually input the approximation to these, which in return will allow us to compute approximate results and to take derivatives. We need to show that we can take derivatives of bivariate and trivariate CDF’s, we present this in the Appendix section. For bivariate normal probabilities we will use the Drenzer and Wesolowsky algorithm, which can be found in Haug (2007) pp.470-486:

\[
N_2[a, b; \rho] = N(a)N(b) + \rho \sum_{j=1}^{5} x_j \exp \left( \frac{2aby_j-a^2-b^2}{2(1-y_j^2\rho^2)} \right) \sqrt{1-y_j^2\rho^2},
\]

where \( N(x) \) is the cumulative normal distribution function and

\[
x_1 = 0.018854042, \quad y_1 = 0.04691008, \\
x_2 = 0.038088059, \quad y_2 = 0.23076534, \\
x_3 = 0.045270739, \quad y_3 = 0.5, \\
x_4 = 0.038088059, \quad y_4 = 0.76923466, \\
x_5 = 0.018854042, \quad y_2 = 0.95308992.
\]

For trivariate normal probabilities, we will use the Drenzer algorithm which is given in Genz (2004)

\[
N_3[a, b, c; \rho_{12}, \rho_{13}, \rho_{23}] = \frac{e^{-a^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp \left( -z^2/2 + za \right) F(a - z)dz,
\]

where

\[
F(a - z) = N_2 \left[ \frac{b - \rho_{12}(a - z)}{\sqrt{1 - \rho_{12}^2}}, \frac{c - \rho_{13}(a - z)}{\sqrt{1 - \rho_{13}^2}}, \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2\rho_{13}^2}} \right].
\]

Both approximations have shown great accuracy. For more discussion of these approximation, please see to Genz (2004).
Chapter 4

Barrier Options

4.1 Exotic Options

Options that depend on the behaviour of the price process during their lifetimes before maturity are known as exotic options. They differ from the common vanilla option such that the final payoff price very on the option's path for prior points in time. There are many types of exotic options, most common ones are Asian options, Barrier options, Chooser options, Forward-start options and Lookback options, each with their own unique characteristics. Exotic options are traded over the counter (OTC) and in most cases offer cheaper premium prices than standard vanilla options. Some but not all exotic options are referred to as path-dependent options, where the option value is heavily monitored before expiration. For example with Asian options, payoffs are determined by some averages of the underlying asset prices during a prespecified period of time before the option expires. As stated by the chapter, we will strictly focus on Barrier options and only for the European case, we also do not consider rebates and dividend yields for simplicity.

4.2 Vanilla Barrier Options

According to Zhang (1997), barrier options are the oldest of all exotic options and have been traded in markets since the late 1960's. As the name suggests, vanilla barrier options (a.k.a standard barrier options) are vanilla options, restricted by the condition that the underlying asset must touch or not touch some barrier $H$ for the time interval 0 to maturity $T$. In financial mathematics we can also refer to them as conditional options.

Vanilla barrier options come in four different types for both call and put options. We must also distinguish whether the strike price $K$ will be greater than or less than the barrier level $H$. Kick-in barrier option may result in a payoff only if the barrier $H$ is touched during the lifetime of the option, otherwise the price will be zero. Kick-out barrier results in a payoff only if the barrier $H$ is not touched during the lifetime of the option, otherwise the price will be zero. For simplicity we use “in” for kick-in barrier options and “out” for kick-out barrier options. An up barrier option is active as soon as the underlying price hits the barrier level from below, a down barrier option is active as soon as the underlying price hits the barrier level from above. Therefore a total of sixteen types of standard barrier
options pricing formulae are available for pricing.

In Figure 4.1, we illustrate an up-and-out barrier, where the barrier is active from 0 to \( T \) and behaves as a kick-out barrier. In this example the underlying price hits the barrier before maturity, therefore the option will have in a zero payoff.

![Standard barrier option](image)

Figure 4.1: Standard barrier option.

Pricing formulae for barrier options first appeared in a paper by Merton (1973), who extended the Black-Scholes model and provided an analytical formula for a down-and-out call option with \( H > K \). This was further extended by Reiner and Rubinstein (1991), with a more detailed paper providing the formulae for eight types of barriers. Zhang (1997) pp 254-256 and Haug (2007) pp. 152-153, both provide formulae for calls and puts for the eight types of barriers, completing the total of 16 pricing formulae for vanilla barrier options.

To demonstrate how to price these options we derive a closed-form formulae for a standard up-and-out put option with \( H > K \) in Chapter 6. We will leave it for the reader to verify all the other possible combinations of standard barrier options using this approach. In Chapter 7 we will verify the accuracy of the formula using simulations, through numerical results for certain specified parameters. An alternative approach for pricing standard barrier options can also be found in Zhang (1997) pp. 201-256.

### 4.2.1 Kick-In-Kick-Out Parity

We saw in Chapter 3 that a put-call parity exist for European vanilla options, where the call option price can be calculated from the put option price with the same parameters. This nice relationship does not hold for barrier options, however there does exist an *in-out parity* which can be very useful when working with long pricing formulae. If we combine an up-and-in call option with a up-and-out call option, we essentially cancel any possibility of a zero payoff before maturity, that is we have an ordinary vanilla call option. Therefore, for the option’s prices we must have
\[ C_{\text{Up and In}} + C_{\text{Up Out}} = C_{\text{Vanilla option}}. \] (4.1)

So if we have a pricing formulae for either of the options, we can find the conjugate by using this relationship. The above relationship can be used for barrier put options and for down barrier options, e.g.

\[ P_{\text{Down and In}} + P_{\text{Down and Out}} = P_{\text{Vanilla option}}. \]

Moreover, we can use the in-out relationship for partial-time and window barrier options to be explained in the next section.

### 4.3 Partial-time Barrier Options

Options where the barrier \( H \) is only considered for some fraction of the option’s lifetime are referred to as *partial-time barrier options*. In this section we discuss early-ending and forward-start barrier options, where the names of the options suggest the type of the barrier during the option’s lifetime. Partial-time barrier options give more flexibility in comparison to vanilla barrier options, they also in general give lower premiums compared to the respective standard vanilla option. Closed form valuation formulae for both of these options were first derived and presented in Heynen and Kat (1994); the same formulae without derivations can also be found in Haug (2007) pp.160-162.

#### 4.3.1 Early-Ending Partial-Time Barrier Option

Partial-time barrier options, where the barrier starts at 0 and ends at some time \( t_1 < T \), are known as *early-ending partial-time barrier options*, or *type A* partial-time barrier options. Figure 4.2 illustrates a situation when an up barrier is effective for the time interval \( t \in [0, t_1) \) for a type A partial-time barrier option.

![Figure 4.2: Early-ending partial-time barrier options.](image)
As we have already discussed for standard barrier options, there can be different combinations of barrier option depending on the barrier direction (up or down) and the barrier variety type (kick-in or kick-out) for both call and put options. Since the barrier will not be active at maturity time $T$, we do not have to distinguish whether $K > H$ or $H < K$. Therefore there will be a total of eight varieties of “type A” partial time barrier options. To avoid showing the long mathematical derivations for all eight barrier option varieties, we will only present the derivation of the pricing formula for a type A kick-out-up partial-time barrier call option which is given in Chapter 6.

### 4.3.2 Forward-Start Partial-Time Barrier Option

Partial-time barrier options, where the barrier becomes active at some time $t_1 > 0$ and ends at maturity time $T$, are known as *forward-start partial-time barrier options* (a.k.a *type B* partial-time barrier options). Figure 4.3 illustrates the situation where a barrier becomes effective for type B partial-time barrier option.

![Figure 4.3: Forward-start partial-time barrier options.](image)

For type B we must specify the direction and variety for both call or puts options. We must also distinguish whether $K > H$ or $H < K$ as the barrier ends at maturity $T$, hence there will be a total of sixteen varieties of type B partial-time barrier options. Again to avoid showing the long mathematical derivations for all sixteen barrier option varieties, we will only present the derivation of the pricing formula for a type B kick-out-up partial-time barrier call option.

### 4.4 Window Options

*Window Options* (a.k.a *limited-time barrier options*) are special kinds of partial-time barrier options, where the barrier starts at some time $t_1 > 0$ and expires before exercise at time $t_2 < T$. We can illustrate this by the following figure, in this example the barrier is not hit (see Figure 4.4). The pricing of window options is somewhat more difficult. They
are more closely related to type A partial-time barrier options as the barrier expiration ends prematurely before exercise. Therefore we do not have to distinguish whether $K < H$ or $K > H$. Also there will be a total of eight different types of window options, that is a combination of up and down, in and out and call or put types. Pricing formulae for window option, can be found in Armstrong (2001), where a general formulae for kick-out window options is given, as well some numerical results using actual call option parameters on the AUD/USD exchange rate. We will derive an explicit closed form expression for an up-and-out window barrier call option in Chapter 6 and present some numerical results in Chapter 7.

![Figure 4.4: Window barrier option.](image)

### 4.5 Up-and-In-Out Barrier Options

Having looked at type A and type B barrier options, we may wish to investigate combinations of both barriers but with different variety type. One such type is the *up-and-in-out barrier* option, where a kick-in barrier $H_{\text{in}}$ is active during $[0, t_1]$ and a kick-out barrier $H_{\text{out}}$ is active during $[t_1, T]$, both barriers being at the same level for the lifetime of the option. In other words, the underlying price process must hit the barrier $H$ between times 0 and $t_1$ and not touch the barrier $H$ from $t_1$ to maturity $T$, we must also distinguish whether $K < H$ or $K > H$. We illustrate this in Figure 4.5, where the kick-in and kick-out regions are shown.

![Figure 4.5](image)

### 4.6 Up-In and Down-Out Barrier Option with Different Barrier levels

Perhaps a more interesting and desirable combination of kick-in and kick-out barrier options is the *up-in-and-down-out barrier* option. Here, a kick-in barrier $H_{\text{in}}$ (or $H_1$) is also active during $[0, t_1]$ and a kick-out barrier $H_{\text{out}}$ (or $H_2$) is active during $[t_1, T]$, however $H_1$
and $H_2$ can be at different. Furthermore, $H_1$ is an up barrier and $H_2$ is a down barrier, so we must distinguish whether $K < H_2$ or $K > H_2$. We illustrate this by the following Figure 4.6.

To the best of our knowledge, both up-and-in-out and up-in and down-out barrier option with different barrier levels have not yet been considered in the literature. In Chapter 6, will derive an explicit closed form expression for an up-and-in-out and an up-in and down-out barrier call option with $H_1 > H_2$ and present some numerical results in Chapter 7.
Chapter 5
Simulation

5.1 Monte Carlo Simulation

When dealing with path dependent payoffs or when there is no analytic expression for the terminal distribution, one generally uses simulation techniques. It has proven to become one of the most useful and important tools used for pricing derivative securities due to the easy theory behind the method. We use Monte Carlo simulation for simulating option price values and compare them with the option’s closed form formulae values. One can also use various variance reduction techniques such as antithetic variables and control variates to improve the efficiency of Monte Carlo simulation.

The Monte Carlo approach for vanilla European options was first introduced by Boyle (1977) under the Black-Scholes setting, where some numerical results and two variance reduction techniques were discussed and compared. Since then the method has received much attention and progress. Boyle, Brodie and Glasserman (1997) discuss more improvements in efficiency by considering more variance reduction techniques and describe the use of Quasi-Monte Carlo simulation. They also summarise some recent applications of the Monte Carlo method to estimating price sensitivities and pricing American options using simulation.

In this section, we will go through the main concepts of Monte Carlo simulation and discuss how we can apply them to pricing options and computing their derivatives. We will also discuss two different approaches that can be used for simulation of price trajectories when using Monte Carlo simulation. Finally, we discuss a variance reduction technique known as Antithetic variates which can be implement to our simulations to reduce the standard error.

5.1.1 Basic Concepts

Consider an integral

\[ \bar{\theta} = \int_{\mathbb{R}^n} g(x)f(x)dx, \]
where \( g(x) \) is some arbitrary function and \( f(x) \geq 0 \) is a probability density function so that
\[
\int f(x)dx = 1.
\]

We can estimate \( \bar{\theta} \) by randomly choosing \( n \) independent sample values \((X_1, ..., X_n)\) of \( X \) from the probability density function \( f(x) \). By the Law of Large Numbers, we can write the estimate of \( \bar{\theta} \) as:
\[
\bar{\theta} = \frac{1}{n} \sum_{j=1}^{n} g(X_j).
\]

Note that this is an unbiased estimator and is consistent i.e \( \bar{\theta} = \mathbb{E}g(\hat{\theta}) \). The standard deviation of the estimate \( \hat{\theta} \) is given by \( \hat{s} \), where:
\[
\hat{s}^2 = \frac{1}{n-1} \sum_{j=1}^{n} (g(x_j) - \hat{\theta})^2. \tag{5.1}
\]

In (5.1), for large enough \( n \) we can replace \( n - 1 \) with \( n \). Notice that the distribution of
\[
\frac{\hat{\theta} - \bar{\theta}}{\sqrt{\hat{s}/n}}
\]
will tend to the standard normal distribution as \( n \to \infty \). That is, the law of large numbers ensures that this estimate converges to the actual value as the number of draws increases and the central limit theorem assures us that the standard error of the estimate tends to zero as \( 1/\sqrt{n} \). For all our simulations will use \( n = 50000 \), hence we can regard the distribution to be normal. So for \( \hat{\theta} \) we will have a standard deviation of \( \hat{s}/\sqrt{n} \), such that if we wanted to reduce the standard deviation by a factor of one hundred, we would need to run ten-thousand simulations trials. Another approach to reducing the standard deviation would be to use a variance reduction technique, where we would solely focus on reducing the size of \( \hat{s} \). One such variance reduction technique is known as \textit{antithetic variables}, which we will discuss more about in the following section.

### 5.1.2 Antithetic variates

The \textit{antithetic variables} method has become one of the most effective variance reduction techniques used for simulating option prices. There are many other variance reduction techniques which can also be used, such as control variates and moment matching methods\(^1\), which have also proved to be very useful in decreasing the standard deviation. Numerical results for regular and antithetic variables, for standard vanilla options under Monte Carlo simulation can be found in Boyle (1977), where a significant reduction in the standard error is observed.

\(^1\)Please refer to Boyle, Brodie and Glasserman (1997), where different types of variance reduction techniques are used for different types of option.
The key idea behind antithetic variables is to use negatively correlated random variables (RV’s) instead of independent X’s. We illustrate this by the following example, which also appears in Lyuu (2002), pp.259-260:

Suppose we want to estimate $E(g(X_1, X_2, ..., X_n))$, where $X_1, X_2, ..., X_n$ are independent RV’s. Let $Y_1$ and $Y_2$ be RV’s with the same distribution as $g(X_1, X_2, ..., X_n)$. Then

$$Var\left(\frac{Y_1 + Y_2}{2}\right) = \frac{Var(Y_1)}{2} + \frac{Cov(Y_1, Y_2)}{2}.$$  

Note that $Var(Y_1)/2$ is the variance of the Monte Carlo method with two independent replications of $Y$. The variance $Var((Y_1 + Y_2)/2)$ is smaller than $Var(Y_1)/2$ when $Y_1$ and $Y_2$ are negatively correlated instead of being independent. In the next section, we will be generating a random draw from the $N(0,1)$ distribution. By symmetry $Z \sim -Z$ where $Z \sim N(0,1)$, so we set $Y_1$ from $X_1, ... X_n i.i.d \sim N(0,1)$ and $Y_2$ from $-X_1, ... -X_n i.i.d \sim N(0,1)$. Thus, we generate a pair of RV’s $(Y_1, Y_2)$, which can then be used to simulate our stochastic process $S_t$.

5.1.3 Monte Carlo Simulation for Option Payoffs

We are interested in finding the price of a derivative security, and this price is given by the expectation of the payoff. So we find an approximation solution to this problem directly applying the Monte Carlo method. The approach consists of the following steps:

- Simulate $n$ independent sample paths of the underlying asset price over the relevant time horizon according to the risk-neutral measure.

- Calculate the payoff of the derivative security for each path.

- Discount payoff at risk-free rate.

- Calculate sample average over paths.

As mentioned, we will need to simulate sample paths of the underlying asset price according to the risk-neutral measure. There are two approaches we can use in simulating the sample paths for the underlying asset price. These are the Brute Force simulation and a more sophisticated approach, based on conditioning of the price process values at the start and end points of the barrier interval. We will discuss both approaches for completeness, but we will only apply the much more economical second approach for simulating sample paths throughout the thesis.

The more common approach found in many papers is the brute force approach, where we simulate the whole trajectory and a new random variable $X_i$ is drawn from the probability density $f(X)$ at each time step $i = 1, 2, ..., N$, where $N$ can be represented as the number of days in the unit time $T$ year/s. Since we are working under the BSM, the underlying asset price is assumed to follow the geometric Brownian motion given in (3.10). Using
the properties of Random walks, this allows us to generate the distribution of stock prices under the geometric Brownian motion:

$$S_{i+1} = S_i \exp \left( (r - \frac{1}{2} \sigma^2) \delta + \sigma \sqrt{\delta} x_i \right)$$

where $i = 1, 2, ..., N$ and $\delta = T/N$.

So essentially, we are simulating the underlying stock price trajectory on $N$ discrete time steps. The $X_i$ are generated from the $N(0, 1)$ distribution, which can be easily simulated by using, say the Box-Müller algorithm. Brute force simulation works very well for barrier options, since each simulated stock price $S_i$ can be monitored individually. For example, when an up-and-out barrier $H$ is active on $[0, T]$, then if the simulated stock process $S_i$ for $i = 1, ..., N$ is ever greater than the barrier $H$, the payoff becomes 0. A major problem with this approach for path-dependent options is that we are using discrete time random walks to simulate a continuous time process, therefore there is a bias. To eliminate it, Anderson and Brotherton-Ratcliffe (1996) present an approach which uses the Brownian bridge property between the time increments for both barrier options and lookback options, whereas Brodie, Glasserman and Kou (1996) introduces a continuity correction in between time increments. Both approaches significantly reduce the bias and should be used when using brute force simulation for path-dependent options.

The second more sophisticated approach, simulates a random vector of end points of the process using the Markov property for Brownian motion $\{W_t\}$, instead of simulating the whole trajectory. It is a more sophisticated approach in the sense that we use an analytical expression in the simulation. The number of points simulated depends on the number of partitions that are used. For example, when simulating a window barrier option, one would partition $[0, t]$ into three sections; from 0 to $t_1$, from $t_1$ to $t_2$ (this is where the barrier is present), and from $t_1$ to $T$. Using the Brownian motion property of independent increments, we can simulate the following:

$$W_{t_1} \sim \sqrt{t_1} N(0, 1),$$
$$W_{t_2} - W_{t_1} \sim \sqrt{t_2 - t_1} N(0, 1),$$
$$W_T - W_{t_2} \sim \sqrt{T - t_2} N(0, 1).$$

We then take the partial sums:

$$W_{t_1} = W_{t_1},$$
$$W_{t_2} = W_{t_1} + (W_{t_2} - W_{t_1}),$$
$$W_T = W_{t_2} + (W_T - W_{t_2}).$$

So we would have a random vector of points $(W_{t_1} W_{t_2} W_T)$. By conditioning on the vector of end and starting points for the barrier duration, we can then use the explicit probability formula (3.6) or (3.7) for the Brownian Motion process hitting or not hitting a boundary. That is, we have the probability that $S_{t_1}$ and $S_{t_2}$ must then be below the barrier $H$ to result in a payoff for a up-and-out barrier option and above the barrier $H$ to
result in a payoff for down-and-out barrier option. Furthermore, we use \( \{W_T\} \) to simulate the payoff price at maturity \( S_T \) using (3.10).

Both approaches have shown to give very good estimates in comparison with the actual value, however the latter approach has a much faster computational time since much fewer RV’s are generated. For example, a window barrier with a maturity time of 1 year, needs \( N = 275 \) simulations for the brute force approach, whereas only 3 simulations are required for the more sophisticated approach. Another benefit in using the more sophisticated approach, is that it is much simpler to work with it and so less effort is needed for programing code, due to its simplicity.

5.2 Forward-Finite Differences

In Chapter 4 we discussed the importance of sensitivities for financial securities. When dealing with barrier options, taking the derivative of pricing formula can be quite difficult and requires great care, therefore Mathematica was used to compute the partial derivatives of pricing formulae. To verify that these values are consistent, we can use Monte Carlo simulation, but need to consider numerical differentiation. The most common method for numerical differentiation is forward-finite differences, see e.g Lyuu (2002) pp.127-130, for other finite difference formulae. The derivative of a function \( f \) at a point \( x \) is defined by the limit:

\[
\frac{d}{dx} \left( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \right).
\]

To see how we can apply this formula to computing sensitivities using Monte Carlo simulation, suppose that the option price \( p \) depends on the parameter of interest \( \theta \) and we wish to estimate \( \frac{dp}{d\theta} \) at \( \theta = \theta_1 \). We denote a Monte Carlo estimator of the price \( p \) at \( \theta = \theta_1 \) by \( P(\theta_1) \). The parameter \( \theta_1 \) is then perturbed by a small value \( h \) to \( \theta_2 = \theta_1 + h \) and the price estimator \( P(\theta_2) \) is computed using the same generated RV’s used for \( P(\theta_1) \). We obtain an estimator of the option’s price derivative by applying the forward finite difference formula:

\[
\frac{dp}{d\theta} \approx \frac{P(\theta_2) - P(\theta_1)}{h}.
\]

We obtain an estimate for the particular sensitivity by taking the sample average over \( n \) independent outcomes of the estimator \( \frac{dp}{d\theta} \). An advantage in using this method is that it involves no programming effort beyond what is required for pricing simulation itself. Brodie and Glasserman (1996) refer to this approach as the re-simulation method. They also consider two more direct approaches known as the path-wise method and a likelihood ratio method for European vanilla options. Both methods give unbiased estimates of derivatives and reduce computational time, however they are more difficult when dealing with barrier options. Therefore, in this thesis we only consider the re-simulation method for calculating the Greeks. The selection of \( h \) is also discussed in Brodie and Glasserman (1996). We avoid this in the thesis and use \( h = 10^{-4} \) for our simulations. In Chapter 7 we present numerical results for the Greeks with respect to \( \Delta \) and \( \eta \), for a selected number of barrier options.
Chapter 6

Pricing Barrier Options

In this section we will derive explicit pricing formulae for some of the barrier options discussed in Chapter 4. To demonstrate how to price these options we have selected seven different kinds of examples: an up-and-out vanilla barrier put option with \( K < H \), a type A up-and-out partial-time barrier call option, a type A up-and-in partial-time barrier call option, a type B up-and-out partial-time barrier call options, an up-and-out window barrier call option, an up-in-and-out barrier call option and a up-in-and-down-out barrier call option with \( H_1 > H_2 \).

6.1 Up-and-Out Barrier Put option with \( H > K \)

Theorem 4.1. The pricing formula for an up-and out barrier put-option is given by:

\[
e^{-rT} \left( N(-h_2) - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N(-a_2) \right) - S_0 \left( N(-h_1) - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2 + 2} N(-a_1) \right),
\]

where

\[
h_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad h_2 = h_1 - \sigma\sqrt{T},
\]

\[
a_1 = \frac{\ln(H^2/S_0K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad a_2 = a_1 - \sigma\sqrt{T}, \quad \mu = r - \frac{1}{2}\sigma^2.
\]

Proof. The expected payoff for an up-and-out put option after discounting can be written as

\[
e^{-rT} \mathbb{E} \left( 1_{S_t < H; t \in [0,T]} (S_T - K)^- \right).
\]

Note that the underlying price must stay below the barrier \( H \), starting at time 0 and ending at maturity time \( T \). Under the BSM (3.10), we have

\[
\{ S_t < H; t \in [0,T] \} \Leftrightarrow \left\{ W_t < \left( \ln(H/S_0) - (r - \frac{1}{2}\sigma^2)t \right)/\sigma ; \ t \in [0,T] \right\}.
\]
Furthermore at \( t = T \), the underlying value must stay below \( K \) for the payoff to be positive. That is, we should have

\[
S_0 \exp((r - \frac{1}{2}\sigma^2)T + \sigma W_T) < K \iff W_T < k,
\]

where \( k = (\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T)/\sigma \). Writing (6.1) in an integral form, we obtain

\[
e^{-rT} \int_{-\infty}^{k} P\left(S_t < H; t \in [0, T] \mid W_T = x\right) \left(K - S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma x}\right) P(W_T \in dx). \quad (6.2)
\]

Using (3.6) we have

\[
P(S_t < H; t \in [0, T]|W_T = x) = 1 - \exp\left(-\frac{2}{T} \left(\frac{\ln(H/S_0)}{\sigma}\right) \left(\frac{\ln(H/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma} - x\right)\right)
\]

\[
= 1 - \exp(c_1 + c_2 x) \quad (6.3)
\]

where

\[
c_1 = -\frac{2(\ln(H/S_0))^2 + 2 \ln(H/S_0)(r - \frac{1}{2}\sigma^2)T}{\sigma^2T} \quad \text{and} \quad c_2 = \frac{2 \ln(H/S_0)}{\sigma T}.
\]

Substituting (6.3) into (6.2) we obtain

\[
e^{-rT} \int_{-\infty}^{k} \left[1 - e^{c_1+c_2x}\right] (K - S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma x}) P(W_T \in dx) \\
= K e^{-rT} \int_{-\infty}^{k} P(W_T \in dx) - S_0e^{-rT} \int_{-\infty}^{k} e^{(r-\frac{1}{2}\sigma^2)T+\sigma x} P(W_T \in dx) \quad (6.4)
\]

\[
- K e^{-rT} \int_{-\infty}^{k} e^{c_1+c_2x} P(W_T \in dx) \quad (6.5)
\]

\[
+ S_0e^{-rT} \int_{-\infty}^{k} e^{c_1+c_2x} e^{(r-\frac{1}{2}\sigma^2)T+\sigma x} P(W_T \in dx). \quad (6.6)
\]

Note that (6.4) can be expressed as the Black-Scholes formula for a European vanilla put option. The integrals (6.5) and (6.6) are very similar in computation, therefore we will only show how to compute (6.5). The reader can follow the same approach for the remaining integral. For (6.5) we have
\[-Ke^{-rT} \int_{-\infty}^{k} e^{c_1+c_2x} P(W_T \in dx) = -Ke^{-rT} \int_{-\infty}^{k} e^{c_1+c_2x} \frac{1}{\sqrt{2\pi T}} e^{-x^2/2T} dx \]
\[= -Ke^{-rT} e^{c_1} \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi T}} e^{c_2x-x^2/2T} dx.\]

Note that \(-\frac{1}{2}(x^2 - 2Tc_2) = -\frac{1}{2}(x - c_2T)^2 + \frac{1}{2}c_2^2T,\) which leads to
\[-Ke^{-rT} e^{c_1} \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi T}} e^{-(x-c_2T)^2/2T} dx.\]

Changing the variable \(x' = x - Tc_2,\) we obtain that the last expression is equal to
\[-Ke^{-rT} e^{c_1} e^{\frac{1}{2}c_2^2T} \int_{-\infty}^{k-c_2T} \frac{1}{\sqrt{2\pi T}} e^{c_2^2x'} dx'. \quad (6.7)\]

Substituting for \(k\) and \(c_2\) in the upper integration limit of (6.7) we obtain
\[k - Tc_2 = \frac{\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T - 2\ln(H/S_0)}{\sigma} = \sqrt{T}(\sigma\sqrt{T} - a_1), \quad (6.8)\]

where \(a_1 = (\ln(H^2/S_0K) + (r + \frac{1}{2}\sigma^2)T) / \sigma\sqrt{T}.\) The product of the exponents in front of the integral in (6.7) equals
\[e^{c_1} e^{\frac{1}{2}c_2^2T} = \exp \left( \frac{-2(\ln(H/S_0))^2 + 2\ln(H/S_0)(r - \frac{1}{2}\sigma^2)T}{\sigma^2T} + \frac{T}{2} \left( \frac{2\ln(H/S_0)}{\sigma T} \right)^2 \right) \]
\[= \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2}, \quad (6.9)\]

where \(\mu = r - \frac{1}{2}\sigma.\) Substituting (6.8) and (6.9) into (6.7) we obtain
\[-Ke^{-rT} \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} \int_{-\infty}^{\sqrt{T}(\sigma\sqrt{T} - a_1)} \frac{1}{\sqrt{2\pi T}} \exp \left( \frac{-x'^2}{2T} \right) dx' \]
\[= -Ke^{-rT} \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} P \left( X' \leq \sqrt{T}(\sigma\sqrt{T} - a_1) \right) \]
\[= -Ke^{-rT} \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} P \left( \sqrt{T}Z \leq \sqrt{T}(\sigma\sqrt{T} - a_1) \right), \quad \text{where } Z \sim N(0, 1) \]
\[= -Ke^{-rT} \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N \left( \sigma\sqrt{T} - a_1 \right) \]
\[= -Ke^{-rT} \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N (-a_2). \]

Integral (6.6) is computed in a similar way. Theorem 6.1 is proved.
6.2 Type A Up-and-Out Partial-Time Barrier Call Option

Theorem 6.2. The pricing formula for an Type A up-and-out partial-time barrier call option is given by:

\[
S_0 \left( N_2 \left[ d_1, -e_1; -\sqrt{\frac{t_1}{T}} \right] - \left( \frac{H}{S_0} \right)^{2(\mu+1)/\sigma^2} N_2 \left[ f_1, -g_1; -\sqrt{\frac{t_1}{T}} \right] \right) \\
- Ke^{-rt_1} \left( N_2 \left[ d_2, -e_2; -\sqrt{\frac{t_1}{T}} \right] - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N_2 \left[ f_2, -g_2; -\sqrt{\frac{t_1}{T}} \right] \right),
\]

where

\[
d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T},
\]

\[
f_1 = \frac{\ln(S_0/K) + 2\ln(H/S_0) + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}, \quad f_2 = f_1 - \sigma \sqrt{T},
\]

\[
e_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2)t_1}{\sigma \sqrt{t_1}}, \quad e_2 = e_1 - \sigma \sqrt{t_1},
\]

\[
g_1 = e_1 + \frac{2\ln(H/S_0)}{\sigma \sqrt{t_1}}, \quad g_2 = g_1 - \sigma \sqrt{t_1}, \quad \mu = r - \frac{1}{2}\sigma^2.
\]

Proof. The discounted expected payoff for a up-and-out partial-time barrier call option can be written as

\[
e^{-rT} E \left( 1_{S_t < H; t \in [0, t_1]} (S_T - K)^+ \right) = e^{-rT} E \left( E \left( 1_{S_t < H; t \in [0, t_1]} (S_T - K)^+ | S_{t_1} \right) \right).
\]

By the Markov property, we obtain that it is equal to

\[
e^{-rT} \int_{-\infty}^{H} P(S_t < H; t \in [0, t_1]|S_{t_1} = s)E \left( (S_T - K)^+ | S_{t_1} = s \right) P(S_{t_1} \in ds). \quad (6.10)
\]

We observe that

\[S_{t_1} \equiv S_0 \exp \left( (r - \frac{1}{2}\sigma^2)t_1 + \sigma W_{t_1} \right) = s\]
is equivalent to

\[ W_{t_1} = x, \]

where \( x = (\ln(s/S_0) - (r - \frac{1}{2} \sigma^2)t_1) / \sigma. \) Also, the integration region \( \{ S_{t_1} < H \} \) can be expressed in terms of \( W_{t_1} \) as \( \{ W_{t_1} < k \} \), where \( k = (\ln(H/S_0) - (r - \frac{1}{2} \sigma^2)t_1) / \sigma. \) Again using (3.6) obtain:

\[ P(S_t < H; t \in [0, t_1]|W_{t_1} = x) = 1 - \exp(c_1 + c_2x), \quad (6.11) \]

where

\[ c_1 = \frac{-2(\ln(H/S_0))^2 + 2 \ln(H/S_0)(r - \frac{1}{2} \sigma^2)t_1}{\sigma^2 t_1} \quad \text{and} \quad c_2 = \frac{2 \ln(H/S_0)}{\sigma t_1}. \]

Substituting (6.11) into (6.10) yields:

\[ e^{-rT} \int_{-\infty}^{k} (1 - \exp(c_1 + c_2x)) E \left( (S_T - K)^+ | W_{t_1} = x \right) P(W_{t_1} \in dx). \quad (6.12) \]

Note that \( E \left( (S_T - K)^+ | W_{t_1} = x \right) \) can be expressed in the form of a European vanilla call option under the Black-Scholes formula for the time interval from \( t_1 \) to \( T \), with the initial asset price \( S_0 \) starting at \( x \):

\[ E \left( (S_T - K)^+ | W_{t_1} = x \right) = S_{t_1}e^{r(T-t_1)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + x}{\sqrt{T-t_1}} \right] - KN \left[ \frac{\sqrt{T}d_2 + x}{\sqrt{T-t_1}} \right], \quad (6.13) \]

where

\[ d_1 = \left( \ln(S_0/K) + (r + \frac{1}{2} \sigma^2)T \right) / \sigma \sqrt{T} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T}. \]

Substituting (6.13) into (6.12) we obtain:

\[ e^{-rT} \int_{-\infty}^{k} (1 - e^{c_1+c_2x}) S_{t_1}e^{r(T-t_1)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + x}{\sqrt{T-t_1}} \right] \]

\[ -KN \left[ \frac{\sqrt{T}d_2 + x}{\sqrt{T-t_1}} \right] P(W_{t_1} \in dx). \quad (6.14) \]

By expanding (6.14) out, we split the integral into four separate parts and evaluate them individually:

\[ e^{-rT} \int_{-\infty}^{k} S_{t_1}e^{r(T-t_1)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + x}{\sqrt{T-t_1}} \right] P(W_{t_1} \in dx) \quad (6.15) \]
\[- Ke^{-rT} \int_{-\infty}^{k} N \left( \frac{\sqrt{T}d_2 + x}{\sqrt{T} - t_1} \right) P(W_{t_1} \in dx) \] (6.16)

\[- e^{-rT} \int_{-\infty}^{k} S_t e^{c_1 + c_2x} e^{(T-t_1)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + x}{\sqrt{T} - t_1} \right] P(W_{t_1} \in dx) \] (6.17)

\[+ Ke^{-rT} \int_{-\infty}^{k} e^{c_1 + c_2x} N \left[ \frac{\sqrt{T}d_2 + x}{\sqrt{T} - t_1} \right] P(W_{t_1} \in dx). \] (6.18)

The integrals (6.15) to (6.18) are again very similar in computation, therefore we will only show how to compute (6.15), The reader can follow the same approach for the remaining integrals.

Using (3.10) with \( t = t_1 \), so that (6.15) is equal to,

\[S_0 e^{-rt_1} \int_{-\infty}^{k} e^{(r - \frac{1}{2} \sigma^2)t_1 + \sigma x} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + x}{\sqrt{T} - t_1} \right] P(W_{t_1} \in dx).\]

Changing the variables \( x' = x/\sqrt{t_1} \), we obtain, with \( X = W_{t_1}/\sqrt{t_1} \sim N(0, 1) \)

\[S_0 e^{-\frac{1}{2} \sigma^2 t_1} \int_{-\infty}^{k/\sqrt{t_1}} e^{\sigma \sqrt{t_1} x'} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + \sqrt{t_1} x'}{\sqrt{T} - t_1} \right] P(X \in dx').\]

\[= S_0 e^{-\frac{1}{2} \sigma^2 t_1} \int_{-\infty}^{k/\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x'^2 + \sigma \sqrt{t_1} x'} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + \sqrt{t_1} x'}{\sqrt{T} - t_1} \right] dx'.\]

\[= S_0 e^{-\frac{1}{2} \sigma^2 t_1} \int_{-\infty}^{k/\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2} \sigma^2 t_1 - \frac{1}{2}(x' - \sigma \sqrt{t_1})^2} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + \sqrt{t_1} x'}{\sqrt{T} - t_1} \right] dx'.\]

Since \(-\frac{1}{2}(x'^2 - 2\sigma \sqrt{t_1}) = -\frac{1}{2}(x' - \sigma \sqrt{t_1})^2 + \frac{1}{2} \sigma^2 t_1 \), Again changing variables \( x'' = x' - \sigma \sqrt{t_1} \), this becomes

\[S_0 \int_{-\infty}^{k/\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x''^2} N \left[ \frac{\sqrt{T}d_1 - \sigma t_1 + \sqrt{t_1} x''}{\sqrt{T} - t_1} \right] dx''. \] (6.19)

The integration limit is given by \( k/\sqrt{t_1} - \sigma \sqrt{t_1} \), so that substituting \( k \) we obtain

\[\frac{k}{\sqrt{t_1}} - \sigma \sqrt{t_1} = \frac{\ln(H/S_0) - (r - \frac{1}{2} \sigma^2)t_1}{\sigma \sqrt{t_1}} - \sigma \sqrt{t_1} = -e_1, \] (6.20)

where \( e_1 = (\ln(S_0/H) + (r + \frac{1}{2} \sigma^2)t_1)/\sigma \sqrt{t_1} \). Substituting (6.20) into (6.19) and introducing a random variable \( Y \sim N(0, 1) \) that is independent of \( X \), yields
\[
S_0 \int_{-\infty}^{-e_1} P(X \in dx'') P\left( Y \leq \frac{\sqrt{T}d_1 + \sqrt{t_1}x''}{\sqrt{T} - t_1} \right)
= S_0 \int_{-\infty}^{-e_1} P(X \in dx'') P\left( \frac{\sqrt{T} - t_1 Y - \sqrt{t_1}x''}{\sqrt{T}} \leq d_1 \right)
= S_0 \int_{-\infty}^{-e_1} P(X \in dx'') P\left( \frac{\sqrt{T} - t_1 Y - \sqrt{t_1}x''}{\sqrt{T}} \leq d_1 \left| X = x'' \right. \right)
= S_0 P \left( \frac{\sqrt{T} - t_1 Y - \sqrt{t_1}X}{\sqrt{T}} \leq d_1, X \leq -e_1 \right) \quad (6.21)
\]

where \( Z_1 = \left( \frac{\sqrt{T} - t_1 Y - \sqrt{t_1}X}{\sqrt{T}} \right) \) and \( Z_2 = X \) are both \( N(0,1) \)-distributed. It remains to find the correlation between \( Z_1 \) and \( Z_2 \),

\[
\rho = \text{Corr}(Z_1, Z_2) = \frac{\text{Cov}(Z_1, Z_2)}{\sqrt{\text{Var}(Z_1)\text{Var}(Z_2)}}.
\]

For the covariance we have

\[
\text{Cov}(Z_1, Z_2) = \mathbb{E}(Z_1 Z_2) - \mathbb{E}(Z_1)\mathbb{E}(Z_2)
= \mathbb{E}(Z_1 Z_2) - 0 \cdot \mathbb{E}(Z_2)
= \mathbb{E} \left( \sqrt{\frac{T - t_1}{T}} XY - \sqrt{\frac{t_1}{T}} X^2 \right).
\]

Since \( X \) and \( Y \) are independent \( N(0,1) \), we have

\[
\mathbb{E} \left( \sqrt{\frac{T - t_1}{T}} XY - \sqrt{\frac{t_1}{T}} X^2 \right) = 0 - \sqrt{\frac{t_1}{T}} \mathbb{E} X^2 = -\sqrt{\frac{t_1}{T}}. \quad (6.22)
\]

Now clearly, \( \text{Var}(Z_2) = \text{Var}(X) = 1 \), so the correlation between \( Z_1 \) and \( Z_2 \) will be \(-\sqrt{\frac{t_1}{T}}\). Therefore (6.21) is equal to

\[
S_0 P \left( Z_1 \leq d_1, Z_2 \leq -e_1 \right) = S_0 N_2 \left[ d_1, -e_1; -\sqrt{\frac{t_1}{T}} \right].
\]

All other integrals (6.16)-(6.18) are computed in a similar way. Theorem 6.2 is proved.
6.3 Type A Up-and-In Partial-Time Barrier Call Option

In Chapter 4, we discussed how pricing formulae for kick-in options can be obtained from the pricing formulae of a kick-out option and vice-versa. We referred to this as the in-out parity. In this section we demonstrate its use.

Using the in-out parity (4.1) for type A partial-time barrier, we have:

\[ C_{\text{Up and In Type A}} = C_{\text{Vanilla option}} - C_{\text{Up Out Type A}}. \]

Substituting for \( C_{\text{Up Out Type A}} \), as given from Theorem 6.2, we obtain:

Theorem 6.3. The pricing formula for a Type A up-and-in partial-time barrier call option is given by:

\[
S_0 \left( N(d_1) - N_2 \left[ d_1, -e_1; -\sqrt{\frac{t_1}{T}} \right] + \left( \frac{H}{S_0} \right)^{2(\mu+1)/\sigma^2} N_2 \left[ f_1, -g_1; -\sqrt{\frac{t_1}{T}} \right] \right) 
+ Ke^{-rt_1} \left( N_2 \left[ d_2, -e_2; -\sqrt{\frac{t_1}{T}} \right] - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N_2 \left[ f_2, -g_2; -\sqrt{\frac{t_1}{T}} \right] - N(d_1 - \sigma\sqrt{T}) \right),
\]

where

\[
d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},
\]

\[
f_1 = \frac{\ln(S_0/K) + 2\ln(H/S_0) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad f_2 = f_1 - \sigma\sqrt{T},
\]

\[
e_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2)t_1}{\sigma\sqrt{t_1}}, \quad e_2 = e_1 - \sigma\sqrt{t_1},
\]

\[
g_1 = e_1 + \frac{2\ln(H/S_0)}{\sigma\sqrt{t_1}}, \quad g_2 = g_1 - \sigma\sqrt{t_1}, \quad \mu = r - \frac{1}{2}\sigma^2.
\]

6.4 Type B Up-and-Out Partial-time Barrier Call Option

Theorem 6.4. The pricing formula for an Type B up-and-out partial-time barrier call option where \( K > H \) is given by:
\[ S_0 \left[ N_2 \left( -e_1, -b_1; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2(\mu/\sigma^2 + 1)} N_2 \left( g_1, -b_3; -\sqrt{\frac{t_1}{T}} \right) \right] \\
-Ke^{-rT} \left[ N_2 \left( -e_2, -b_2; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N_2 \left( g_2, -b_4; -\sqrt{\frac{t_1}{T}} \right) \right] \\
-S_0 \left[ N_2 \left( -e_1, -d_1; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2(\mu/\sigma^2 + 1)} N_2 \left( -f_1, g_1; -\sqrt{\frac{t_1}{T}} \right) \right] \\
+Ke^{-rT} \left[ N_2 \left( -e_2, -d_2; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N_2 \left( -f_2, g_2; -\sqrt{\frac{t_1}{T}} \right) \right] , \]

where

\[
d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T},
\]

\[
f_1 = \frac{\ln(S_0/K) + 2\ln(H/S_0) + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}, \quad f_2 = f_1 - \sigma \sqrt{T},
\]

\[
e_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2)t_1}{\sigma \sqrt{t_1}}, \quad e_2 = e_1 - \sigma \sqrt{t_1},
\]

\[
g_1 = e_1 + \frac{2\ln(H/S_0)}{\sigma \sqrt{t_1}}, \quad g_2 = g_1 - \sigma \sqrt{t_1},
\]

\[
b_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}, \quad b_2 = b_1 - \sigma \sqrt{T},
\]

\[
b_3 = b_1 + \frac{2\ln(H/S_0)}{\sigma \sqrt{t_1}}, \quad b_4 = b_3 - \sigma \sqrt{T}, \quad \mu = r - \frac{1}{2}\sigma^2.
\]

Proof. The discounted expected payoff for a Type B up-and-out partial-time barrier call option with \( K < H \) can be written as

\[ e^{-rT} \mathbb{E} \left( 1_{S_{t_1} < H; t \in [t_1, T]} (S_T - K)^+ \right). \]

By the Markov property and conditioning on \( S_{t_1} \) and \( S_T \) we obtain for this expression
\[ e^{-rT} E \left( E \left( 1_{S_t < H; t \in [t_1, T]} \bigg| S_{t_1}, S_T \right) E \left( (S_T - K)^+ \bigg| S_{t_1}, S_T \right) \right) \]

\[ = e^{-rT} \int_{-\infty}^{K} \int_{-\infty}^{H} P \left( S_t < H; t \in [t_1, T] \bigg| S_{t_1} = s_1, S_T = s_2 \right) (s_2 - K) \times P (S_{t_1} \in ds_1, S_T \in ds_2) \]

\[ = e^{-rT} \int_{-\infty}^{K} \int_{-\infty}^{H} P \left( S_t < H; t \in [t_1, T] \bigg| S_{t_1} = s_1, S_T = s_2 \right) (s_2 - K) \times P (S_{t_1} \in ds_1, S_T \in ds_2) \]

\[-e^{-rT} \int_{-\infty}^{K} \int_{-\infty}^{H} P \left( S_t < H; t \in [t_1, T] \bigg| S_{t_1} = s_1, S_T = s_2 \right) \times (s_2 - K) P (S_{t_1} \in ds_1, S_T \in ds_2). \]

(6.23)

As before, we observe that \( S_{t_1} \equiv S_0 \exp( (r - \frac{1}{2}\sigma^2)t_1 + \sigma W_{t_1} ) = s_1 \) is equivalent to \( W_{t_1} = x_1 \), where \( x_1 = (\ln(s_1/S_0) - (r - \frac{1}{2}\sigma^2)t_1)/\sigma \). Similarly, \( S_T \equiv S_0 \exp( (r - \frac{1}{2}\sigma^2)T + \sigma W_T ) = s_2 \) is equivalent to \( W_T = x_2 \), where \( x_2 = (\ln(s_2/S_0) - (r - \frac{1}{2}\sigma^2)T)/\sigma \).

The event \( \{ S_{t_1} < H \} \) can be expressed as \( \{ W_{t_1} < k_1 \} \), where

\[ k_1 = \left( \ln(H/S_0) - (r - \frac{1}{2}\sigma^2)t_1 \right)/\sigma. \]

Following the same approach, we have \( \{ S_T < H \} = \{ W_T < k_2 \} \) and \( \{ S_T < K \} = \{ W_T < k_3 \} \), where \( k_2 = (\ln(H/S_0) - (r - \frac{1}{2}\sigma^2)T)/\sigma \) and \( k_3 = (\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T)/\sigma \). Thus (6.23) becomes

\[ e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} P \left( S_t < H; t \in [t_1, T] \bigg| W_{t_1} = x_1, W_T = x_2 \right) \left( S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma x_2} - K \right) \]

\[ \times P (W_{t_1} \in dx_1, W_T \in dx_2) - e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_3} P \left( S_t < H; t \in [t_1, T] \bigg| W_{t_1} = x_1, W_T = x_2 \right) \]

\[ \times \left( S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma x_2} - K \right) P (W_{t_1} \in dx_1, W_T \in dx_2). \]

(6.24)

Using (3.6), we have

\[ P (S_t < H; t \in [t_1, T] \big| W_{t_1} = x_1, W_T = x_2) = 1 - \exp(c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1 x_2), \]

(6.25)

where

\[ c_1 = \frac{-2(\ln(H/S_0))^2 + 2 \ln(H/S_0) \left( r - \frac{1}{2}\sigma^2 \right) t_1 + 2 \ln(H/S_0) \left( r - \frac{1}{2}\sigma^2 \right) T - 2 \left( r - \frac{1}{2}\sigma^2 \right)^2 t_1 T}{\sigma^2 (T - t_1)}. \]
\[ c_2 = \frac{2 \ln(H/S_0) - 2(r - \frac{1}{2} \sigma^2)T}{\sigma(T - t_1)}, \quad c_3 = \frac{2 \ln(H/S_0) - 2(r - \frac{1}{2} \sigma^2)t_1}{\sigma(T - t_1)} \text{ and } c_4 = \frac{-2}{T - t_1} \]

Substituting (6.25) into (6.24) we obtain

\[
e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \left( 1 - \exp(c_1 + c_2x_1 + c_3x_2 + c_4x_1x_2) \right) \left( S_0 e^{(r - \frac{1}{2} \sigma^2)T + \sigma x_2} - K \right) \times P(W_{t_1} \in dx_1, W_T \in dx_2) - e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \left( 1 - \exp(c_1 + c_2x_1 + c_3x_2 + c_4x_1x_2) \right) \times \left( S_0 e^{(r - \frac{1}{2} \sigma^2)T + \sigma x_2} - K \right) P(W_{t_1} \in dx_1, W_T \in dx_2). \tag{6.26}
\]

Note that

\[
P(W_{t_1} \in dx_1, W_T \in dx_2) = f_{(W_{t_1}, W_T)}(x_1, x_2)dx_1dx_2, \tag{6.27}
\]

where

\[
f_{(W_{t_1}, W_T)} = f_{W_{t_1}}f_{W_T|W_{t_1}} = \frac{1}{2\pi \sqrt{t_1(T - t_1)}} \exp \left( -\frac{x_1^2}{2t_1} \right) \exp \left( -\frac{(x_2 - x_1)^2}{2(T - t_1)} \right).
\]

To make computations easier we can further divide (6.26) into eight individual integrals. Therefore, we have

\[
S_0 e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \frac{1}{2\pi \sqrt{t_1(T - t_1)}} \exp \left( (r - \frac{1}{2} \sigma^2)T + \sigma x_2 \right) \exp \left( -\frac{x_1^2}{2t_1} \right) dx_1dx_2 \tag{6.28}
\]

\[
- Ke^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \frac{1}{2\pi \sqrt{t_1(T - t_1)}} \exp \left( \frac{-x_1^2}{2t_1} \right) \exp \left( \frac{- (x_2 - x_1)^2}{2(T - t_1)} \right) dx_1dx_2 \tag{6.29}
\]

\[
- S_0 e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \frac{1}{2\pi \sqrt{t_1(T - t_1)}} \exp \left( c_1 + c_2x_1 + c_3x_2 + c_4x_1x_2 + (r - \frac{1}{2} \sigma^2)T + \sigma x_2 \right) \\
\times \exp \left( \frac{-x_1^2}{2t_1} \right) \exp \left( -\frac{(x_2 - x_1)^2}{2(T - t_1)} \right) dx_1dx_2 \tag{6.30}
\]

\[
+ Ke^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \frac{1}{2\pi \sqrt{t_1(T - t_1)}} \exp(c_1 + c_2x_1 + c_3x_2 + c_4x_1x_2) \exp \left( -\frac{x_1^2}{2t_1} \right) \\
\times \exp \left( \frac{- (x_2 - x_1)^2}{2(T - t_1)} \right) dx_1dx_2 \tag{6.31}
\]
- $S_0e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_3} \frac{1}{2\pi\sqrt{t_1(T-t_1)}} \exp \left( (r - \frac{1}{2}\sigma^2)T + \sigma x_2 \right) \\
\times \exp \left(-\frac{x_1^2}{2t_1}\right) \exp \left(-\frac{(x_2-x_1)^2}{2(T-t_1)}\right) \, dx_1 \, dx_2$

+ $Ke^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_3} \frac{1}{2\pi\sqrt{t_1(T-t_1)}} \exp \left(-\frac{x_1^2}{2t_1}\right) \exp \left(-\frac{(x_2-x_1)^2}{2(T-t_1)}\right) \, dx_1 \, dx_2$  \hspace{1cm} (6.32)

+ $S_0e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_3} \frac{1}{2\pi\sqrt{t_1(T-t_1)}} \exp \left( c_1 + c_2x_1 + c_3x_2 + c_4x_1x_2 + (r - \frac{1}{2}\sigma^2)T + \sigma x_2 \right) \\
\times \exp \left(-\frac{x_1^2}{2t_1}\right) \exp \left(-\frac{(x_2-x_1)^2}{2(T-t_1)}\right) \, dx_1 \, dx_2$  \hspace{1cm} (6.34)

- $Ke^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_3} \frac{1}{2\pi\sqrt{t_1(T-t_1)}} \exp \left( c_1 + c_2x_1 + c_3x_2 + c_4x_1x_2 \right) \exp \left(-\frac{x_1^2}{2t_1}\right) \\
\times \exp \left(-\frac{(x_2-x_1)^2}{2(T-t_1)}\right) \, dx_1 \, dx_2$  \hspace{1cm} (6.35)

The integrals (6.28) to (6.35) are again very similar in computation, therefore we will only show how to compute (6.28). This approach will also be used as a central example, when dealing with linear combinations of exponents and bivariate normal densities. The expression in (6.28) can be written as

$$S_0e^{-\frac{1}{2}\sigma^2T} \int_{-\infty}^{k_1} \int_{-\infty}^{k_3} \frac{1}{2\pi\sqrt{t_1(T-t_1)}} \exp(\sigma x_2) \exp \left(-\frac{x_1^2}{2t_1}\right) \exp \left(-\frac{(x_2-x_1)^2}{2(T-t_1)}\right) \, dx_1 \, dx_2$$ \hspace{1cm} (6.36)

The exponential part of (6.36) is

$$\exp(\sigma x_2) \exp \left(-\frac{x_1^2}{2t_1}\right) \exp \left(-\frac{(x_2-x_1)^2}{2(T-t_1)}\right) = \exp \left( \sigma x_2 + \frac{x_1x_2}{T-t_1} - \frac{x_2^2}{2(T-t_1)} - \frac{x_1^2T}{2t_1(T-t_1)} \right).$$ \hspace{1cm} (6.37)

By equating the inside of the exponential of (6.37) to the expanded “complete square” form (3.15), we obtain:

$$\sigma x_2 + \frac{x_1x_2}{T-t_1} - \frac{x_2^2}{2(T-t_1)} - \frac{x_1^2T}{2t_1(T-t_1)} = -\frac{1}{2}\gamma_{11}x_1^2 + (\gamma_{11}\mu_1 + \gamma_{12}\mu_2)x_1 - \gamma_{12}x_1x_2$$
$$+ (\gamma_{22}\mu_2 + \gamma_{12}\mu_1)x_2 - \frac{1}{2}(2\gamma_{12}\mu_1\mu_2 + \gamma_{22}\mu_2^2 + \gamma_{11}\mu_1^2) - \frac{1}{2}\gamma_{22}x_2^2.$$
We can easily find the constants on the right-hand side equating the respective terms. Substituting $\gamma_{11}, \gamma_{22}, \gamma_{12}, \mu_1$ and $\mu_2$ into the exponent form of (3.14) we obtain

$$-\frac{1}{2} \left( \frac{T}{t_1(T-t_1)}(x_1 - \sigma t_1)^2 + \frac{2}{(T-t_1)}(x_2 - \sigma T)(x_1 - \sigma t_1) + \frac{1}{(T-t_1)}(x_2 - \sigma T)^2 \right).$$

Substituting $\gamma_{11}, \gamma_{22}, \gamma_{12}, \mu_1$ and $\mu_2$ into covariance matrix $\Sigma$, the correlation is given by $\rho = \sqrt{\frac{t_1}{T}}$. Note that $1 - \rho^2 = \frac{T-t}{T}$ and since $\sigma_1 = \sqrt{t_1}$ and $\sigma_2 = \sqrt{T}$, we have:

$$-\frac{1}{2} \left( \frac{(x_1 - \sigma t_1)^2}{\sigma_1^2}(1 - \rho^2) + \frac{2(x_2 - \sigma T)(x_1 - \sigma t_1)\rho}{(1 - \rho^2)\sigma_1\sigma_2} + \frac{(x_2 - \sigma T)^2}{\sigma_2^2} \right).$$

Replacing with this the exponential part of (6.36) and using $1 - \rho^2 = \frac{T-t}{T}$, we obtain after some cancellation:

$$S_0 \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \frac{(2\pi)^{-1}}{\sigma_1\sigma_2\sqrt{(1 - \rho^2)}} \exp \left( -\frac{1}{2(1 - \rho^2)} \left( \frac{(x_1 - \sigma t_1)^2}{\sigma_1^2} \right. \right.$$

$$\left. + \frac{2\rho(x_2 - \sigma T)(x_1 - \sigma t_1)}{\sigma_1\sigma_2} + \frac{(x_2 - \sigma T)^2}{\sigma_2^2} \right) \right) dx_1 dx_2.$$

$$= S_0 \int_{-\infty}^{k_1 - \sigma t_1} \int_{-\infty}^{k_2 - \sigma T} \frac{(2\pi)^{-1}}{\sigma_1\sigma_2\sqrt{(1 - \rho^2)}} \exp \left( -\frac{1}{2(1 - \rho^2)} \left( \frac{x_1^2}{\sigma_1^2} + \frac{2\rho x_2 x_1'}{\sigma_1\sigma_2} + \frac{x_2'^2}{\sigma_2^2} \right) \right) dx_1 dx_2. \quad (6.38)$$

Changing the variables $x_1' = x_1 - \sigma t_1$ and $x_2' = x_2 - \sigma T$. The first integration limiting bound for (6.38) is given by $k_1 - \sigma t_1$. Substituting the value of for $k_1$ we obtain for it

$$k_1 - \sigma t_1 = \left( \ln(H/S_0) - (r - \frac{1}{2}\sigma^2)t_1 \right) / \sigma - \sigma t_1 = \sqrt{t_1}(-e_1),$$

where

$$e_1 = \left( \ln(S_0/H) + (r + \frac{1}{2}\sigma^2)t_1 \right) / \sigma \sqrt{t_1}.$$

The second integration limiting bound for (6.38) is given by

$$k_2 - \sigma \sqrt{T} = \left( \ln(H/S_0) - (r - \frac{1}{2}\sigma^2)T \right) / \sigma - \sigma T = \sqrt{T}(-b_1),$$

where

$$b_1 = \left( \ln(S_0/H) + (r + \frac{1}{2}\sigma^2)T \right) / \sigma \sqrt{T}.$$
The pricing formula for an up-and-out window barrier call option is given by:

\[
S_0 \int_{-\infty}^{\sqrt{t_1}(-e_1)} \int_{-\infty}^{\sqrt{t_2}(-b_1)} \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp \left( \frac{-1}{2(1-\rho^2)} \left( \frac{x_1^2}{\sigma_1^2} + 2\rho x_1 x_2 \frac{x_2^2}{\sigma_2^2} \right) \right) dx_1 dx_2
= S_0 P \left( X_1' \leq \sqrt{t_1}(-e_1), X_2' \leq \sqrt{T}(-b_1) \right)
= S_0 P \left( \sqrt{t_1}Z_1 \leq \sqrt{t_1}(-e_1), \sqrt{T}Z_2 \leq \sqrt{T}(-b_1) \right) \quad \text{where} \quad Z_1, Z_2 \sim N(0,1), \quad \text{Corr}(Z_1, Z_2) = \rho,
= S_0 P \left( Z_1 \leq (-e_1), Z_2 \leq (-b_1) \right)
= S_0 \int_{-\infty}^{-e_1} \int_{-\infty}^{-b_1} \frac{(2\pi)^{-1}}{\sqrt{1-\rho^2}} \exp \left( \frac{-1}{2(1-\rho^2)} \left( z_1^2 + 2\rho z_1 z_2 + z_2^2 \right) \right) dz_1 dz_2.
= S_0 N_2 \left( -e_1, -b_1; \sqrt{\frac{t_1}{T}} \right).
\]

All other integrals (6.29)-(6.35) are computed in a similar way. Theorem 6.4 is proved.

\[\square\]

### 6.5 Up-and-Out Window Barrier Call Options

**Theorem 6.5.** The pricing formula for an up-and-out window barrier call option is given by:

\[
S_0 \left( N_3 \left[ -e_1, -a_1, d_1; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_1}{T}}, -\sqrt{\frac{t_2}{T}} \right] - \left( \frac{H}{S_0} \right)^{2(\mu+1)/\sigma^2} N_3 \left[ g_1, -b_1, f_1; -\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_1}{T}}, -\sqrt{\frac{t_2}{T}} \right] \right)
- Ke^{-rt_1} \left( N_3 \left[ -e_2, -a_2, d_2; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_1}{T}}, -\sqrt{\frac{t_2}{T}} \right] - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N_3 \left[ g_2, -b_2, f_2; -\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_1}{T}}, -\sqrt{\frac{t_2}{T}} \right] \right),
\]

where

\[
e_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2)t_1}{\sigma \sqrt{t_1}}, \quad e_2 = e_1 - \sigma \sqrt{t_1},
\]

\[
g_1 = e_1 + \frac{2 \ln(H/S_0)}{\sigma \sqrt{t_1}}, \quad g_2 = g_1 - \sigma \sqrt{t_1},
\]

\[
a_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2)t_2}{\sigma \sqrt{t_2}}, \quad a_2 = a_1 - \sigma \sqrt{t_2},
\]

\[
b_1 = a_1 + \frac{2 \ln(H/S_0)}{\sigma \sqrt{t_2}}, \quad b_2 = b_1 - \sigma \sqrt{t_1},
\]

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The discounted expected payoff for a up-and-out window barrier call option after 3.6
6.39
(6.41)

Using the same approach as in the case of type A and B options, we have
expressed in terms as
where
disc can be written as:

Proof. The discounted expected payoff for a up-and-out window barrier call option after
discount can be written as:

\[ e^{-rT} \mathbb{E}(1_{S_t < H; t \in [t_1, t_2]}(S_T - K)^+) \]

Using the same approach as in the case of type A and B options, we have

\[ e^{-rT} \mathbb{E}(1_{S_t < H; t \in [t_1, t_2]}|S_{t_1}, S_{t_2}) \mathbb{E}((S_T - K)^+|S_{t_1}, S_{t_2})) \]

\[ = e^{-rT} \int_{-\infty}^{H} \int_{-\infty}^{H} \mathbb{P}(S_t < H; t \in [t_1, T]|S_{t_1} = s_1, S_{t_2} = s_2) \mathbb{E}((S_T - K)^+|S_{t_1} = s_1, S_{t_2} = s_2) \times \mathbb{P}(S_{t_1} \in ds_1, S_T \in ds_2). \] (6.39)

We again note that \( S_{t_i} \equiv S \exp((r - \frac{1}{2}\sigma^2)t_i + \sigma W_{t_i}) = s_1 \) is equivalent to \( W_{t_i} = x_i \), where \( x_i = (\ln(s_i/S_0) - (r - \frac{1}{2}\sigma^2)t_i)/\sigma, \ i = 1, 2 \). Furthermore, the event \( \{S_{t_i} < H\} \), can be expressed in terms as \( \{W_{t_i} < k_i\} \), where

\[ k_i = \left( \ln(H/S_0) - (r - \frac{1}{2}\sigma^2)t_i \right)/\sigma. \ i = 1, 2. \]

Therefore the price from (6.39) can be written as

\[ e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \mathbb{P}(S_t < H; t \in [t_1, t_2]|W_{t_1} = x_1, W_T = x_2) \mathbb{E}((S_T - K)^+|W_{t_1} = x_1, W_T = x_2) \times \mathbb{P}(W_{t_1} \in dx_1, W_{t_2} \in dx_2). \] (6.40)

Using (3.6), we have

\[ \mathbb{P}(S_t < H; t \in [t_1, t_2]|W_{t_1} = x_1, W_{t_2} = x_2) = 1 - \exp(c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1 x_2), \] (6.41)

where

\[ c_1 = -2(\ln(H/S_0))^2 + 2 \ln(H/S_0)(r - \frac{1}{2}\sigma^2)t_1 + 2 \ln(H/S_0)(r - \frac{1}{2}\sigma^2)t_2 - 2(r - \frac{1}{2}\sigma^2)^2 t_1 t_2, \]

\[ \sigma^2(t_2 - t_1). \]
Substituting (6.41) into (6.40) we obtain:

\[
e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} (1 - \exp(c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1 x_2)) \mathbf{E} \left( (S_T - K)^+ \left| W_{t_1} = x_1, W_T = x_2 \right. \right) \times \mathbf{P}(W_{t_1} \in dx_1, W_{t_2} \in dx_2).
\]

Here,

\[
\mathbf{E} \left( (S_T - K)^+ \left| W_{t_1} = x_1, W_{t_2} = x_2 \right. \right) = S_{t_2} e^{r(T-t_2)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_2 + x_2}{\sqrt{T - t_2}} \right] - K N \left[ \frac{\sqrt{T}d_2 + x}{\sqrt{T - t_2}} \right],
\]

where

\[
d_1 = \left( \ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T \right) /\sigma\sqrt{T} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.
\]

Substituting (6.43) into (6.42), we obtain

\[
e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} S_{t_2} e^{r(T-t_2)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_2 + x_2}{\sqrt{T - t_2}} \right] \mathbf{P}(W_{t_1} \in dx_1, W_{t_2} \in dx_2),
\]

Expanding the integral into four separate parts and evaluating them individually, we have

\[
e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} S_{t_2} e^{r(T-t_2)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_2 + x_2}{\sqrt{T - t_2}} \right] \mathbf{P}(W_{t_1} \in dx_1, W_{t_2} \in dx_2),
\]

\[
- K e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} N \left[ \frac{\sqrt{T}d_2 + x}{\sqrt{T - t_2}} \right] \mathbf{P}(W_{t_1} \in dx_1, W_{t_2} \in dx_2),
\]

\[
- e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} S_{t_1} \exp(c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1 x_2) S_{t_2} e^{r(T-t_2)} N \left[ \frac{\sqrt{T}d_1 - \sigma t_2 + x_2}{\sqrt{T - t_2}} \right] \times \mathbf{P}(W_{t_1} \in dx_1, W_{t_2} \in dx_2)
\]
\[ +K e^{-rT} \int_{-\infty}^{k_2} \int_{-\infty}^{k_1} \exp(c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1 x_2) N \left( \frac{\sqrt{T}d_2 + x}{\sqrt{T} - t_2} \right) \]

\[ \times \mathbf{P}(W_{t_1} \in dx_1, W_{t_2} \in dx_2). \]  

(6.47)

The integrals (6.44) and (6.47) can be computed in the same way as partial-time barrier options. Therefore we will omit the remainder of the proof and encourage the remainder of the proof.

\[ \Box \]

### 6.6 Up-and-In-Out Barrier Call Options

**Theorem 6.6.** The pricing formula for an up-and-in-out barrier call option is given by:

\[
S_0 \left[ N_2 \left( e_1, -d_1; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2(\mu/\sigma^2 + 1)} N_2 \left( -e_3, -f_1; \sqrt{\frac{t_1}{T}} \right) \right] - Ke^{-rT} \left[ N_2 \left( e_2, -d_2; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N_2 \left( -e_3, -f_2; \sqrt{\frac{t_1}{T}} \right) \right] - S_0 \left[ N_2 \left( e_1, g_1; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2(\mu/\sigma^2 + 1)} N_2 \left( -e_3, -g_3; \sqrt{\frac{t_1}{T}} \right) \right] + Ke^{-rT} \left[ N_2 \left( e_2, -g_2; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H}{S_0} \right)^{2\mu/\sigma^2} N_2 \left( -e_3, g_4; \sqrt{\frac{t_1}{T}} \right) \right],
\]

where

\[
d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T},
\]

\[
f_1 = \frac{\ln(S_0/K) + 2 \ln(H/S_0) + (r + \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}}, \quad f_2 = f_1 - \sigma \sqrt{T},
\]

\[
e_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2) t_1}{\sigma \sqrt{t_1}}, \quad e_2 = e_1 - \sigma \sqrt{t_1},
\]

\[
g_1 = e_1 + \frac{2 \ln(H/S_0)}{\sigma \sqrt{t_1}}, \quad g_2 = g_1 - \sigma \sqrt{t_1},
\]

\[
b_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}}, \quad b_2 = b_1 - \sigma \sqrt{T},
\]
\[ b_3 = b_1 + \frac{2 \ln(H/S_0)}{\sigma \sqrt{t}}, \quad b_4 = b_3 - \sigma \sqrt{T}, \quad \mu = r - \frac{1}{2} \sigma^2. \]

**Proof.** The discounted expected payoff for a up-and-in-out barrier call option can be written as

\[ e^{-rT} E \left( 1_{S_t > H; t \in [0, t_1]} \cap 1_{S_t < H; t \in [t_1, T]} (S_T - K)^+ \right). \]

Here, the kick-in barrier is active from 0 to \( t_1 \) and the kick-out barrier is active at times from \( t_1 \) to \( T \).

\[ e^{-rT} E \left( E \left( 1_{S_t > H; t \in [0, t_1]} \cap 1_{S_t < H; t \in [t_1, T]} (S_T - K)^+ \mid S_{t_1}, S_T \right) \right). \]

By the Markov property and by independent increments, we have the following representation for the above expression:

\[ e^{-rT} \int_{-\infty}^{H} \int_{-\infty}^{H} P \left( S_t > H; t \in [0, t_1] \mid S_{t_1} = s_1, S_T = s_2 \right) P \left( S_t < H; t \in [t_1, T] \mid S_{t_1} = s_1, S_T = s_2 \right) \]

\[ \times (s_2 - K) P (S_{t_1} \in ds_1, S_T \in ds_2). \]

\[ = e^{-rT} \int_{-\infty}^{H} \int_{-\infty}^{H} P \left( S_t < H; t \in [t_1, T] \mid S_{t_1} = s_1, S_T = s_2 \right) P \left( S_t > H; t \in [0, t_1] \mid S_{t_1} = s_1, S_T = s_2 \right) \]

\[ \times (s_2 - K) P (S_{t_1} \in ds_1, S_T \in ds_2) - e^{-rT} \int_{-\infty}^{H} \int_{-\infty}^{H} P \left( S_t > H; t \in [0, t_1] \mid S_{t_1} = s_1, S_T = s_2 \right) \]

\[ \times P \left( S_t < H; t \in [t_1, T] \mid S_{t_1} = s_1, S_T = s_2 \right) (s_2 - K) P (S_{t_1} \in ds_1, S_T \in ds_2). \]

(6.48)

Note that \( \{S_{t_1} < H\} = \{W_{t_1} < k_1\} \), \( \{S_T\} = \{W_T < k_2\} \) and \( \{S_T\} = \{W_T < k_3\} \),

where \( k_1 = (\ln(H/S_0) - (r - \frac{1}{2} \sigma^2)t_1) / \sigma \), \( k_2 = (\ln(H/S_0) - (r - \frac{1}{2} \sigma^2)T) / \sigma \) and \( k_3 = (\ln(K/S_0) - (r - \frac{1}{2} \sigma^2)T) / \sigma \), thus (6.48) becomes

\[ e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} P(S_t > H; t \in [0, t_1] \mid W_{t_1} = x_1, W_T = x_2) P(S_t < H; t \in [t_1, T] \mid W_{t_1} = x_1, W_T = x_2) \]

\[ \times \left( S_0 e^{(r - \frac{1}{2} \sigma^2)T + \sigma x_2} - K \right) P(W_{t_1} \in dx_1, W_T \in dx_2) \]

\[ - e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_3} P(S_t > H; t \in [0, t_1] \mid W_{t_1} = x_1, W_T = x_2) P(S_t < H; t \in [t_1, T] \mid W_{t_1} = x_1, W_T = x_2) \]

\[ \times \left( S_0 e^{(r - \frac{1}{2} \sigma^2)T + \sigma x_2} - K \right) P(W_{t_1} \in dx_1, W_T \in dx_2). \]

(6.49)
Using (3.6), we have:

\[ P(S_t > H; t \in [0, t_1] | W_{t_1} = x_1, W_T = x_2) = \exp(c_1 + c_2 x_1) \]  
\[ (6.50) \]

where

\[ c_1 = \frac{-2(\ln(H/S_0))^2 + 2 \ln(H/S_0)(r - \frac{1}{2} \sigma^2)t_1}{\sigma^2 t_1} \quad \text{and} \quad c_2 = \frac{2 \ln(H/S_0)}{\sigma t_1}, \]

also

\[ P(S_t < H; t \in [t_1, T] | W_{t_1} = x_1, W_T = x_2) = 1 - \exp(c_3 + c_4 x_1 + c_5 x_2 + c_6 x_1 x_2) \]  
\[ (6.51) \]

where

\[ c_3 = \frac{-2(\ln(H/S_0))^2 + 2 \ln(H/S_0)(r - \frac{1}{2} \sigma^2)t_1 + 2 \ln(H/S_0)(r - \frac{1}{2} \sigma^2)T - 2(r - \frac{1}{2} \sigma^2)^2 t_1 T}{\sigma^2 (T - t_1)}, \]
\[ c_4 = \frac{2 \ln(H/S_0) - 2(r - \frac{1}{2} \sigma^2)T}{\sigma (T - t_1)}, \quad c_5 = \frac{2 \ln(H/S_0) - 2(r - \frac{1}{2} \sigma^2)t_1}{\sigma (T - t_1)} \quad \text{and} \quad c_6 = \frac{-2}{T - t_1}. \]

Substituting (6.50) and (6.51) into (6.49) we obtain:

\[
e^{-rT} \int_{-\infty}^{k_1} \int_{-\infty}^{k_2} \exp(c_1 + c_2 x_1) (1 - \exp(c_3 + c_4 x_1 + c_5 x_2 + c_6 x_1 x_2)) \times \left(S_0 e^{(r - \frac{1}{2} \sigma^2)T + \sigma x_2} - K\right) P(S_{t_1} \in ds_1, S_T \in ds_2) \\
- e^{-rT} \int_{-\infty}^{k_3} \int_{-\infty}^{k_3} \exp(c_1 + c_2 x_1) (1 - \exp(c_3 + c_4 x_1 + c_5 x_2 + c_6 x_1 x_2)) \times \left(S_0 e^{(r - \frac{1}{2} \sigma^2)T + \sigma x_2} - K\right) P(S_{t_1} \in ds_1, S_T \in ds_2). \]  
\[ (6.52) \]

Expanding (6.52) into eight separate integrals where we can evaluate them individually. Furthermore we have seen this type of computation when considering type B options, where we simply deal with a linear combination of exponentials and try to express them in terms of normal CDF’s.

\[ \square \]

### 6.7 Up-In and Down-Out Barrier Call Option with \( H_1 > H_2 \)

**Theorem 6.7.** The pricing formula for an up-in and down-out barrier call option where \( K < H \) is given by:
\[ S_0 \left[ N_2 \left( e_1, b_1; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H_2}{S_0} \right)^{2(\mu/\sigma^2+1)} N_2 \left( -g_1, b_3; -\sqrt{\frac{t_1}{T}} \right) \right] \]

\[-Ke^{-rT} \left[ N_2 \left( e_2, b_2; \sqrt{\frac{t_1}{T}} \right) - \left( \frac{H_2}{S_0} \right)^{2\mu/\sigma^2} N_2 \left( -g_2, b_4; -\sqrt{\frac{t_1}{T}} \right) \right] \]

\[ +S_0 \left( \frac{H_1}{S_0} \right)^{2(\mu/\sigma^2+1)} \left[ N_2 \left( n_1, m_1; \sqrt{\frac{t_1}{T}} \right) - N_2 \left( g_1, m_1; \sqrt{\frac{t_1}{T}} \right) \right] \]

\[-S_0 \left( \frac{H_1}{S_0} \right)^{-2(\mu/\sigma^2+1)} \left( \frac{H_2}{S_0} \right)^{2(\mu/\sigma^2+1)} \left[ N_2 \left( -n_3, m_3; -\sqrt{\frac{t_1}{T}} \right) - N_2 \left( -e_1, m_3; -\sqrt{\frac{t_1}{T}} \right) \right] \]

\[-Ke^{-rT} \left( \frac{H_1}{S_0} \right)^{2(\mu/\sigma^2)} \left[ N_2 \left( n_2, m_2; \sqrt{\frac{t_1}{T}} \right) - N_2 \left( g_2, m_2; \sqrt{\frac{t_1}{T}} \right) \right] \]

\[ +Ke^{-rT} \left( \frac{H_1}{S_0} \right)^{-2(\mu/\sigma^2)} \left( \frac{H_2}{S_0} \right)^{2(\mu/\sigma^2)} \left[ N_2 \left( -n_4, m_4; -\sqrt{\frac{t_1}{T}} \right) - N_2 \left( -e_2, m_1; -\sqrt{\frac{t_1}{T}} \right) \right], \]

where

\[ n_1 = \frac{\ln(H_1^2/S_0H_2) + (r + \frac{1}{2}\sigma^2)t_1}{\sigma \sqrt{T}}, \quad n_2 = n_1 - \sigma \sqrt{t_1}, \]

\[ n_3 = \frac{\ln(S_0H_2/H_1^2) + (r + \frac{1}{2}\sigma^2)t_1}{\sigma \sqrt{T}}, \quad n_4 = n_3 - \sigma \sqrt{t_1}, \]

\[ m_1 = \frac{\ln(H_1^2/S_0H_2) + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}, \quad m_2 = m_1 - \sigma \sqrt{T}, \]

\[ m_3 = \frac{\ln(S_0H_2/H_1^2) + (r + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}, \quad m_4 = m_3 - \sigma \sqrt{T}, \]

\[ e_1 = \frac{\ln(S_0/H) + (r + \frac{1}{2}\sigma^2)t_1}{\sigma \sqrt{T}}, \quad e_2 = e_1 - \sigma \sqrt{t_1}. \]
\[ g_1 = e_1 + \frac{2 \ln(H_1/S_0)}{\sigma \sqrt{t_1}}, \quad g_2 = g_1 - \sigma \sqrt{t_1}, \]

\[ b_1 = \frac{\ln(S_0/H_2) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}}, \quad b_2 = b_1 - \sigma \sqrt{T}, \]

\[ b_3 = b_1 + \frac{2 \ln(H_2/S_0)}{\sigma \sqrt{t_1}}, \quad b_4 = b_3 - \sigma \sqrt{T}, \quad \mu = r - \frac{1}{2} \sigma^2. \]

**Proof.** The following derivation will follow the same approach as Theorem 6.6, however we now have two barrier \( H_1 \) and \( H_2 \) at different levels, with \( H_1 \) an up and \( H_2 \) as an down barrier. The expected payoff price for a up-and-in-out barrier call option after discount can be written as,

\[ e^{-rT} E \left( 1_{S_t > H_1; t \in [0,t_1]} \cap 1_{S_t > H_2; t \in [t_1,T]} (S_T - K)^+ \right) \]

By the Markov property and by independent increments, we obtain:

\[
e^{-rT} E \left( E \left(1_{S_t > H_1; t \in [0,t_1]} | S_{t_1}, S_T \right) \cdot E \left(1_{S_t > H_2; t \in [t_1,T]} | S_{t_1}, S_T \right) \cdot E \left((S_T - K)^+ | S_{t_1}, S_T \right) \right)
\]

\[ = e^{-rT} \int_{H_1}^{H_2} \int_{H_2}^{H_2} \mathbb{P}(S_t > H; t \in [0, t_1] | S_{t_1} = s_1, S_T = s_2) \mathbb{P}(S_t > H; t \in [t_1, T] | S_{t_1} = s_1, S_T = s_2) \times (s_2 - K) \mathbb{P}(S_{t_1} \in ds_1, S_T \in ds_2) + e^{-rT} \int_{H_1}^{H_2} \int_{H_2}^{H_2} \mathbb{P}(S_t > H; t \in [t_1, T] | S_{t_1} = s_1, S_T = s_2) (s_2 - K) \times \mathbb{P}(S_{t_1} \in ds_1, S_T \in ds_2) \times \left( s_0 e^{(r - \frac{1}{2} \sigma^2)T + \sigma x_2} - K \right) \mathbb{P}(S_{t_1} \in ds_1, S_T \in ds_2) - e^{-rT} \int_{H_1}^{H_2} \int_{H_2}^{H_2} \mathbb{P}(S_t > H; t \in [0, t_1] | S_{t_1} = s_1, S_T = s_2) \times (s_2 - K) \mathbb{P}(S_{t_1} \in ds_1, S_T \in ds_2) + e^{-rT} \int_{H_1}^{H_2} \int_{H_2}^{H_2} \mathbb{P}(S_t > H; t \in [t_1, T] | S_{t_1} = s_1, S_T = s_2) (s_2 - K) \mathbb{P}(S_{t_1} \in ds_1, S_T \in ds_2) \]

Again, note that the events \( \{S_{t_1} > H_1\} \), \( \{S_{t_1} > H_2\} \) and \( \{S_T > H_2\} \) can be expressed in terms of \( \{W_{t_1}\} \) and \( \{W_T\} \) as \( \{W_{t_1} > k_1\} \), \( \{W_{t_1} > k_2\} \) and \( \{W_T > k_3\} \), where

\[ k_1 = \left( \ln(H_1/S_0) - (r - \frac{1}{2} \sigma^2) t_1 \right) / \sigma, \quad k_2 = \left( \ln(H_2/S_0) - (r - \frac{1}{2} \sigma^2) t_1 \right) / \sigma \]
and
\[ k_3 = \left( \ln(H_2/S_0) - (r - \frac{1}{2}\sigma^2)T \right) / \sigma. \]

Using (3.6) and (3.7), we have:
\[ P(S_t > H_1; t \in [0, t_1] | W_{t_1} = x_1, W_T = x_2) = \exp(c_1 + c_2x_1) \]
where
\[ c_1 = \frac{-2(\ln(H_1/S_0))^2 + 2 \ln(H_1/S_0)(r - \frac{1}{2}\sigma^2)t_1}{\sigma^2 t_1} \quad \text{and} \quad c_2 = \frac{2 \ln(H_1/S_0)}{\sigma t_1}, \]
also
\[ P(S_t > H_2; t \in [t_1, T] | W_{t_1} = x_1, W_T = x_2) = 1 - \exp(c_3 + c_4x_1 + c_5x_2 + c_6x_1x_2) \]
where
\[ c_3 = \frac{-2(\ln(H_2/S_0))^2 + 2 \ln(H_2/S_0)(r - \frac{1}{2}\sigma^2)t_1 + 2 \ln(H_2/S_0)(r - \frac{1}{2}\sigma^2)T - 2(r - \frac{1}{2}\sigma^2)^2 t_1 T}{\sigma^2(T - t_1)} \]
\[ c_4 = \frac{2 \ln(H_2/S_0) - 2(r - \frac{1}{2}\sigma^2)T}{\sigma(T - t_1)}, \quad c_5 = \frac{2 \ln(H_2/S_0) - 2(r - \frac{1}{2}\sigma^2)t_1}{\sigma(T - t_1)} \quad \text{and} \quad c_6 = \frac{-2}{T - t_1}. \]
So we now have
\[ e^{-rT} \int_{k_1}^{\infty} \int_{k_3}^{\infty} \exp(c_1 + c_2x_1) \left( 1 - \exp(c_3 + c_4x_1 + c_5x_2 + c_6x_1x_2) \right) \left( S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x_2} - K \right) \]
\[ \times P(W_{t_1} \in x_1, W_T \in x_2) - e^{-rT} \int_{k_2}^{\infty} \int_{k_3}^{\infty} \exp(c_1 + c_2x_1) \left( 1 - \exp(c_3 + c_4x_1 + c_5x_2 + c_6x_1x_2) \right) \]
\[ \times \left( S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x_2} - K \right) P(S_{t_1} \in ds_1, S_T \in ds_2) + e^{-rT} \int_{k_1}^{\infty} \int_{k_3}^{\infty} \left( 1 - \exp(c_3 + c_4x_1 + c_5x_2 + c_6x_1x_2) \right) \]
\[ \times \left( S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x_2} - K \right) P(W_{t_1} \in dx_1, W_T \in dx_2). \]

(6.53)

Expanding (6.53) into twelve separate integrals where we can evaluate them individually. Furthermore we have seen this type of computation when considering type B options, where we simply deal with a linear combination of exponentials and try to express them in terms of normal CDF’s.
Chapter 7

Computational Results

In this chapter, we compare numerical results obtained from the derived formulae with results of Monte Carlo simulations for options’ prices. We give computational results for; up-and-out type A partial-time barrier call options, up-and-out window barrier call options, up-in-and-out barrier call options and up-in and down-out barrier options with $H_1 > H_2$. The comparisons appear in tables where prices and their respective standard errors are presented for a range of different initial asset values $S_0$ and barrier levels $H$. For each of the barrier options considered, we also present numerical results for the Delta and Eta using the numerical differentiation techniques discussed in Chapter 5. We compare these with the explicit formula for Delta and Eta values, with different levels of $S_0$ and $H$. Explicit formula for Delta and Eta were obtained using Mathematica. As an example, Delta and Eta formulae for a up-and-out type A partial-time barrier call option are presented in the Appendix section. Antithetic variables were used for simulations of expected payoffs and for the Greeks. $N = 50000$ trials were used for each estimate obtained.

7.1 Up-Out Type A Partial-time Barrier Call Option

Price values for a range of $S_0$ values and $H$ levels

<table>
<thead>
<tr>
<th>Initial value $S_0$</th>
<th>Barrier value $H$</th>
<th>Formula value</th>
<th>Simulated value (Standard error)</th>
<th>Using Anithetic variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>130</td>
<td>5.67180</td>
<td>5.69559 (0.04436)</td>
<td>5.67147 (0.02563)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>15.39934</td>
<td>15.29553 (0.06524)</td>
<td>15.42201 (0.01205)</td>
</tr>
<tr>
<td>120</td>
<td>130</td>
<td>8.10874</td>
<td>8.10451 (0.04463)</td>
<td>8.10413 (0.02128)</td>
</tr>
<tr>
<td>100</td>
<td>105</td>
<td>1.53276</td>
<td>1.53561 (0.01391)</td>
<td>1.542166 (0.00860)</td>
</tr>
<tr>
<td>100</td>
<td>110</td>
<td>4.01429</td>
<td>4.03262 (0.02839)</td>
<td>4.00829 (0.01702)</td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td>19.42605</td>
<td>19.46891 (0.07906)</td>
<td>19.39761 (0.02370)</td>
</tr>
</tbody>
</table>

Table 1.1: $K = 90$, $r = 0.1$, $\sigma = 0.2$, barrier $H$ starting at 0 and finishing at $t_1 = 0.5$ and maturity time $T = 1$ year.
Delta Greeks and Eta Greeks for a range of $S_0$ values and $H$ levels.

<table>
<thead>
<tr>
<th>Initial value $S_0$</th>
<th>Barrier level $H$</th>
<th>Formula value</th>
<th>Simulated value (Standard error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>130</td>
<td>0.48466</td>
<td>0.48388 (0.00179)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>0.22343</td>
<td>0.2267 (0.00548)</td>
</tr>
<tr>
<td>120</td>
<td>130</td>
<td>-0.81315</td>
<td>-0.81659 (0.00429)</td>
</tr>
<tr>
<td>$\eta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>110</td>
<td>0.57147</td>
<td>0.57109 (0.00287)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>0.39704</td>
<td>0.39260 (0.003708)</td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td>0.06797</td>
<td>0.06966 (0.00219)</td>
</tr>
</tbody>
</table>

Table 1.2: $K = 90$, $r = 0.1$, $\sigma = 0.2$, barrier $H$ starting at 0 and finishing at $t_1 = 0.5$ and maturity time $T = 1$ year.

### 7.2 Up-and-Out Window Barrier Call Option

Price values for a range of $S_0$ value and $H$ levels.

<table>
<thead>
<tr>
<th>Initial value $S_0$</th>
<th>Barrier level $H$</th>
<th>Formula payoff value</th>
<th>Simulated (Standard error)</th>
<th>Using Anithetic variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>130</td>
<td>5.21984</td>
<td>5.21603 (0.03974)</td>
<td>5.25575 (0.01725)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>11.08011</td>
<td>11.12401 (0.04986)</td>
<td>11.11003 (0.03088)</td>
</tr>
<tr>
<td>120</td>
<td>130</td>
<td>5.77673</td>
<td>5.72093 (0.03935)</td>
<td>5.77796 (0.02135)</td>
</tr>
<tr>
<td>100</td>
<td>105</td>
<td>0.99234</td>
<td>1.00332 (0.01235)</td>
<td>0.98487 (0.00815)</td>
</tr>
<tr>
<td>100</td>
<td>110</td>
<td>2.20433</td>
<td>2.18649 (0.01963)</td>
<td>2.18605 (0.01222)</td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td>17.51395</td>
<td>17.54131 (0.06894)</td>
<td>17.52885 (0.02368)</td>
</tr>
</tbody>
</table>

Table 2.1: $K = 90$, $r = 0.1$, $\sigma = 0.2$, barrier $H$ starting at $t_1 = 0.25$ and finishing at $t_2 = 0.75$ and maturity time $T = 1$ year.
Delta and Eta Greeks for a range of $S_0$ and $H$ levels

<table>
<thead>
<tr>
<th></th>
<th>Initial value $S_0$</th>
<th>Barrier level $H$</th>
<th>Formula value</th>
<th>Simulated sensitivity value</th>
<th>(Standard error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>80</td>
<td>130</td>
<td>0.40069</td>
<td>0.40416</td>
<td>(0.00255)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>130</td>
<td>0.01732</td>
<td>0.01759</td>
<td>(0.00459)</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>130</td>
<td>-0.38469</td>
<td>-0.38249</td>
<td>(0.00334)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>100</td>
<td>110</td>
<td>0.30394</td>
<td>0.30293</td>
<td>(0.00211)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>130</td>
<td>0.45616</td>
<td>0.45554</td>
<td>(0.00311)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>150</td>
<td>0.18575</td>
<td>0.18404</td>
<td>(0.00281)</td>
</tr>
</tbody>
</table>

Table 2.2: $K = 90$, $r = 0.1$, $\sigma = 0.2$, barrier $H$ starting at $t_1 = 0.25$ and finishing at $t_2 = 0.75$ and maturity time $T = 1$ year.

### 7.3 Up-and-In-Out Barrier Call Option

Price values for a range of $S_0$ values and $H$ levels

<table>
<thead>
<tr>
<th>Initial value $S_0$</th>
<th>Barrier level $H$</th>
<th>Formula payoff value</th>
<th>Simulated value (Standard error)</th>
<th>Using Anithetic variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>130</td>
<td>0.00182</td>
<td>0.00197 (0.00018)</td>
<td>0.00168 (0.00010)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>0.173045</td>
<td>0.17488 (0.00211)</td>
<td>0.17368 (0.00138)</td>
</tr>
<tr>
<td>120</td>
<td>130</td>
<td>1.44639</td>
<td>1.46772 (0.01013)</td>
<td>1.43598 (0.00542)</td>
</tr>
<tr>
<td>100</td>
<td>105</td>
<td>0.22574</td>
<td>0.22592 (0.00248)</td>
<td>0.22646 (0.00160)</td>
</tr>
<tr>
<td>100</td>
<td>110</td>
<td>0.31930</td>
<td>0.32048 (0.00276)</td>
<td>0.31986 (0.00168)</td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td>0.02209</td>
<td>0.02079 (0.00079)</td>
<td>0.02254 (0.00057)</td>
</tr>
</tbody>
</table>

Table 3.1: $K = 90$, $r = 0.1$, $\sigma = 0.2$, barrier $H$ starting at 0 and finishing at $t_1 = 0.5$ and maturity time $T = 1$ year.
## Delta and Eta Greeks for a range of $S_0$ levels and $H$ levels

<table>
<thead>
<tr>
<th>Initial value $S_0$</th>
<th>Barrier level $H$</th>
<th>Formula sensitivity price value</th>
<th>Simulated sensitivity value (Standard error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>130</td>
<td>0.00058</td>
<td>0.00059 (0.00006)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>0.02676</td>
<td>0.02718 (0.00046)</td>
</tr>
<tr>
<td>120</td>
<td>130</td>
<td>0.00482</td>
<td>0.00465 (0.00023)</td>
</tr>
<tr>
<td>$\eta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>110</td>
<td>-0.01239</td>
<td>0.01206 (0.00048)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>-0.01343</td>
<td>-0.01336 (0.00035)</td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td>-0.00271</td>
<td>-0.00277 (0.00016)</td>
</tr>
</tbody>
</table>

Table 3.2: $K = 90$, $r = 0.1$, $\sigma = 0.2$, barrier $H$ starting at 0 and finishing at $t_1 = 0.5$ and maturity time $T = 1$ year.

### 7.4 Up-In & Down-Out Barrier Call Option with $H_1 > H_2$

Price values for a range of initial asset prices levels and barrier levels

<table>
<thead>
<tr>
<th>Initial value $S_0$</th>
<th>Barrier value $H_1$</th>
<th>Barrier value $H_2$</th>
<th>Formula value</th>
<th>Simulated value (Standard error)</th>
<th>Using Anithetic variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>130</td>
<td>110</td>
<td>0.05966</td>
<td>0.05638 (0.00657)</td>
<td>0.06046 (0.00501)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>110</td>
<td>4.15305</td>
<td>4.14798 (0.05982)</td>
<td>4.15906 (0.04035)</td>
</tr>
<tr>
<td>120</td>
<td>130</td>
<td>110</td>
<td>27.57351</td>
<td>27.54667 (0.12902)</td>
<td>27.57796 (0.04325)</td>
</tr>
<tr>
<td>100</td>
<td>115</td>
<td>110</td>
<td>7.88777</td>
<td>7.86196 (0.07775)</td>
<td>7.87246 (0.04876)</td>
</tr>
<tr>
<td>100</td>
<td>150</td>
<td>110</td>
<td>0.55865</td>
<td>0.54189 (0.02389)</td>
<td>0.56437 (0.01755)</td>
</tr>
<tr>
<td>100</td>
<td>180</td>
<td>110</td>
<td>0.00915</td>
<td>0.00800 (0.00360)</td>
<td>0.00912 (0.00246)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>95</td>
<td>4.57109</td>
<td>4.58032 (0.06062)</td>
<td>4.57571 (0.04051)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>120</td>
<td>3.06799</td>
<td>3.06693 (0.05489)</td>
<td>3.02863 (0.03737)</td>
</tr>
<tr>
<td>100</td>
<td>130</td>
<td>125</td>
<td>2.23573</td>
<td>2.23551 (0.04805)</td>
<td>2.21549 (0.03331)</td>
</tr>
</tbody>
</table>

Table 4.1: $K = 90$, $r = 0.1$, $\sigma = 0.2$, $H_1$ in between times $[0, t_1]$ and $H_2$ in between times $[t_1, T]$ and $T = 1$ year.
Delta and Eta Greeks for a range of $S_0$ values and barrier levels for $H_1$.

<table>
<thead>
<tr>
<th></th>
<th>Initial value $S_0$</th>
<th>Barrier value $H_1$</th>
<th>Formula value</th>
<th>Simulated sensitivity value (Standard error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>80</td>
<td>130</td>
<td>0.01793</td>
<td>0.01773 (0.00141)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>130</td>
<td>0.58351</td>
<td>0.58596 (0.00529)</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>130</td>
<td>1.61056</td>
<td>1.61160 (0.00535)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>100</td>
<td>130</td>
<td>-0.07541</td>
<td>-0.07413 (0.00130)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>130</td>
<td>-0.30847</td>
<td>-0.30626 (0.00360)</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>180</td>
<td>-0.06695</td>
<td>-0.06734 (0.00216)</td>
</tr>
</tbody>
</table>

Table 4.2: $K = 90$, $H_2 = 110$, $r = 0.1$, $\sigma = 0.2$, $H_1$ in between times $[0, t_1]$ and $H_2$ in between times $[t_1, T]$ and $T = 1$ year.

### 7.5 Analyses

Monte Carlo simulation gave very good price estimates with close proximity to the values given by the derived pricing formulae. Note that, the use of antithetic variables had given smaller standard errors for prices, in most case they were cut by almost 50%. For sensitivities, Monte Carlo simulation also showed results consistent with formulae obtained by taking the partial derivatives to find both Delta and Eta. Overall, Monte Carlo simulation does show consistency’s with expected payoff values and Greeks for the barrier options specified.

For Figures 7.1-7.4, four plots (a)-(d) are presented within each Figure. The following barrier options and their respective fixed parameter values are given in each Figure:

(a) up-and-out type A partial-time barrier call options, with $K = 90$, $r = 0.1$, $\sigma = 0.2$ and $t_1 = 0.5$.

(b) up-and-out window barrier call options, with $K = 90$, $r = 0.1$, $\sigma = 0.2$ and $t_1 = 0.25$ and $t_2 = 0.75$,

(c) up-in-and-out barrier call options $K = 90$, $r = 0.1$, $\sigma = 0.2$ and $t_1 = 0.5$

(d) up-in and down-out barrier call options $K = 90$, $r = 0.1$, $H_2 = 110$, $\sigma = 0.2$ and $t_1 = 0.5$. 

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Comparison of prices for different barrier options, with different $S_0$ values

Figure 7.1: Here, fixed barrier levels $H_1 = 130$ and $H_2 = 110$ were used. The up-in-and-out barrier call option (c) has a the cheapest price value for increasing $S_0$, whereas the up-in and down-out barrier call option (d) keeps increasing in price value as $S_0$ is increased. Both the up-and-out type A partial-time barrier call option (a) and up-and-out window barrier call options (d) behave the same for increasing $S_0$, with the up-and-out window call barrier option (d) being slightly cheaper for the same $S_0$ values.
Comparison of prices for different barrier options, with different $H$ levels

Figure 7.2: Here, a fixed $S_0 = 100$ was used. Note that (c) has a different price scale in comparison to (a),(b) and (d). Here, the up-and-out type A partial-time barrier call option (a) and up-and-out window barrier call options (d) behave similarly for increasing $H$. As $H$ is increased the (a) and (b) behave like standard vanilla call options, this is expected since kick-out barrier is less to be hit for large $H$. Both up-in-and-out barrier call option (c) and up-in and down-out barrier call option (d) appear to be much cheaper in price value, also the price becomes worthless for larger $H$ and $H_1$, since the kick-in contribution for both options have small probabilities of being hit.
Comparison of Delta for different barrier options, with different $S_0$ values

Figure 7.3: Here, fixed barrier levels $H_1 = 130$ and $H_2 = 110$ were used. Note that Delta values are not scaled the same size for all (a)-(d). Here, we observe a sin like curve for the up-and-out type A partial-time barrier call option (a) and up-and-out window barrier call options (d) for increasing $S_0$. So, as $S_0$ is increased, small positive changes are observed for Delta, this begins to drop to 0 at $S_0 = 100$. Small negative values are then observed onwards. Both Up-in-and-out barrier call option (c) and up-in and down-out barrier call option (d) behave the same, where only positive values are observed for increasing Delta with a peak at approx. $S_0 = 120$ for both. Note that the Up-in-and-out barrier call option (c) give much smaller values for Delta as $S_0$ is increased.
Comparison of Eta for different barrier options, with different $H$ levels

Figure 7.4: Here, a fixed $S_0 = 100$ was used. Again note that Eta values are not scaled the same size for all (a)-(d). Here, the up-in-and-out barrier call option (c) has very small Eta values for larger $H$, the limiting value becomes 0 for $H > 160$. Up-and-out window barrier call options (d) gave negative values for Eta as $H_1$ becomes larger. Type A partial-time barrier call option (a) and up-and-out window barrier call options (b) appear to follow the same trend, giving positive values for Eta with a peak value at approx. $H = 120$ for (a) and $H = 125$ for (b).
Chapter 8

Conclusions and Further Developments

8.1 Conclusions

Our prime focus in the present thesis was on deriving closed-form formulae for a selection of European barrier options that are used in the financial market. We demonstrated that, by using properties of a Brownian motion and the Brownian Bridge process we were able to obtain probabilities of boundary hitting by the Brownian motion. Using techniques and key ideas from standard vanilla options pricing, we were able to apply them in a more restrictive fashion in the option’s lifetime. We found that most pricing formulae for barrier options resulted in being very long, this of course depended on the barrier characteristics; such as the type, length and direction. This was also the case with the Greeks, where Mathematica was used to compute the closed-form formulae for Delta and Eta. Although unappealing in presentation, we found that most barrier options can be priced and their respective Greeks can also be obtained by using a general approach.

We also delved into unknown territory, constructing two new varieties of barrier options, the up-in-and-out barrier call option and the up-in-and-down-out barrier call options with $H_1 > H_2$. Here a combination of kick-in and kick-out barriers were considered in a partial-time sense. We found that these options were much cheaper in comparison to the other common barrier options due to the presence of the two restrictive barriers.

We used Monte Carlo simulation for the price estimation for certain barrier options, where a much faster approach for simulating price trajectories was applied. We found that the performance of simulation for a range of barrier options very well agreed with the formula prices. Furthermore, we found that the variance reduction technique of antithetic variables substantially improved the precision of Monte Carlo estimates. As a consequence this was indeed the case, where smaller standard errors were recognised. For option sensitivities, we experimented with the behaviour of the barrier options price, where the parameters were perturbed. We found some interesting results. In particular, for Delta, where small changes in the initial asset price behave as a sin curve for a up-and-in type A partial-time barrier option and a up-and-in window barrier options. Using simulations, we
found that the finite difference technique for numerical differentiation gives very accurate results for a range of parameters tested, with very consistent standard errors.

8.2 Further Developments

A barrier option which we did not consider for pricing is the double barrier option. Here, the underlying price is sandwiched between two kick-out barriers, one from above and one from below. We could introduce combinations of knock-in and knock-out barriers, also we could investigate Eta for the two barriers. These options are much more difficult to price, where Laplace transform techniques can be used.

Other Greeks for barrier options can also be investigated, such as Rho, Vega, Gamma, Theta and for different time lengths $t_i$, depending on the barrier activation and dis-activation times. Other approaches for computing Greeks can also be considered, such as the already mentioned piece-wise and likelihood ratio. Both of these techniques have not been fully investigated for barrier options and may provide more accurate results. Other possibilities for estimating Greeks is the use of Malliavin Calculus, which has not yet been fully investigated for barrier options.

Monte Carlo simulation could also be modified to improve estimation and performance. This includes different variance reduction techniques and the use of Quasi Monte Carlo simulation. We only considered one type of numerical differentiation technique, however some other can also be considered. A more robust technique that can be used other than finite differences is numerical differentiation by Finite-Dimensional regularization. This techniques involves more in-depth theory, so we avoided using this approach, that may however lead to more accurate results.
Chapter 9

Appendix

In the first section, we present the computation of derivatives for the bivariate and trivariate normal CDF’s. As an example of the R programming code used, we present the simulation code for an up-and-out window barrier call option, for both the price and Greeks in the second section. Note that simulation code is similar for the other barrier options. Finally, as an example of the Mathematica output, we present the closed-form formula of Delta and Eta for an Up-and-out Type A partial-time barrier call option.

9.1 Derivatives of Bivariate and Trivariate CDF’s

Here, we compute the derivatives of the bivariate and trivariate CDF’s. For the bivariate normal CDF we have

\[ N_2[x, y; \rho] = \mathbf{P}(X \leq x, Y \leq y) = \int_{-\infty}^{y} \mathbf{P}(X \leq x \mid Y = z) \mathbf{P}(y \in dz) \]

Note that

\[ \frac{\partial}{\partial x} N_2[x, y; \rho] = \frac{\partial}{\partial y} N_2[x, y; \rho] \]

by symmetry and provided that we swap the variables in the function on the RHS (after differentiation). Thus we only need to compute \( \frac{\partial}{\partial x} N_2[x, y; \rho] \). Also note that

\[ \frac{\partial}{\partial x} \mathbf{P}(X \leq x \mid Y = z) = f_{X|Y}(x \mid z) \]

Taking the derivative with respect to \( x \) we have

\[ \frac{\partial}{\partial x} \mathbf{P}(X \leq x, Y \leq y) = \int_{-\infty}^{y} f_{X|Y}(x \mid z) \mathbf{P}(Y \in dz) \]

\[ = \int_{-\infty}^{y} f_{X|Y}(x \mid z) f_Y(z) dz \]

\[ = \frac{1}{(2\pi)^{\frac{3}{2}}(1-\rho^2)} \int_{-\infty}^{y} \exp \left(-\frac{(x - \rho z)^2}{2\sqrt{1-\rho^2}}\right) \exp \left(-\frac{z^2}{2}\right) \]

\[ = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{1-\rho^2}} \int_{-\infty}^{y} \exp \left(-\frac{1}{2} \left[ z^2 + \frac{\rho^2}{1-\rho^2} z^2 - \frac{2\rho x z}{1-\rho^2} + \frac{x^2}{1-\rho^2} \right] \right) \]
Using the “complete square” form and a change of variables, we obtain the following closed-form expression for the standard normal CDF.

\[
\frac{\partial}{\partial x} P(X \leq x, Y \leq y) = \frac{\exp(\alpha)}{4\pi^2(1 - \rho^2)} N[\beta],
\]

where

\[
\alpha = x^2 - \rho^2 \quad \text{and} \quad \beta = y - \rho.
\]

Thus, the derivative of the bivariate normal CDF can be expressed as a standard normal CDF.

For the trivariate normal CDF we have

\[
N_3[x, y, z; \rho_{12}, \rho_{13}, \rho_{23}] = P(X \leq x, Y \leq y, Z \leq y) = \int_{-\infty}^{z} \int_{-\infty}^{y} P(X \leq x \mid Y = y_1, Z = z_1) P(y \in dy_1, Z \in dz_1)
\]

Note that we have

\[
\frac{\partial}{\partial x} P(X \leq x \mid Y = y_1, Z = z_1) = f_{X|Y,Z}(x \mid y_1, z_1)
\]

and

\[
P(Y \in dz) = f_{Y,Z}(y_1, z_1)dy_1dz_1
\]

By taking the derivative with respect \(x\) we have

\[
\frac{\partial}{\partial x} P(X \leq x, Y \leq y, Z \leq z) = \int_{-\infty}^{z} \int_{-\infty}^{y} f_{X|Y,Z}(x \mid y_1, z_1)f_{Y,Z}(y_1, z_1)dy_1dz_1
\]

By substituting the densities and obtaining a “complete square form”, this can be expressed in closed-form expression for the bivariate normal CDF. We will leave it for the reader to verify this.

### 9.2 Simulation Code for Chapter 7 Computations of Prices

Code for Monte Carlo simulation for estimating the expected payoff prices with and without antithetic variables, for the an up-and-out window barrier call option.

```
# The approach for Monte Carlo simulation of the expected payoff price, using conditioning and the Brownian Bridge process for an up-and-out window barrier call option.

Direct.window.Bar<- function(S0,K,H,r,sigma,t1,t2,T){
  Wt1<- sqrt(t1)*rnorm(1,0,1);
  # Code for Monte Carlo simulation for estimating the expected payoff prices with and without antithetic variables, for the an up-and-out window barrier call option.
```

69
Wt2 <- Wt1 + sqrt(t2-t1)*rnorm(1,0,1);
WT <- Wt2 + sqrt(T-t2)*rnorm(1,0,1);
Ht1 <- (log(H/S0)-(r-0.5*sigma^2)*t1)/sigma;
Ht2 <- (log(H/S0)-(r-0.5*sigma^2)*t2)/sigma;
if(Wt1 < Ht1 & Wt2 < Ht2){
c1 <- (-2*(log(H/S0))^2 + 2*log(H/S0)*(r-0.5*sigma^2)*t1 + 2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
       / (sigma^2*(t2-t1));
c2 <- (-2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
c3 <- (-2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
c4 <- -2/(t2-t1);
bar.cond <- (1-exp(c1 + c2*Wt1 + c3*Wt2 + c4*Wt1*Wt2)); # Barrier probability
ST <- S0*exp((r-0.5*sigma^2)*T + sigma*WT);
payoff <- exp(-r*T)*max((ST-K),0)*bar.cond;
}
else{
payoff <- 0;
}
return(payoff)

########################################################
# Average payoff after n simulations
########################################################
Direct.Ave.window.Bar <- function(S0,K,H,r,sigma,t1,t2,T,n){
  Call.price <- c();
  for(i in 1:n){
    sim <- Direct.window.Bar(S0,K,H,r,sigma,t1,t2,T);
    Call.price[i] <- sim[[1]];
  }
  expected.payoff <- mean(Call.price);
  standard.error <- sd(Call.price)/sqrt(n);
  cat("Expected payoff for a for an up-and-out window barrier call option,
      using sophisticated approach and anithetic variables.", "\n",
      "\nExpected payoff: ", expected.payoff, "\n",
      "\nStandard error of estimate: ", standard.error, "\n")
}

Direct.Ave.window.Bar(120,90,130,0.1,0.2,0.25,0.75,1,50000)

########################################################
# The approach for Monte Carlo simulation of the expected
# payoff price, using conditioning and the Brownian Bridge
# process for an up-and-out window barrier call option
# and using antithetic variables.
########################################################

Anti.Direct.window.Bar <- function(S0,K,H,r,sigma,t1,t2,T){
  Wt1a <- sqrt(t1)*rnorm(1,0,1);
Wt2a <- Wt1a + sqrt(t2-t1)*rnorm(1,0,1);
WTa <- Wt2a + sqrt(T-t2)*rnorm(1,0,1);
Wt1b <- -Wt1a;
Wt2b <- -Wt2a;
WTb <- -WTa;
Ht1 <- (log(H/S0)-(r-0.5*sigma^2)*t1)/sigma;
Ht2 <- (log(H/S0)-(r-0.5*sigma^2)*t2)/sigma;
if(Wt1a < Ht1 && Wt2a < Ht2){
c1 <- (-2*(log(H/S0))^2 + 2*log(H/S0)*(r-0.5*sigma^2)*t1 +
2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
/(sigma^2*(t2-t1));
c2 <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
c3 <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
c4 <- -2/(t2-t1);
bar.conda <- (1-exp(c1 + c2*Wt1a + c3*Wt2a +
c4*Wt1a*Wt2a)); #Barrier probability
STa <- S0*exp((r-0.5*sigma^2)*T + sigma*WTa);
payoff1 <- exp(-r*T)*max((STa-K),0)*bar.conda;
}
else{
payoff1 <- 0;
}
if(Wt1b < Ht1 && Wt2b < Ht2){
c1 <- (-2*(log(H/S0))^2 + 2*log(H/S0)*(r-0.5*sigma^2)*t1 +
2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
/(sigma^2*(t2-t1));
c2 <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
c3 <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
c4 <- -2/(t2-t1);
bar.condb <- (1-exp(c1 + c2*Wt1b + c3*Wt2b +
c4*Wt1b*Wt2b)); #Barrier probability
STb <- S0*exp((r-0.5*sigma^2)*T + sigma*WTb);
payoff2 <- exp(-r*T)*max((STb-K),0)*bar.condb;
}
else{
payoff2 <- 0;
}
payoff <- (payoff1 + payoff2)/2;
return(payoff)
}

# Average payoff after n simulations

Anti.Direct.Ave.window.Bar <- function(S0,K,H,r,sigma,t1,t2,T,n){
  Call.price <- c();
  for(i in 1:n){
    sim <- Anti.Direct.window.Bar(S0,K,H,r,sigma,t1,t2,T);
    Call.price[i] <- sim[[1]];
  }
  return(mean(Call.price));
}
9.3 Simulation Code for Chapter 7 of Sensitivities Computations

R Code for Monte Carlo simulation with finite-differencing for estimating the Delta and Eta Greeks with antithetic variables for an up-and-out window barrier call option with $H > K$.

```r
# Re simulation approach for finding Delta for up-and-out window # barrier call option, with antithetic variables.

Ant.Window.Bar.Direct.Sen.wrtS0<- function(S0,K,H,r,sigma,t1,t2,T,h){
  Wt1a<- sqrt(t1)*rnorm(1,0,1);
  Wt2a<-Wt1a + sqrt(t2-t1)*rnorm(1,0,1);
  WTa<- Wt2a + sqrt(T-t2)*rnorm(1,0,1);
  Wt1b<- -Wt1a;
  Wt2b<- -Wt2a;
  W Tb<- -WTa;
  Ht11<- (log(H/S0)-(r-0.5*sigma^2)*t1)/sigma;
  Ht21<- (log(H/S0)-(r-0.5*sigma^2)*t2)/sigma;
  if(Wt1a < Ht11 && Wt2a < Ht21){
    c11a<-(2*log(H/S0)^2 + 2*log(H/S0)*(r-0.5*sigma^2)*t1 +
           2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
         /(sigma^2*(t2-t1));
    c21a<-(2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
    c31a<-(2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
    c41a< -2/(t2-t1); 
    bar.cond1a<- (1-exp(c11a + c21a*Wt1a + c31a*Wt2a +
                       c41a*Wt1a*Wt2a)); #Barrier probability 
    ST1a<- S0*exp((r-0.5*sigma^2)*T + sigma*WTa);
    payoff1a<- exp(-r*T)*max((ST1a-K),0)*bar.cond1a;
  }else{
    payoff1a<-0; 
  }
  if(Wt1b < Ht11 && Wt2b < Ht21){
    c11b<-(2*log(H/S0)^2 + 2*log(H/S0)*(r-0.5*sigma^2)*t1 +
           2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
         /(sigma^2*(t2-t1));
    c21b<-(2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
    c31b<-(2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
    c41b<- -2/(t2-t1); 
    bar.cond1b<- (1-exp(c11b + c21b*Wt1b + c31b*Wt2b +
                       c41b*Wt1b*Wt2b)); #Barrier probability 
    ST1b<- S0*exp((r-0.5*sigma^2)*T + sigma*WTb);
    payoff1b<- exp(-r*T)*max((ST1b-K),0)*bar.cond1b;
  }else{
    payoff1b<-0; 
  }
}
```
2*\log(H/S0)*(r-0.5*\sigma^2)*t2-2*(r-0.5*\sigma^2)^2*t2*t1)/(\sigma^2*(t2-t1));
c12b<-(-2*(\log(H/S0) - 2*(r-0.5*\sigma^2)*t2)/(\sigma*(t2-t1));
c32b<-(-2*(\log(H/S0) - 2*(r-0.5*\sigma^2)*t1)/(\sigma*(t2-t1));
c42b<- -2/(t2-t1);
bar.cond2b<- (1-exp(c12b + c22b*Wt1b + c32b*Wt2b + c42b*Wt1b*Wt2b));  #Barrier probability
ST2b<- S0*exp((r-0.5*\sigma^2)*T + \sigma*WTb);
payoff2b<- exp(-r*T)*max((ST2b-K),0)*bar.cond2b;
}
else{
payoff2b<-0;
}
payoff2<-(payoff2a+payoff2b)/2;
S0<-S0+h;
Ht12<- (\log(H/S0)-(r-0.5*\sigma^2)*t1)/\sigma;
Ht22<- (\log(H/S0)-(r-0.5*\sigma^2)*t2)/\sigma;
if(Wt1a < Ht12 && Wt2a < Ht22){
c12a<-(-2*(\log(H/S0))^2 + 2*\log(H/S0)*(r-0.5*\sigma^2)*t1 + 2*\log(H/S0)*(r-0.5*\sigma^2)*t2-2*(r-0.5*\sigma^2)^2*t2*t1)/(\sigma^2*(t2-t1));
c22a<-(-2*(\log(H/S0) - 2*(r-0.5*\sigma^2)*t2)/(\sigma*(t2-t1));
c32a<-(-2*(\log(H/S0) - 2*(r-0.5*\sigma^2)*t1)/(\sigma*(t2-t1));
c42a<- -2/(t2-t1);
bar.cond2a<- (1-exp(c12a + c22a*Wt1a + c32a*Wt2a + c42a*Wt1a*Wt2a));  #Barrier probability
ST2a<- S0*exp((r-0.5*\sigma^2)*T + \sigma*WTa);
payoff2a<- exp(-r*T)*max((ST2a-K),0)*bar.cond2a;
}
else{
payoff2a<-0;
}
if(Wt1b < Ht12 && Wt2b < Ht22){
c12b<-(-2*(\log(H/S0))^2 + 2*\log(H/S0)*(r-0.5*\sigma^2)*t1 + 2*\log(H/S0)*(r-0.5*\sigma^2)*t2-2*(r-0.5*\sigma^2)^2*t2*t1)/(\sigma^2*(t2-t1));
c22b<-(-2*(\log(H/S0) - 2*(r-0.5*\sigma^2)*t2)/(\sigma*(t2-t1));
c32b<-(-2*(\log(H/S0) - 2*(r-0.5*\sigma^2)*t1)/(\sigma*(t2-t1));
c42b<- -2/(t2-t1);
bar.cond2b<- (1-exp(c12b + c22b*Wt1b + c32b*Wt2b + c42b*Wt1b*Wt2b));  #Barrier probability
ST2b<- S0*exp((r-0.5*\sigma^2)*T + \sigma*WTb);
payoff2b<- exp(-r*T)*max((ST2b-K),0)*bar.cond2b;
}
else{
payoff2b<-0;
}
payoff2<-(payoff2a+payoff2b)/2;
if(payoff1 > 0 && payoff2 > 0){


dc.dS0 <- (payoff2 - payoff1) / h; # Forward finite-difference formula
}
else{
  dc.dS0 <- 0;
}
return(dc.dS0)
}

########################################################
# Derivative price after n simulations.
########################################################
Ant.Window.Bar.Sen.Direct.wrtS0 <- function(S0, K, H, r, sigma, t1, t2, T, h, n) {
  average <- c();
  for (i in 1:n) {
    average[i] <- Ant.Window.Bar.Direct.Sen.wrtS0(S0, K, H, r, sigma, t1, t2, T, h)
  }
  dc.dS0 <- mean(average)
  Standard.error <- sd(average) / sqrt(n);
  cat("Delta for a up-out window barrier call option using re-simulation using antithetic variables.", "\n",
  "\nForward finite-difference formula i.e. f(x+h)-f(x)/h =", dc.dS0, "\n",
  "\nStandard error of estimate: ", Standard.error, "\n")
}
Ant.Window.Bar.Sen.Direct.wrtS0(80, 90, 130, 0.1, 0.2, 0.25, 0.75, 1, 0.0001, 50000)

###############################################################
# Re simulation approach for finding Eta for up-and-out window
# barrier call option, where H > K, with antithetic variables.
###############################################################
Ant.Window.Bar.Direct.Sen.wrtH <- function(S0, K, H, r, sigma, t1, t2, T, h) {
  Wt1a <- sqrt(t1) * rnorm(1, 0, 1);
  Wt2a <- Wt1a + sqrt(t2 - t1) * rnorm(1, 0, 1);
  WTa <- Wt2a + sqrt(T - t2) * rnorm(1, 0, 1);
  Wt1b <- -Wt1a;
  Wt2b <- -Wt2a;
  W Tb <- -WTa;
  Ht11 <- (log(H/S0) - (r - 0.5 * sigma^2) * t1) / sigma;
  Ht21 <- (log(H/S0) - (r - 0.5 * sigma^2) * t2) / sigma;
  if (Wt1a < Ht11 && Wt2a < Ht21) {
    c11a <- (-2 * (log(H/S0))^2 + 2 * log(H/S0) * (r - 0.5 * sigma^2) * t1 +
    2 * log(H/S0) * (r - 0.5 * sigma^2) * t2 - 2 * (r - 0.5 * sigma^2) * t1)
    / (sigma^2 * (t2 - t1));
    c21a <- (-2 * log(H/S0) - 2 * (r - 0.5 * sigma^2) * t2) / (sigma * (t2 - t1));
    c31a <- (-2 * log(H/S0) - 2 * (r - 0.5 * sigma^2) * t1) / (sigma * (t2 - t1));
    c41a <- -2 / (t2 - t1);
    bar.cond1a <- (1 - exp(c11a + c21a * Wt1a + c31a * Wt2a +
    c41a * Wt1a * Wt2a)); # Barrier probability
ST1a <- S0*exp((r-0.5*sigma^2)*T + sigma*WTa);
payoff1a <- exp(-r*T)*max((ST1a-K),0)*bar.cond1a;
else{
payoff1a <- 0;
}

if(Wt1b < Ht11 && Wt2b < Ht21){
c11b <- (-2*(log(H/S0))^2 + 2*log(H/S0)*r-0.5*sigma^2)*t1 +
2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
/((sigma^2)*(t2-t1));
c21b <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
c31b <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
c41b <- -2/(t2-t1);
bar.cond1b <- (1-exp(c11b + c21b*Wt1b + c31b*Wt2b +
c41b*Wt1b*Wt2b)); #Barrier probability
ST1b <- S0*exp((r-0.5*sigma^2)*T + sigma*WTb);
payoff1b <- exp(-r*T)*max((ST1b-K),0)*bar.cond1b;
}
else{
payoff1b <- 0;
}

payoff1 <- (payoff1a + payoff1b)/2;
H <- H+h;
Ht12 <- (log(H/S0)-(r-0.5*sigma^2)*t1)/sigma;
Ht22 <- (log(H/S0)-(r-0.5*sigma^2)*t2)/sigma;

if(Wt1a < Ht12 && Wt2a < Ht22){
c12a <- (-2*(log(H/S0))^2 + 2*log(H/S0)*r-0.5*sigma^2)*t1 +
2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
/((sigma^2)*(t2-t1));
c22a <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
c32a <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
c42a <- -2/(t2-t1);
bar.cond2a <- (1-exp(c12a + c22a*Wt1a + c32a*Wt2a +
c42a*Wt1a*Wt2a)); #Barrier probability
ST2a <- S0*exp((r-0.5*sigma^2)*T + sigma*WTa);
payoff2a <- exp(-r*T)*max((ST2a-K),0)*bar.cond2a;
}
else{
payoff2a <- 0;
}

if(Wt1b < Ht12 && Wt2b < Ht22){
c12b <- (-2*(log(H/S0))^2 + 2*log(H/S0)*r-0.5*sigma^2)*t1 +
2*log(H/S0)*(r-0.5*sigma^2)*t2-2*(r-0.5*sigma^2)^2*t2*t1)
/((sigma^2)*(t2-t1));
c22b <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t2)/(sigma*(t2-t1));
c32b <- (2*log(H/S0) - 2*(r-0.5*sigma^2)*t1)/(sigma*(t2-t1));
c42b <- -2/(t2-t1);
bar.cond2b <- (1-exp(c12b + c22b*Wt1b + c32b*Wt2b +
c42b*Wt1b*Wt2b)); #Barrier probability
ST2b <- S0*exp((r-0.5*sigma^2)*T + sigma*WTb);
payoff2b <- exp(-r*T)*max((ST2b-K),0)*bar.cond2b;
}
else{
    payoff2b <- 0;
}
payoff2 <- (payoff2a + payoff2b)/2;
if(payoff1 > 0 && payoff2 > 0){
    dc.dH <- (payoff2 - payoff1)/h; # Forward finite-difference formula
}
else{
    dc.dH <- 0;
}
return(dc.dH)

# Derivative price after n simulations.
Ant.Window.Bar.Sen.Direct.wrtH <- function(S0,K,H,r,sigma,t1,t2,T,h,n){
    average <- c();
    for(i in 1:n){
        average[i] <- Ant.Window.Bar.Direct.Sen.wrtH(S0,K,H,r,sigma,t1,t2,T,h)
    }
    dc.dH <- mean(average)
    Standard.error <- sd(average)/sqrt(n);
    cat("Eta for a up-out window barrier call option using re-simulation and using antithetic variables.", "\n", "\nForward finite-difference formula i.e. f(x+h)-f(x)/h =", dc.dH, "\n", "\nStandard error of estimate: ", Standard.error, "\n")
}
Ant.Window.Bar.Sen.Direct.wrtH(100,90,110,0.1,0.2,0.25,0.75,1,0.0001,50000)
9.4 Delta and Eta Closed-form Formula for a Up-and-Out Type A Partial-time Barrier Option

\[
\text{BivariateNorm}(H/s) = \frac{T(r + 0.5 \sigma^2) + \log(\frac{H}{S})}{\sqrt{T \sigma}} - \frac{1}{\sqrt{H/T}} - 1. \left[ \frac{(t_1 r + 0.5 t_1 \sigma^2) + \log(\frac{H}{S})}{\sqrt{T \sigma}} - 1. \sqrt{\frac{H}{T}} \right] - 1. \sqrt{\frac{H}{T}}
\]

\[
1. \left( \frac{H}{S} \right)^{1.5} \frac{r + 0.5 \sigma^2 + 2 \log(\frac{H}{S})}{\sqrt{T \sigma}} - \frac{1}{\sqrt{H/T}} - 1. \left[ \frac{(t_1 r + 0.5 t_1 \sigma^2) + \log(\frac{H}{S})}{\sqrt{T \sigma}} - 1. \sqrt{\frac{H}{T}} \right] - 1. \sqrt{\frac{H}{T}}
\]

\[
\frac{1}{S \sqrt{T \sigma}} \left( \frac{H}{S} \right)^{1.5} \frac{r + 0.5 \sigma^2 + 2 \log(\frac{H}{S})}{\sqrt{T \sigma}} - \frac{1}{\sqrt{H/T}} - 1. \left[ \frac{(t_1 r + 0.5 t_1 \sigma^2) + \log(\frac{H}{S})}{\sqrt{T \sigma}} - 1. \sqrt{\frac{H}{T}} \right] - 1. \sqrt{\frac{H}{T}}
\]

Figure 9.1: Closed-form formula of Delta for a up-and-out type A partial-time barrier option. We will only
Figure 9.2: Closed-form formula of Eta for an up-and-out type A partial-time barrier option.
Bibliography


