THREE LECTURES ON THE MODULARITY OF $\bar{\rho}_{E,3}$ AND THE LANGLANDS RECIPROCITY CONJECTURE

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WILES' work on Fermat's Last Theorem is based on methods due to FALTINGS, FREY, LANGLANDS, MAZUR, RIBET, SERRE, TAYLOR, and others. My purpose in these Lectures is to explain how the (automorphic representation theoretic methods and) results of LANGLANDS come into the proof, and how these results themselves are proved. An Introduction to each of the Lectures describes more of the topics discussed; but the titles already speak for themselves:

Lecture I: "The Modularity of $\bar{\rho}_{E,3}$ and Automorphic Representations of Weight One"

Lecture II: "The Langlands Program: Some Results and Methods"

Lecture III: "Proof of the Langlands-Tunnell Theorem"

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Lecture I
The Modularity of $\bar{\rho}_{E,3}$ and Automorphic Representations of Weight One

Abstract
The following result plays a small but key step in Wiles' proof of the Shimura-Taniyama-Weil Conjecture:

Proposition 1.4. *For an elliptic curve E over $\mathbb{Q}$, let*

$$\bar{\rho}_{E,p} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_p)$$

denote the natural representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the points of $E(\bar{\mathbb{Q}})$ of order p. Then if $p = 3$, and $\bar{\rho}_{E,3}$ is irreducible, it must also follow that $\bar{\rho}_{E,3}$ is modular, i.e., there exists a normalized eigen-cuspform

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

of weight two, and a prime λ of $\bar{\mathbb{Q}}$ containing 3, such that

$$a_q \equiv \mathrm{trace}(\bar{\rho}_{E,3}(\mathrm{Fr}_q)) \pmod{\lambda}$$

for almost all primes q. (Fr_q is explained below.)

Our main purpose in this Lecture is to explain how this result follows from the following special case of Langlands' Reciprocity Conjecture for Artin L-functions:

Theorem 1.3. (cf. [La1] and [Tu]). *Suppose that the continuous representation*

$$\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

is "odd," irreducible, and has solvable image in $\mathrm{PGL}_2(\mathbb{C})$. *(Here odd means that if τ denotes complex conjugation in $G_{\mathbb{Q}}$, then $\det(\sigma(\tau)) = -1$.) Then there exists a normalized eigen-cuspform*

$$g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$$

of weight one such that

$$b_q = \mathrm{trace}(\sigma(\mathrm{Fr}_q))$$

for all but finitely many primes q.

As we shall see, the *proof* of this result requires working not only over an arbitrary number field, but also with automorphic cuspidal *representations* (in place of classical cusp forms). Thus the second half of this Lecture will be devoted to recalling the basic representation theory required to reformulate Theorem 1.3 as follows:

Theorem 2.6. *For each irreducible representation*

$$\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

which is odd and solvable, there is an automorphic "weight one" cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$*, call it* $\pi(\sigma)$*, with the property that*

$$\mathrm{trace}(t_{\pi_q}) = \mathrm{trace}(\sigma(\mathrm{Fr}_q))$$

for almost every q*. (Here* t_{π_q} *denotes the Langlands class in* $\mathrm{GL}_2(\mathbb{C})$ *associated to the unramified local component* π_q *of* $\pi(\sigma) = \otimes\pi_p$*, and "weight one" means that* π_∞ *is the principal series representation of* $\mathrm{GL}_2(\mathbb{R})$ *induced from the characters 1 and* sgn.*)*

§1. The Modularity of $\bar{\rho}_{E,3}$

1.1. Galois Representations mod p

Let E denote a fixed elliptic curve defined over $\mathbb{Q}$. For a chosen prime p, let $E[p]$ denote the subgroup of $E(\bar{\mathbb{Q}})$ consisting of points of order p. Then $E[p] \cong \mathbb{F}_p^2$, regarded as a two-dimensional vector space over $\mathbb{F}_p$. The natural action of the Galois group

$$G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

on $E[p]$ consequently gives rise to a continuous representation

$$\bar{\rho}_{E,p} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_p) \approx \mathrm{Aut}(E[p]),$$

which is uniquely defined up to its isomorphism class. That $\bar{\rho}_{E,p}$ encodes much of the arithmetic of E is clear from the two following crucial properties of $\bar{\rho}_{E,p}$:

(a) Write

$$\omega_p : G_{\mathbb{Q}} \longrightarrow \mathbb{F}_p^\times$$

for the character giving the action of $G_{\mathbb{Q}}$ on the p-th roots of unity μ_p. Then

$$\det(\bar{\rho}_{E,p}) = \omega_p; \tag{1.1.1}$$

this results from the existence of a "Weil pairing" $E[p] \times E[p] \longrightarrow \mu_p$, compatible with the action of $G_{\mathbb{Q}}$ and such that $\bigwedge^2_{\mathbb{F}_p}(E[p]) \approx \mu_p$ (cf. §V.2 of [Silv]).

(b) If q is any prime number, and Q is a prime of $\bar{\mathbb{Q}}$ dividing q, let Fr_q denote the canonical *Frobenius* conjugacy class in D_Q/I_Q (the quotient of the decomposition group at Q by the inertia group at Q). Then

$$\mathrm{trace}\,\bar{\rho}_{E,p}(\mathrm{Fr}_q) \equiv q + 1 - \#(E(\mathbb{F}_g)) \pmod p \tag{1.1.2}$$

for almost all primes q, namely those where $\bar{\rho}_{E,p}$ is trivial on (any) I_Q (i.e. those q where $\bar{\rho}_{E,p}$ is *unramified*).

N.B. (i) The invariants $\operatorname{trace} \bar{\rho}_{E,p}(\mathrm{Fr}_q)$ (and also $\det \bar{\rho}_{E,p}(\mathrm{Fr}_q)$) are well-defined elements of $\mathbb{F}_p$ precisely when $\bar{\rho}_{E,p}$ is unramified at q.

(ii) The identity (1.1.2) essentially amounts to the Riemann hypothesis for elliptic curves over finite fields (proved by Hasse; cf. §V.2 of [Silv]).

(iii) Alternatively, the primes q for which (1.1.2) holds can be characterized as those which are different from p and such that E has "good reduction mod q." Equivalently, let K be the kernel of $\bar{\rho}_{E,p}$, and $\bar{\mathbb{Q}}^K := \mathbb{Q}(E[p])$ the corresponding finite Galois extension of $\mathbb{Q}$; then

$$\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q}) \approx \mathrm{Im}\, \bar{\rho}_{E,p},$$

and (1.1.2) holds exactly for those q which are unramified in $\mathbb{Q}(E[p])$ (equivalently, those q such that $\bar{\rho}_{E,p}$ is trivial on I_q).

1.2. The Modularity of $\bar{\rho}_{E,p}$

Let $S_k(\Gamma_0(N), \varepsilon)$ denote the vector space of modular cusp forms $f(z)$ of weight $N \geq 1$ and character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$.

Definition. We call $\bar{\rho}_{E,p}$ *modular* if there exists some (normalized) eigenform

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_2(\Gamma_0(N), \varepsilon)$$

(for some N and ε), and a prime λ of $\bar{\mathbb{Q}}$ containing p, such that

$$a_q \equiv q + 1 - \#(E(\mathbb{F}_q)) \pmod{\lambda}$$

for almost all primes q.

Recall that Wiles' goal was to prove that E itself is *modular*, i.e., for some weight two $f(z)$ as above, the *identity* (as opposed to congruence)

$$a_q = q + 1 - \#(E(\mathbb{F}_q))$$

holds for almost all q. As discussed elsewhere, what Wiles actually proves is Mazur's "Modular Lifting Conjecture":

If p is a prime such that
(i) $\bar{\rho}_{E,p}$ is irreducible, and
(ii) $\bar{\rho}_{E,p}$ is modular,
THEN E ITSELF IS MODULAR.

More precisely, Wiles proves that (a) the Modular Lifting Conjecture is true for $p = 3$ and 5 when E is a semistable elliptic curve, and (b) the Modular Lifting Conjecture for $p = 3$ and 5 already implies the Taniyama-Shimura-Weil Conjecture (that E is modular).

Our modest goal is to explain how the theory of automorphic forms is used to prove that for $p = 3$, the second hypothesis of the Modular Lifting Conjecture automatically follows from the first, i.e., if $\bar{\rho}_{E,3}$ is irreducible, then it is modular.

1.3. The Theorem of Langlands-Tunnell

The crucial ingredient in proving the modularity of $\bar{\rho}_{E,3}$ is the following:

Theorem 1.3. (cf. [La1] and [Tu]) *Suppose*

$$\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

is a continuous, irreducible two dimensional representation whose image in $\mathrm{PGL}_2(\mathbb{C})$ *is a solvable group. Suppose moreover that* σ *is "odd" in the sense that*

$$\det(\sigma(\tau)) = -1.$$

(τ *is an automorphism in* $G_{\mathbb{Q}}$ *defined by complex conjugation.) Then there exists a (normalized)*

$$g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z} \in S_1(\Gamma_0(N), \psi)$$

(for some N *and* ψ*), such that* f *is an eigenform for all the Hecke operators, and*

$$b_q = \mathrm{trace}(\sigma(\mathrm{Fr}_q)) \tag{1.3.1}$$

for almost all primes q.

Remarks. (1) Because *any* continuous representation

$$\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

factors through some finite Galois group $\mathrm{Gal}(K/\mathbb{Q})$, its image in $\mathrm{GL}_2(\mathbb{C})$ is finite, and its image *in* $\mathrm{PGL}_2(\mathbb{C})$ is just one of the symmetry groups of a regular polyhedron in $\mathbb{R}^3$ (cf. section 13 of [Shaf]). From this it is deduced that the image of any *irreducible* σ in $\mathrm{PGL}_2(\mathbb{C})$ is either A_5 (the *icosahedral* case), S_4 (the *octahedral* case), A_4 (the *tetrahedral* case), or D_{2n} (the *dihedral* case). As we shall recall in §5.3, in the *dihedral* case the existence of the required weight one from $g(z)$ above is essentially due to much earlier work of Hecke and Maass. Hence in dealing with "solvable" σ, the theorem of Langlands and Tunnell is ultimately concerned with "just" the tetrahedral and octahedral possibilities.
(2) The relevant theorems of [La] and [Tu] do not actually produce the required modular form $g(z)$, but rather a certain automorphic representation $\pi(\sigma)$. Using the fact that $\det(\sigma(\tau)) = -1$, we shall explain in §4.2 of Lecture II how this automorphic representation produces $g(z)$ itself. In the meantime, we take the above theorem as given, and use it to prove the modularity of $\bar{\rho}_{E,3}$.

1.4. Proof of the Modularity of $\bar{\rho}_{E,3}$

More precisely, we need to prove:

Proposition 1.4. *If $\bar{\rho}_{E,3}$ is irreducible, then it is modular, i.e., there exists a normalized eigenform*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

of weight one, and a prime λ of $\bar{\mathbb{Q}}$ containing 3, such that

$$a_q \equiv q + 1 - \#E(\mathbb{F}_q) \pmod{\lambda}$$

for almost all primes q.

The *strategy of proof* is simple. First one "lifts" $\bar{\rho}_{E,3}$ to a complex representation $\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ to which the Theorem of Langlands-Tunnell is applicable; this produces a modular form $g(z)$ of weight *one* whose Fourier coefficients b_q are almost everywhere equal to trace(Fr_q). Then one multiplies g by an Eisenstein series of weight one, whose (non-trivial) Fourier coefficients are all congruent to 0 mod 3; this essentially produces the required form of weight *two* whose Fourier coefficients are *congruent* to trace(Fr_q) modulo some divisor of 3 (and hence also congruent to $q + 1 - \#E(\mathbb{F}_q)$, by virtue of (1.1.2)).

Because of the importance of Proposition 1.4, we shall go through its proof carefully (expanding on the single paragraph allotted it in Chapter V of [W1]). We note that the idea of applying "Langlands-Tunnell" in this context goes back to Serre (cf. [Se], §5.3, page 220).

Step 1. Extend $\bar{\rho}_{E,3} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_3)$ to a complex representation

$$\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

by composing $\bar{\rho}_{E,3}$ with a specific (injective) homomorphism

$$\Psi : \mathrm{GL}_2(\mathbb{F}_3) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}(\sqrt{-2})) \subset \mathrm{GL}_2(\mathbb{C})$$

described below.

Following [RuSi], we introduce Ψ directly through the formulas

$$\Psi \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\Psi \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -\sqrt{-2} & -1 + \sqrt{-2} \end{pmatrix}.$$

Here $\alpha = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ are two convenient generators of $\mathrm{GL}_2(\mathbb{F}_3)$. Once it is checked that the above formulas indeed preserve the required relations, it is immediately seen that the resulting homomorphism

$$\Psi : \mathrm{GL}_2(\mathbb{F}_3) \longrightarrow \mathrm{GL}_2(\mathbb{Z}(\sqrt{-2})) \subset \mathrm{GL}_2(\mathbb{C})$$

is the identity upon reduction $\bmod(1+\sqrt{-2})$. In particular,

$$\text{trace}(\Psi(g)) \equiv \text{trace}(g) \ \big(\text{mod } 1+\sqrt{-2}\big) \tag{1.4.1}$$

and

$$\det(\Psi(g)) \equiv \det(g) \ \big(\text{mod } 3 = (1+\sqrt{-2})(1-\sqrt{-2})\big). \tag{1.4.2}$$

N.B. This representation

$$\Psi : \mathrm{GL}_2(\mathbb{F}_3) \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

is really just one of the (three) so-called *cuspidal* representations of the group $\mathrm{GL}_2(\mathbb{F}_3)$; compare, for example, [PS1] §10.

Step 2. Check that

$$\sigma = \Psi \circ \bar{\rho}_{E,3} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

is "odd," irreducible and solvable.

Let us first check that $\bar{\rho}_{E,3}$ itself has odd determinant. On the one hand, (1.1.1) implies

$$\det(\rho_{E,3}(\tau)) = \omega_3(\tau),$$

and it is clear that $\omega_3(\tau) = -1$. On the other hand, $\det \sigma(\tau)$ is *a priori* ± 1, since $\tau^2 = 1$, and (1.4.2) implies $\det(\sigma(\tau)) \equiv \det \bar{\rho}_{E,3}(\tau) \pmod 3$. So since $-1 \not\equiv 1 \pmod 3$, we must have $\det(\sigma(\tau)) = -1$, as required. As for the "solvable" assertion, just recall that

$$\mathrm{PGL}_2(\mathbb{F}_3) \approx S_4;$$

this says that the image of $\sigma = \Psi \circ \bar{\rho}_{E,3}$ in $\mathrm{PGL}_2(\mathbb{C})$ is a subgroup of S_4, hence itself solvable.

Now what about irreducibility? From the fact that $\det \bar{\rho}_{E,3}(\tau) = -1$, it follows that $\bar{\rho}_{E,3}$ has distinct eigenvalues in $\mathbb{F}_3$ (namely 1 and -1). We claim this implies $\bar{\rho}_{E,3}$ is *absolutely* irreducible, i.e., irreducible over $\overline{\mathbb{F}}_3$ as well as $\mathbb{F}_3$. Indeed, the only matrices in $M_2(\overline{\mathbb{F}}_3)$ which can commute with $\bar{\rho}_{E,3}(G_{\mathbb{Q}})$ (in particular $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* some non-diagonal matrix $\bar{\rho}_{E,3}(g)$) are the scalar matrices λI themselves. Hence by Schur's Lemma, $\bar{\rho}_{E,3}$ is absolutely irreducible.

Now suppose that the *complex* representation $\sigma = \Psi \circ \bar{\rho}_{E,3}$ is *not* irreducible. We claim this implies its image in $\mathrm{GL}_2(\mathbb{C})$ must be abelian. Indeed, any complex representation of a finite (or compact) group is completely reducible. In the case of σ, this means σ is the sum of two characters, and this clearly implies that its image in $\mathrm{GL}_2(\mathbb{C})$ is abelian.

On the other hand, as $\bar{\rho}_{E,3}$ is absolutely irreducible, the only matrices commuting with its image in $\mathrm{GL}_2(\overline{\mathbb{F}}_3)$ must be scalar ones (again by

Schur's Lemma). So pulling back through the embedding Ψ, we conclude $\bar{\rho}_{E,3}$ has both an abelian and irreducible image in $\mathrm{GL}_2(\overline{\mathbb{F}}_3)$, an obvious contradiction. Thus $\sigma = \Psi \circ \bar{\rho}_{E,3}$ must after all be irreducible.

Step 3. Apply Theorem 1.3 to get a normalized eigenform

$$g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z} \quad \text{in some} \quad S_1(\Gamma_0(N_1), \varepsilon_1)$$

with

$$b_q = \operatorname{trace}(\sigma(\mathrm{Fr}_q)) \quad \text{for almost all primes } q. \tag{1.4.3}$$

Remark. Recall that for any normalized new form of weight $k \geq 1$ (and character ψ), the Fourier coefficients a_n (together with the values $\psi(n)$) lie in the ring of integers O_K of some number field K (of finite degree over $\mathbb{Q}$). In particular, for our form $g(z)$ above, it makes sense to discuss congruence conditions on the coefficients b_n modula a prime ideal $\mathfrak{p}$ of (the appropriate) O_K.

Now recall that (1.4.1) implies

$$\begin{aligned}\operatorname{trace}(\sigma(\mathrm{Fr}_q)) &= \operatorname{trace}(\Psi \cdot \sigma(\mathrm{Fr}_q)) \\ &\equiv \operatorname{trace}(\bar{\rho}_{E,3}(\mathrm{Fr}_q)) \left(\bmod 1 + \sqrt{-2}\right)\end{aligned}$$

So by (1.1.2), (1.4.3), and the above Remark, we have

$$b_q \equiv q + 1 - \#E(\mathbb{F}_q) \pmod{\mathfrak{p}} \tag{1.4.4}$$

for almost every q, and $\mathfrak{p}$ some prime of $\bar{\mathbb{Q}}$ containing $(1+\sqrt{-2})$ (and hence 3). In other words, we've proven that $\bar{\rho}_{E,3}$ is modular, but with the *hitch* that $\sum_{n=1}^{\infty} b_n e^{2\pi i n z}$ is of weight *one* instead of *two*! To remedy this, we follow two ideas, going back respectively to Shimura, and Deligne-Serre.

Step 4. Pick a modular (non-cuspidal) form E of weight 1, such that $E \equiv 1$ (3); the product

$$g(z)E(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}$$

is of weight 2 (for some level N and character ψ), and

$$c_n \equiv b_n \pmod{\mathfrak{p}}$$

(for $\mathfrak{p}$ a prime of $\bar{\mathbb{Q}}$ lying above 3...).

More explicitly, take

$$E(z) = E_{1,\chi}(z) = 1 + 6 \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) e^{2\pi i n z}$$

where

$$\chi(d) = \begin{cases} 0 & \text{if } d \equiv 0 \pmod 3 \\ 1 & \text{if } d \equiv 1 \pmod 3 \\ -1 & \text{if } d \equiv -1 \pmod 3 \end{cases}$$

is an *odd* Dirichlet character mod 3 (i.e., $\chi(-1) = -1$). Then $E_{1,\chi} \in M_1(\Gamma_0(3), \chi)$, and $gE_{1,\chi}$ belongs to $S_2(\Gamma_0(3N_1), \varepsilon_1\chi)$. Furthermore, each "non-constant" Fourier coefficient of $E_{1,\chi}$ is divisible by 3 (in fact 6), so it easily follows that

$$c_n \equiv b_n \pmod{\mathfrak{p}}.$$

Note that $g(z)E(z)$ is the *product* of a normalized eigenform with an Eisenstein series, but not itself such an eigenform. If it were, we would (by (1.4.4)) have completed our task of proving $\bar{\rho}_{E,3}$ modular. To finish the job, we need the following result of Deligne and Serre:

Lemma. (cf. §6.10 of [DS]) *Suppose*

$$f_1(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}$$

is a normalized element of $S_k(\Gamma_0(N), \psi)$, and K is a finite extension of $\mathbb{Q}$ whose ring of integers contains the coefficients c_n and $\psi(n)$. Suppose $\mathfrak{p}$ is a prime ideal of O_K containing 3, and that f_1 is a mod $\mathfrak{p}$ eigenform, i.e., there exists b_n such that $T_n f_1 - b_n f_1 \equiv 0 \pmod{\mathfrak{p}}$ for all n. Then there exists an f in $S_k(\Gamma_0(N), \psi)$, and d_n, such that for all n,

$$T_n f = d_n f,$$

and

$$d_n \equiv c_n \pmod{\mathfrak{p}'}$$

for some prime $\mathfrak{p}'$ dividing $\mathfrak{p}$.

We want to apply this Lemma to our modular form $g(z)E_1(z)$ of weight 2. Since the constant term in the Fourier expansion of $E_1(z)$ is 1, this gE is indeed normalized, i.e., $c_1 = 1$. Let K be its corresponding number field with prime ideal $\mathfrak{p}$ (dividing 3). Since $E \equiv 1 \pmod 3$, we have

$$T_n f_1 \equiv b_n f_1 \pmod{\mathfrak{p}}$$

(since $T_n g = b_n g$) for all n. Thus the Lemma is indeed applicable (taking $f_1 = g(z)E_1(z)$), and it produces a normalized form $f \in S_2(\Gamma_0(N), \psi)$ such that $T_p f = a_p f$ for all p, and $a_p \equiv c_p \equiv b_p \pmod{p}$. In particular, for almost all q,

$$a_q \equiv q + 1 - \#E(\mathbb{F}_q) \pmod{\mathfrak{p}'},$$

as required.

§2. Automorphic Representations of Weight One

2.1. For σ an irreducible, "odd," solvable two-dimensional representation of $G_{\mathbb{Q}}$, the relevant results of Langlands and Tunnell do not directly produce the required modular form $g(z)$, but rather — as already suggested — a certain automorphic cuspidal representation $\pi(\sigma)$, which is shown to *correspond* to such a form $g(z)$. An honest explanation of how this works requires (at least) the exposition of representation theory given in the pages below. Roughly speaking, a classical eigen-cusp form g, of weight one, and fixed level and character, generates a *collection* of irreducible representations

$$\{\pi_p\}_{p\leq\infty}$$

of the "local" groups $\mathrm{GL}_2(\mathbb{Q}_p)$ — each of which is uniquely determined by the data attached to $g(z)$; moreover this collection comprises what is called an *automorphic cuspidal representation* π of the adele group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$.

Once the notion of "automorphic form" is liberated from its classical (upper half-plane) setting, it seems completely natural to take the further step of replacing $\mathbb{Q}$ by an arbitrary number field F, and GL_2 by any GL_n, $n \geq 1$. Thus one ultimately views "Dirichlet characters mod n," Hecke characters of a number field, classical modular forms, and even "Maass cusp forms" (cf. Remark 2.5.5) as manifestations of one and the same kind of global object, namely, an *automorphic representation of* GL_n *over a number field.* It is this language which Langlands used to formulate the following general *Langlands Reciprocity Conjecture* (LRC): for any n-dimensional representation σ of $\mathrm{Gal}(\bar{F}/F)$ (or more generally the *Weil group* W_F), there is a corresponding automorphic representation

$$\pi(\sigma) = \otimes_v \pi_v$$

of $\mathrm{GL}_n(\mathbb{A}_F)$ such that for almost all the primes v of F,

$$\operatorname{trace}\sigma(\mathrm{Fr}_v) = \operatorname{trace}(t_{\pi_v})$$

(with t_{π_v} the Langlands class of π_v in $\mathrm{GL}_n(\mathbb{C})$); moreover, if σ is irreducible, then $\pi(\sigma)$ is *cuspidal.*

Our main goal in Lectures II and III will be to describe the ideas behind the statement and the *proof* of this "Strong Artin Conjecture" in case $n = 2$ and σ is solvable, namely:

Theorem 2.1. *To each irreducible*

$$\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

with solvable image in $\mathrm{PGL}_2(\mathbb{C})$, *there corresponds an automorphic cuspidal representation* $\pi(\sigma) = \otimes\pi_v$ *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *with the property that (its central character equals* $\det\sigma$ *and)*

$$\operatorname{trace}(t_{\pi_v}) = \operatorname{trace}\sigma(\mathrm{Fr}_v) \quad \textit{for almost every prime } v \textit{ of } F.$$

Our more modest goal in the rest of this Lecture is to explain how Theorem 2.1 gives the required classical result (Theorem 1.3) *when* $F = \mathbb{Q}$, σ *factors through the Galois group* $G_{\mathbb{Q}}$, *and* $\det(\sigma)$ *is odd.* One step is to show that for such σ, the corresponding $\pi(\sigma)$'s are "automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ of weight one." This step will actually be postponed until we discuss the correspondence $\sigma \longrightarrow \pi(\sigma)$ in earnest in Lecture II; cf. Proposition 4.2. A second step is to show there is a one-one correspondence between these "automorphic representations of weight one" and the classical new forms of weight one required in Theorem 1.3 (cf. Proposition 2.5).

2.2. Archimedean Representation Theory (GL_2)

Let $G = G_\infty$ denote $\mathrm{GL}_2(\mathbb{R})$, and $\mathfrak{g}$ its complexified Lie algebra. Let $K = K_\infty = O(2,\mathbb{R})$ denote the real 2×2 orthogonal group, a maximal compact subgroup of G.

Definition. An *admissible representation* of G on a Hilbert space H is a homomorphism

$$\pi : G \longrightarrow \mathrm{GL}(H)$$

such that (i) the map $(g, v) \longrightarrow \pi(g)v$ from $G \times H \longrightarrow H$ is continuous, and (ii) ("admissibility") the restriction of π to K contains each irreducible unitary representation of K with *finite* multiplicity (recall that each such representation is automatically *finite* dimensional).

Remark. For an admissible representation $\pi : G \longrightarrow \mathrm{Aut}(H)$, let $V = V_K \subset H$ denote the subspace of *K-finite vectors*, i.e., those v in H whose translates under $\pi(K)$ span a finite-dimensional space. Such vectors are *not* preserved by the action of $\pi(G)$, but they are by the corresponding (differentiated) action of the *Lie algebra* $\mathfrak{g}$ *of* G. In fact, as a representation space jointly for the action of $\mathfrak{g}$ and K, V enjoys certain "compatibility" properties which ensure that it is (what's called) a $(\mathfrak{g}, K)$-module. The advantage of these modules is that they are "algebraic" linear objects as opposed to "analytic" Lie group theoretic objects, and yet they accurately reflect the nature and properties of π. For example, π is *irreducible* in the usual sense (that H has no *Hilbert space* subspaces invariant under $\pi(G)$) if and only if V is irreducible in the algebraic sense that it has no *vector space* subspaces invariant under both $\pi(K)$ and $d\pi(\mathfrak{g})$). This leads to an equivalence between the natural categories of irreducible admissible representations of G and irreducible $(\mathfrak{g}, K)$ modules. In particular, whenever convenient, we allow ourselves to confuse π with the $(\mathfrak{g},\ K)$ module V.

Example 2.1. Let μ_1 and μ_2 denote two characters of $\mathbb{R}^\times$ (i.e., *not* necessarily "unitary" homomorphisms from $\mathbb{R}^\times$ to $\mathbb{C}^\times$). Let $H = H(\mu_1, \mu_2)$ denote the Hilbert space of functions $f : G \longrightarrow \mathbb{C}$ such that

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left| \frac{a_1}{a_2} \right|^{1/2} \mu_1(a_1)\mu_2(a_2)f(g)$$

for all $\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}$ in the Borel subgroup B consisting of upper triangular matrices of G, and such that

$$\|f\|^2 = \int_K |f(k)|^2 dk < \infty \ .$$

(Strictly speaking, one first looks at *continuous* such f, and then defines $H(\mu_1, \mu_2)$ to be the closure of such functions with respect to the norm $\|f\| \ldots$.) Then the right regular action of G on functions f in $H(\mu_1, \mu_2)$ defines an *admissible* representation of G which is denoted $\pi(\mu_1, \mu_2)$ and called the representation of G *induced from the character* $\mu_1\mu_2$ (of B). The corresponding $(\mathfrak{g}, K)$ module V_K consists of finite linear combinations of smooth functions ϕ_k defined by

$$\phi_k(g) = \mu_1(a_1)\mu_2(a_2)e^{-ik\theta}$$

if g has "Iwasawa decomposition"

$$g = \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} r(\theta).$$

(Here $r(\theta)$ denotes the rotation element $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in K, and there is an obvious compatibility condition coming from the elements $\pm I$ in $B \cap K$, namely $\mu_1\mu_2(-1) = e^{-ik\pi}$.) From this picture it is a simple matter to prove that $\pi(\mu_1, \mu_2)$ is *irreducible* if and only if $\mu_1\mu_2^{-1}(x) \neq x^p \operatorname{sgn}(x)$ with p a non-zero integer. Slightly less transparent is the proof of the following:

Fact. *Every* irreducible admissible representation π of G is (equivalent to) a $\pi(\mu_1, \mu_2)$, or an irreducible subrepresentation thereof. For example, suppose $\mu_1\mu_2^{-1}(x) = x^p \operatorname{sgn}(x)$ with p a positive integer. Then $H(\mu_1, \mu_2)$ contains exactly one *invariant* subspace, namely

$$\{\ldots, \phi_{-p-3}, \phi_{-p-1}, \phi_{p+1}, \phi_{p+3}, \ldots\},$$

and the restriction of $\pi(\mu_1, \mu_2)$ to this subspace realizes an *irreducible discrete series representation* of "lowest weight $p+1$."

Concluding Remarks. (a) All the above notions, and most of the results, hold for an arbitrary semisimple or reductive Lie group; in particular, every irreducible admissible representation of such a group is still realizable as a subrepresentation of the analogous induced representation. (The theory for $\mathrm{GL}_2(\mathbb{R})$ is essentially Bargmann's, and the general theory essentially due to Harish-Chandra; for many more details, see the recent survey paper of Knapp [Kn].)

(b) For any irreducible admissible representation π, an analogue of Schur's Lemma implies that there is a character of $\mathbb{R}^\times$, denoted ω_π and called *the central character of* π, such that

$$\pi\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \omega_\pi(r)I \quad \text{for all } r \in \mathbb{R}^\times.$$

2.3. *P*-adic Representation Theory (GL_2)

In this Section, F is a p-adic field with a ring of integers O_F, and $G = \mathrm{GL}_2(F)$. As in the real case, *most* of the notions and facts we review below extend not only to GL_n, but to an arbitrary reductive p-adic group; however, we recall here only those facts (even for GL_2) which are really needed in the sequel.

Definition 2.3.1. An *admissible representation* of G on a *vector space* V is a homomorphism

$$\pi : G \longrightarrow \mathrm{GL}(V)$$

such that (i) the stabilizer in G of any v in V is open, and (ii) ("admissibility") for any compact open subgroup $K^0 \subset G$, the space

$$\{v \in V \, : \, \pi(k)v = v \text{ for all } k \in K^0\}$$

is *finite*-dimensional.

Remark. Suppose π is an *irreducible unitary* representation of G in some Hilbert space H (same definition as in the case of Lie groups), and V_K is its subspace of K-finite vectors (for any compact open K, say $K = \mathrm{GL}_2(O_F)$). Then (by a Theorem of J. Bernstein) the restriction of $\pi(G)$ to $V = V_K$ produces an *admissible* representation of G in the above sense. Thus the p-adic notion of an admissible representation (on a vector space V) is a natural analogue of the archimedean notion of a $(\mathfrak{g}, K)$ module. What's special in the p-adic case is that G itself acts on V_K (and there is no need to go to the Lie algebra...).

Example (of $\pi(\mu_1, \mu_2)$). For each pair of characters $\mu_i : F^\times \longrightarrow \mathbb{C}^\times$, the induced representations $\pi(\mu_1, \mu_2)$ in $H(\mu_1, \mu_2)$ are defined just as in the archimedean case. But now reducibility is possible *only* if $\mu_1\mu_2^{-1}(x) = |x|^{\pm 1}$, i.e., there is no room for many "discrete series" representations to appear as subrepresentations of $\pi(\mu_1, \mu_2)$. This reflects the fact that there are now representations which are absolutely *cuspidal*, i.e., they cannot be constructed in this simple way.

Fortunately, we shall not need to discuss the cuspidal representations in these Lectures, but rather only those representations which are as far from being cuspidal as possible!

Definition 2.3.1. An irreducible admissible representation π of G is *unramified* (or *of conductor zero*) if its space of $K = \mathrm{GL}_2(O_F)$ fixed vectors is non-empty.

In this case, it is known that the space of K-fixed vectors is *one-dimensional*, and that π is either one-dimensional (of the form $\chi \circ \det$, for some *unramified* character χ of $F^\times$), or an irreducible $\pi(\mu_1, \mu_2)$ with μ_1 and μ_2 unramified characters of $F^\times$. As we shall recall in the next paragraph, it is these latter representations which typically play a crucial role in the adelic theory of automorphic representations.

Concluding Remark. Even if some π does not have "conductor zero," there will still exist a smallest (positive) integer N (called the *conductor of* π) such that the space

$$\{v \in V : \pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \omega_\pi(a)v \text{ for all } k \in K^N\}$$

is non-empty. (Here K^N denotes the Hecke congruence subgroup of K consisting of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \pmod{\mathfrak{p}^N}$.) This fact was first proved in [Cas], and then generalized to GL_n in [J-PS-S1]; if $N = \mathrm{conductor}(\pi)$, then it is also known (as in the case $N = 0$) that the space V^{K^N} is automatically one-dimensional.

2.4. Adelic Representations (GL_2)

This is mostly a matter of putting together the local representation theory.

Suppose we are given a collection of irreducible admissible representations $\{\pi_p\}$ of the local groups $G_v = \mathrm{GL}_2(\mathbb{Q}_p)$, such that π_p is unramified for all but finitely many p. Then the *restricted* tensor product

$$\bigotimes_{p \leq \infty} \pi_p$$

makes sense as a representation of the restricted product $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \prod'_{p\leq\infty} \mathrm{GL}_2(\mathbb{Q}_p)$, and defines an irreducible "admissible" representation π in a sense that can be made precise; cf. §4.c of [Gel] or §9 of [JL] for more details. Conversely, any irreducible "admissible" (and in particular any irreducible *unitary*) representation π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is uniquely factorizable as

$$\pi = \otimes \pi_p,$$

with almost every π_p unramified. Moreover, the obvious analogues of these statements hold for an arbitrary number field F.

Example $G = \mathrm{GL}_1$. In this case, we consider a collection of *characters* χ_v of $F_v^\times$, one for each prime of F, such that almost all the χ_v's are unramified, i.e., trivial on $O_v^\times$. Then

$$\chi = \prod_v \chi_v$$

defines a nice (continuous) character of the ideles $\mathbb{A}_F^\times = \mathrm{GL}_1(\mathbb{A}_F)$, and every such character thus arises (cf. Proposition 7-1-12 of [Gold]). Of course,

in number theory one is primarily concerned with characters χ of $\mathbb{A}_F^x$ which are trivial on $F^\times$, i.e., with "grossencharacters," or characters of the *idele class group*

$$\mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A}_F^\times) = F^\times \backslash \mathbb{A}^\times.$$

These are the "automorphic representations" of GL(1) over F.

Special Example. Since $\mathbb{Q}$ has class number one,

$$(*) \qquad \mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot \mathbb{R}^+ \cdot \prod_{p<\infty} \mathbb{Z}_p^\times.$$

Thus a *Dirichlet character* $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ determines a *grossencharacter* χ_ψ as follows: for any $p < \infty, \psi$ can be pulled back to a character χ_p of $\mathbb{Z}_p^\times$ through the canonical homomorphism from $\mathbb{Z}_p^\times$ to $(\mathbb{Z}/N\mathbb{Z})^\times$; the product $\prod_{p<\infty} \chi_p$ then defines a character of $\prod_{p<\infty} \mathbb{Z}_p^\times$, and hence by $(*)$ a character χ_ψ of $\mathbb{A}_\mathbb{Q}^\times$ *trivial on* $\mathbb{R}^+$ *as well as* $\mathbb{Q}^\times$. In this way one obtains a grossencharacter of $\mathbb{A}_\mathbb{Q}^\times$ of *finite order*, and all such grossencharacters arise in this way for suitably large N. (Note that χ_p is unramified for the primes p not dividing N.)

Following this lead for GL_1, it is clear that one should define *automorphic representations* for $G = \mathrm{GL}_n$ in terms of the quotient space

$$\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}).$$

For simplicity, we give a precise definition only for $G = \mathrm{GL}_2$ and *cuspidal* automorphic representations.

Definition. Fix a grossencharacter ω of F, and let $L_0^2(G(F) \backslash G(\mathbb{A}), \omega)$ denote the (closure of the) space of all continuous $\varphi : G(\mathbb{A}) \longrightarrow \mathbb{C}$ such that

(i) $\varphi(\gamma g z) = \omega(z)\varphi(g)$
for all $\gamma \in G(F)$ and $z \in Z(\mathbb{A}) = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \right\}$ (the center...);

(ii) $\displaystyle\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} |\varphi(g)|^2 dg < \infty$; and

(iii) φ is *cuspidal*, i.e., for any g in $G(\mathbb{A})$,

$$\int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

Then an irreducible admissible (necessarily unitary) representation $\pi = \otimes\pi_v$ of $G(\mathbb{A})$ is called *automorphic cuspidal* if there is some ω such that π is equivalent to an irreducible summand of the right regular representation of $G(\mathbb{A})$ in $L_0^2(G(F) \backslash G(\mathbb{A}), \omega)$.

Remarks. (i) In a completely similar way, the notion of an automorphic cuspidal representation π can be defined for $G = \mathrm{GL}_n$, and more generally, for an arbitrary reductive algebraic group G. In this generality, it is a Theorem of Gelfand and Piatetski-Shapiro (cf. [GGPS]) that the right regular representation R_0 of $G(\mathbb{A})$ in $L_0^2(G(F) \setminus G(\mathbb{A}), \omega)$ decomposes *discretely*, and with finite multiplicities, i.e.,

$$R_0 = \bigoplus m_\pi \pi, \quad \text{with} \quad m_\pi < \infty.$$

For $G = \mathrm{GL}_n$, it is actually known that this multiplicity is *one* (cf. [JL], [GK], and [Shal]), but for other groups this need no longer hold. For general G, it is at least still true that any automorphic cuspidal representation has a factorization of the type

$$\pi = \otimes \pi_v$$

discussed above, with π_v almost everywhere an unramified representation of $G(F_v)$.

(ii) There is also the notion of an *automorphic* (*not* necessarily cuspidal) representation of $G(\mathbb{A})$, but as we shall not focus on these in the sequel, we refrain from giving a precise definition. Suffice it to say that such π are the irreducible admissible representations of $G(\mathbb{A})$ which are built out of cuspidal representations of the Levi components of G by way of *induction* (and taking of quotients); for example, for GL_2, the automorphic (non-cuspidal) representations are the quotients of the induced representations $\pi(\mu_1, \mu_2)$ with μ_1 and μ_2 grossencharacters of F (viewed as "cuspidal representations" of the diagonal subgroup of GL_2). For a thorough discussion, see [BoJa] and [La2].

2.5. A Dictionary (Between the Classical and Modern Theories for GL_2)

In the last paragraph, we recalled how classical Dirichlet characters correspond to certain automorphic representations of GL(1), namely those grossencharacters which are of finite order. We now describe an analogous result for GL(2).

Proposition. *There is a 1-1 correspondence between normalized new forms*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

in $S_k(\Gamma_0(N), \psi)$, and irreducible automorphic cuspidal representations

$$\pi = \otimes_p \pi_p$$

of $G(\mathbb{A})$ such that:

(a) the central character of π is χ_ψ,

(b) π_p is unramified for all $p \nmid N$, and

(c) π_∞ *has "lowest weight* k."
Moreover, if $N = \prod p_i^{\alpha_i}$, *and* $p|N$, *then* π_p *has conductor* α_i; *but if* $p \nmid N$, *then* $\pi_p = \pi(\mu_1, \mu_2)$, *with* μ_1, μ_2 *unramified characters of* $\mathbb{Q}_p^\times$ *such that*

$$p^{\frac{k-1}{2}}(\mu_1(p) + \mu_2(p)) = a_p. \tag{2.5.1}$$

Remark 2.5.2. In case $k > 1$, then π_∞ is a *discrete series* representation of "lowest weight k," sitting inside some $\pi_\infty(\mu_1, \mu_2)$ (cf. Fact 2.2); this discrete series representation has central character *trivial* on $\mathbb{R}^+$, and is denoted D_k. However, if $k = 1$, then π_∞ is *not* a *sub*representation of any $\pi_\infty(\mu_1, \mu_2)$, but rather equals the full induced ("principal series") representation π_∞ $(1, \mathrm{sgn})$ (see the proof below). In this case, the corresponding representation $\pi = \otimes\pi_p$ (with $\pi_\infty = \pi_\infty$ $(1, \mathrm{sgn})$) is called an *automorphic cuspidal representation of weight one.*

Corollary of the Proposition. *A normalized new form in* $S_1(\Gamma_0(N), \psi)$, *with eigenvalues* $\{a_p\}$, *is one and the same thing as an automorphic cuspidal representation* $\otimes\pi_p$ *of weight one such that (cf. (2.5.1))*

$$a_p = \mu_1(p) + \mu_2(p), \quad \textit{for all } p \nmid N.$$

N.B. Here μ_1 and μ_2 are the two unramified characters of $\mathbb{Q}_p^\times$ inducing the unramified representation $\pi_p = \pi_p(\mu_1, \mu_2)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. From now on, we rewrite the above relation in the more suggestive form

$$a_p = \text{trace } (t_{\pi_p}) \tag{2.5.3}$$

with

$$t_{\pi_p} = \begin{pmatrix} \mu_1(p) & 0 \\ 0 & \mu_2(p) \end{pmatrix} \quad \text{in} \quad \mathrm{GL}_2(\mathbb{C})$$

the so-called *Langlands class of* π_p.

Sketch of the Proof (of the Proposition). The *first step* is at the level of *functions.* Using the decomposition

$$G(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})\, \mathrm{GL}_2^*(\mathbb{R}) \prod_{p<\infty} K_p^N,$$

(analogous to $(*)$ for GL_1), one defines for any f in $S_k(\Gamma_0(N), \psi)$ a function φ_f on $G(\mathbb{A}_\mathbb{Q})$ by

$$\varphi_f(\gamma g_\infty k_0) = f(g_\infty(i)) j(g_\infty, i)^{-k} \chi_\psi(k_0); \tag{2.5.4}$$

here

$$K_p^N = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in K_p = \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv \sigma(N) \right\}$$

and

$$j(g_\infty, z) = (cz+d)(\det\ g_\infty)^{-\frac{1}{2}} \quad \text{if} \quad g_\infty = \begin{pmatrix} a & b \\ c & a \end{pmatrix}.$$

It is now a standard matter (cf. [Cas] and [De]) to check that this map $f \longrightarrow \varphi_f$ is an isomorphism from $S_k(\Gamma_0(N), \psi)$ to the space of smooth functions $\{\varphi\}$ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that
(i) $\varphi(\gamma g) = \varphi(g)$ for all $g \in \mathrm{GL}_2(\mathbb{Q})$;
(ii) $\varphi(gk) = (\chi_\psi)(k)\varphi$ for all $k \in K_p^N$;
(iii) $\varphi(gr(\theta)) = e^{-ik\theta}\varphi(g)$ for all $r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in G_\infty$;
(iv) $\varphi(zg) = \chi_\psi(z)\varphi(g)$ for all $z \in Z(\mathbb{A})$;
(v) If X denotes the "pushing down" differential operator corresponding to the element $\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ of $\mathfrak{g}$, then

$$X \cdot \varphi = 0$$

(this is the condition reflecting the holomorphy of f);
(vi) φ is of "moderate growth" on $G(\mathbb{A})$ (relecting the holomorphy of f "at the cusps"); and
(vii) φ is cuspidal, i.e.,

$$\int_{\mathbb{F}\diagdown\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx \equiv 0$$

(reflecting the cuspidality of $f\ldots$).

Conditions (i)–(vii) imply that φ_f belongs to $L_0^2(G(\mathbb{Q})\diagdown G(\mathbb{A}), \chi_\psi)$. The *second step* of the proof is then to show that φ_f generates (under right translation by $G(\mathbb{A})$) an *irreducibly* invariant subspace π_f (of $L_0^2(\chi_\psi)$) whose corresponding representation $\pi_f = \otimes\pi_p$ is as claimed; one must also show that all such representations are thus obtained. These arguments are explained in §5 of [Gel], but only really for the case $k > 1$, where π_∞ is the discrete series representation D_k. In case $k = 1$, one must argue as follows (in order to identify π_∞).

Suppose $\mu_i(x) = |x|^{s_i}\,\mathrm{sgn}(x)^{\varepsilon_i}$, and $\pi_\infty = \pi(\mu_1, \mu_2)$. Then $H(\mu_1, \mu_2)$ consists only of functions ϕ_k with k of the same parity as $\varepsilon_1 + \varepsilon_2$. (In particular, $\pi(1, \mathrm{sgn})$ consists only of "odd" functions....) A straightforward computation (á la Bargmann...) also shows that

$$X \cdot \phi_k = \left(\frac{s_1 - s_2 + 1}{2} - \frac{k}{2}\right)\phi_k$$

and

$$\phi_k\left(g\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\right) = r^{s_1+s_2}\phi_k(g), \quad \text{for } r > 0.$$

This means that if ω_{π_∞} is to be trivial on $\mathbb{R}^+$, we must have $s_1 = -s_2 = s$, and if $X \cdot \phi_1$ is to be 0, we must have $s_1 = s_2 = s = 0$, i.e., $\pi_\infty = \pi(1, \mathrm{sgn})$ as claimed.

Concerning the converse direction, we recall the following. Suppose $\pi = \otimes\pi_p$ is an irreducible subrepresentation of $L_0^2(G(\mathbb{Q} \setminus G(\mathbb{A}_\mathbb{Q}), \chi_\psi)$ (for some grossencharacter χ_ψ of finite order), and π_∞ is "of weight k" ($k \geq 1$). If the *conductor* $c(\pi) = \prod_{p<\infty} c(\pi_p)$ of π is N, let φ_π denote the function in the space H_π of π which is right K_p^N invariant for all p. (This function is uniquely determined up to a scalar; cf. the Concluding Remark of §2.3.) Then via the correspondence $\varphi_\pi \longrightarrow f_\pi$ (inverse to (2.5.4)), we obtain from π the required *new form* of weight k.

Remark 2.5.5. The fact that the principal series $\pi(1, \mathrm{sgn})$ leads to a *holomorphic* form $f(z)$ (of weight one...) relies on the critical confluence of conditions

$$X\phi = 0$$

and

$$\phi(gr(\theta)) = e^{-i\theta}\phi(g).$$

Indeed, these two conditions together imply that the function f defined by

$$f(z) = \phi(g_\infty(i))j(g_\infty, i)$$

will be holomorphic on the upper half-plane $\mathfrak{h}$. The delicacy of this point can be appreciated by examining what would happen if we took π_∞ to be the principal representation $\pi(1,1)$ (or $\pi(\mathrm{sgn}, \mathrm{sgn})$) in place of $\pi(1, \mathrm{sgn})$. In this case, only ϕ_k's of even parity occur in $H(\mu_1, \mu_2)$, $X \cdot \phi_k = \left(\frac{1-k}{2}\right)\phi_k$ is *never* zero, and no *holomorphic* f's arise in $\mathfrak{h}$. The crucial point now is that ϕ_0 will be K_∞-*invariant*, and hence directly define a function $f(z)$ on $\mathfrak{h}$, with the property that

$$\Delta f = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 1/4f.$$

Indeed, let D denote the standard *Casimir operator* in the center of the universal enveloping algebra of $\mathfrak{g}$, which for K_∞-invariant functions corresponds exactly to the *Laplace-Beltrami* operator Δ above. Then the action of D in $H(\mu_1, \mu_2)$ (with $\mu_2(x) = |x|^{s_i}\,\mathrm{sgn}(x)^{\varepsilon_i}$) is given by the formula

$$D \cdot \phi_k = \frac{(s_1 - s_2)^2 - 1}{4}\,\phi_k$$

(cf. Lemma 5.6 of [JL], page 166, keeping in mind that our Casimir is 1/2 theirs). Thus the same reasoning as used above (to show that an automorphic cuspidal representation $\otimes\pi_p$ with $\pi_\infty = \pi(1, \mathrm{sgn})$ corresponds to a classical cusp form of weight 1) shows also that *a cuspidal representation* $\otimes\pi_p$ *of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ *with* $\pi_\infty = \pi(1,1)$ *(or* $\pi(\mathrm{sgn}, \mathrm{sgn})$*) corresponds to a Maass cusp form of "eigenvalue* 1/4."

2.6. Reformulation of Theorem 1.3

Suppose

$$\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

is a continuous, irreducible, two dimensional "odd" representation whose image in $\mathrm{PGL}_2(\mathbb{C})$ is solvable. Then there exists an automorphic cuspidal representation $\pi(\sigma) = \otimes_p \pi_p$ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ which is of weight one, central character $\det \sigma$, and such that for almost all p, $\pi_p = \pi(\mu_1, \mu_2)$ is unramified with

$$\operatorname{trace} \sigma(\mathrm{Fr}_p) = \operatorname{trace}(t_{\pi_p}) = \mu_1(p) + \mu_2(p). \tag{2.6.1}$$

Remarks. **(2.6.2).** Recall that the matrix

$$\begin{pmatrix} \mu_1(p) & 0 \\ 0 & \mu_2(p) \end{pmatrix} = t_{\pi_p}$$

is the *Langlands class* in $\mathrm{GL}_2(\mathbb{C})$ attached to the unramified representation π_p.

(2.6.3) According to Corollary 2.5, the existence of an automorphic cuspidal $\pi = \otimes \pi_p$ as above implies the existence of a *new* form $f = \sum a_n e^{2\pi i n z}$ in some $S_1(\Gamma_0(N), \psi)$ with

$$a_p = \operatorname{trace}\ \sigma(\mathrm{Fr}_p)$$

for almost all p. Thus this representation theoretic reformulation of Theorem 1.3 indeed implies Theorem 1.3.

(2.6.4) In the next few lectures we shall explain how the more general Theorem 2.1 is proved; this will imply Theorem 2.6, in case $F = \mathbb{Q}$, σ factors through $G_{\mathbb{Q}}$, and $\det \sigma$ is odd, for it is a simple matter to see that $\pi(\sigma)$ is then actually of weight one, i.e., $\pi_\infty = \pi(1, \mathrm{sgn})$ (cf. Proposition 4.2 of Lecture II).

(2.6.5) *"Strong Multiplicity One"* for GL(2) asserts that two automorphic cuspidal representations π and π' are equivalent as soon as they are equivalent almost everywhere, i.e.,

$$\pi_p \cong \pi'_p \quad \text{for almost all } p.$$

This fact is explicitly used, together with *multiplicity one* for GL(2), in the proof that a new form f generates an *irreducible* subspace π_f of L_0^2. In the statement of Theorem 2.6 (or 2.1), it also implies that $\pi(\sigma)$ (if it exists at all) is unique. Indeed, once the central character is fixed, condition (2.6.1), which holds almost everywhere, uniquely determines π_p. Similarly, in the classical version (Theorem 1.3) of Langlands-Tunnell, the new form $g(z)$ is uniquely determined by the condition (1.3.1) which fixes its eigenvalues almost everywhere; this reflects the fact that the theory of *new* forms is one and the same thing as the strong multiplicity one result coupled with the notion of conductors! (See [Cas] or [Gel] for a further explanation of this point.)

Lecture II
The Langlands Program: Some Results and Methods

Abstract
We start this lecture by describing the *Local Langlands Conjecture (LLC) for* GL(n) over a local field F. In case n is a prime, this is a *Theorem* over any field F, known as the "Local Langlands Correspondence." Thus we can (and will) describe the resulting correspondence in some detail for $n = 2$, and apply it to *refine* (and generalize) the two-~~dimen~~sional Langlands Reciprocity Conjecture (LRC) as follows:

To each continuous irreducible representation

$$\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

of the *Weil* group of a number field F, there is associated an automorphic cuspidal representation $\pi(\sigma) = \otimes\pi_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$ with the property that

$$\pi_v \longleftrightarrow \sigma_v$$

for every place v. (Here σ_v, a two-dimensional representation of the local Weil group W_{F_v}, is the "Langlands parameter" of the corresponding representation π_v of $\mathrm{GL}_2(F_v)$; for almost every v, σ_v and π_v are unramified, and the relation $\sigma_v \longleftrightarrow \pi_v$ reduces to the more familiar relation

$$\sigma_v(\mathrm{Fr}_v) \sim t_{\pi_v}$$

in $\mathrm{GL}_2(\mathbb{C})$.)

As we shall see, the "classical version" of the Langlands-Tunnell Theorem (Theorem 1.3 of 2.6) follows immediately from the proof of the general Reciprocity Conjecture in the solvable case; indeed, when $F = \mathbb{Q}$, and σ factors through $G_{\mathbb{Q}}$ and is "odd," $\pi(\sigma)$ must be automorphic of weight one, i.e., $\pi(\sigma_\infty) = \pi_\infty(1, \mathrm{sgn})$ (cf. Proposition 4.2).

In the second half of this lecture, we also begin to collect the automorphic results required for proving the global LRC in the two-dimensional solvable case. As we shall see, all these results, as well as the LRC itself, are but special realizations of a "Principle of Functoriality with respect to the L-group," namely:

Langlands' Functoriality Conjecture

Given two reductive F-groups G and G' (with G' quasi-split), and a morphism

$$\rho : {}^L G \longrightarrow {}^L G'$$

between their L-groups, there is a corresponding mapping of automorphic representations

$$\pi \longrightarrow \pi(\rho) = \otimes\pi'_v$$

from G to G', such that for almost every v, ρ takes the Langlands class t_{π_v} in ${}^LG$ to the Langlands class $t_{\pi'_v} = t_{\pi_v(\rho)}$ in ${}^LG'$.

§3. The Local Langlands Correspondence for $GL(2)$

3.1. The Archimedean Case

We assume $F = \mathbb{R}$. (For the simpler case of $F = \mathbb{C}$, which we do not need, see Remark 3.1.2 below). In this case, the *Weil Group* W_F is an extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z}$ given by

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times,$$

where $j^2 = -1$, and $jcj^{-1} = \bar{c}$, and the natural surjection

$$\varphi : W_{\mathbb{R}} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

is given by $\varphi(\mathbb{C}^\times) = 1$ and $\varphi(j\mathbb{C}^\times) = \tau$ (complex conjugation). We are interested in the set of equivalence classes $\Phi(\mathrm{GL}_n/\mathbb{R})$ of n-dimensional complex representations σ of $W_{\mathbb{R}}$ whose images consist of semisimple elements in $\mathrm{GL}_n(\mathbb{C})$.

Example 3.1.1. The *one-dimensional* representations of $W_{\mathbb{R}}$ are of the form $\mu \sim (t, \varepsilon)$, taking z in $\mathbb{C}^\times$ to $|z|_{\mathbb{C}}^t$, $t \in \mathbb{C}$, and j to $\varepsilon = \pm 1$. (Indeed, if $\mu(j) = w$, then on $\mathbb{C}^\times, \mu(\bar{z}) = \mu(jzj^{-1}) = w\mu(z)w^{-1} = \mu(z) = z^t\bar{z}^s$ with $t = s$, i.e., $\mu(z) = r^{2t} = |z|^t$; also $\mu(-1) = 1 = \mu(j^2) = w^2 \Longrightarrow w = \pm 1$.) On the other hand, the *two-dimensional irreducible* representations of $W_{\mathbb{R}}$ are all induced from some character

$$z \longrightarrow |z|_{\mathbb{C}}^t \left(\frac{z}{|z|}\right)^m$$

of $\mathbb{C}^\times$, with t arbitrary in $\mathbb{C}$, *and $m \geq 1$ an integer.* Clearly these representations are "semisimple." It is also easy to show that every n-dimensional semisimple representation σ of $W_{\mathbb{R}}$ is a direct sum of these one and two-dimensional irreducible representations.

Theorem. The Local Langlands Correspondence for $\mathrm{GL}_n(\mathbb{R})$. *There is a well defined bijection*

$$\sigma \longleftrightarrow \pi(\sigma)$$

between $\Phi(\mathrm{GL}_n/\mathbb{R})$, the set of classes of n-dimensional semisimple complex representations σ of $W_{\mathbb{R}}$, and $\Pi(\mathrm{GL}_n/\mathbb{R})$, the set of classes of irreducible admissible representations π of $\mathrm{GL}_n(\mathbb{R})$; moreover, the L and ε factors assigned to σ and π are preserved by this correspondence.

Remarks. The existence of this correspondence, formulated and proved more generally by Langlands for an arbitrary reductive Lie group, is the subject matter of [La3]; the fact that L and ε factors may be defined for σ

and π *in the context of* GL_n, and then preserved by this correspondence, is discussed in [Ja1].

Example of $\mathrm{GL}_2(\mathbb{R})$. Suppose first that σ is the sum of two one-dimensional representations (i.e., characters) $\mu_i \sim (t_1, \varepsilon_1)$ as in Example (3.1.1). Then $\pi(\sigma)$ is taken to be the unique irreducible *quotient* of the (induced representation) $\pi(\mu_1, \mu_2)$, where $\mu_i(x) = |x|_{\mathbb{R}}^{t_1}(\mathrm{sgn}(x))^{\varepsilon_1}$, and the order of t_1, t_2 is arranged so that $\mathrm{Re}(t_1) \geq \mathrm{Re}(t_2)$. For example, if $\sigma = \left(\frac{1}{2}, 0\right) \oplus \left(-\frac{1}{2}, 0\right)$, then $\pi(\sigma)$ is the trivial representation, whereas if $\sigma = (0,0) \oplus (0,1)$ (resp. $(0,0) \oplus (0,0)$) then $\pi(\sigma)$ is the irreducible principal series representation $\pi(1, \mathrm{sgn})$ (of "lowest weight 1") (resp. the class 1 principal series representation $\pi(1,1)$, with Casimir eigenvalue $\lambda = -\frac{1}{4}$). On the other hand, suppose now that σ is the irreducible two-dimensional representation of Ex. 3.1.1, with parameters t and $m \geq 1$. Then $\pi(\sigma)$ is taken to be the *discrete series representation* $D_{m+1} \otimes |\det(\ )|_{\mathbb{R}}^{t}$, with D_{m+1} of lowest weight $m+1$ and trivial central character.

Remark 3.1.2.. In case $F = \mathbb{C}$, the Weil group is just $\mathbb{C}^\times$ and each n dimensional semisimple representation σ is just a sum of characters μ_i of the form $\left(\frac{z}{\bar{z}}\right)^{m_i} |z|_{\mathbb{C}}^{t_i}$ with $m_i \in \mathbb{Z}$. In this case, there are *no* discrete series, and to each σ as above, the corresponding $\pi(\sigma)$ is just the unique irreducible quotient of $\mathrm{Ind}\, \mu_1 \mu_2 \cdots \mu_n$, with the μ_i's arranged so that

$$\mathrm{Re}(t_1) \geq \cdots \geq \mathrm{Re}(t_n).$$

3.2. The p-adic case

In case F is a p-adic field, its *Weil group* W_F is a dense *subgroup* of $\mathrm{Gal}(\bar{F}/F)$, equipped with an isomorphism

$$F^\times \xleftarrow{\sim} W_F^{ab}.$$

In particular, the *one*-dimensional (complex) representations of W_F are again identified with the irreducible admissible representations (i.e. characters) of $F^\times = \mathrm{GL}(1, F)$, just as in the archimedean case. However, unlike in the archimedean case, there are now irreducible representations of W_F of *arbitrary* dimension (reflecting the existence of extensions of F of arbitrary degree...). This fact considerably complicates the representation theory — and concommitent local Langlands correspondence — for $\mathrm{GL}(n)$. Fortunately, for our purposes, we don't need to describe the full Langlands correspondence; instead, we need only the following:

Theorem 3.2. *For each two-dimensional "semisimple" representation σ of W_F, there is exactly one irreducible admissible representation $\pi = \pi(\sigma)$ of $\mathrm{GL}_2(F)$ with*

$$\omega_\pi(\alpha) := \pi \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \det \sigma(\alpha) I, \tag{3.2.1}$$

and such that for all characters χ *of* $F^\times$, *and* ψ_F *of* F,

$$L(s, \pi \otimes \chi) = L(s, \sigma \otimes \chi),$$
$$L(s, \tilde{\pi} \otimes \chi^{-1}) = L(s, \tilde{\sigma} \otimes \chi^{-1}),$$
$$\varepsilon(s, \pi \otimes \chi, \psi_F) = \varepsilon(s, \sigma \otimes \chi, \psi_F).$$

Moreover, all irreducible admissible π *thus arise. (Here the* L *and* ε *factors on the left-hand side are those of Jacquet-Langlands, and those on the right-hand side the local factors of [La4]; "$\sim$" denotes the contragredient representation.)*

Remark 3.2.2. The *existence* parts and exhaustion of the Theorem are easy, except for the case of irreducible σ (which is due — for arbitrary F — to Kutzko [Kut]). The *uniqueness* part is Corollary 2.19 of [JL], and the resulting bijection

$$\sigma \longleftrightarrow \pi(\sigma)$$

amounts to *the Langlands correspondence for* GL(2).

Caution. Missing in the image of the map

$$\sigma \longrightarrow \pi(\sigma)$$

just described are the "special representations" of $\mathrm{GL}_2(F)$. Although they can be obtained by considering representations of the Weil-Deligne group W_F' in place of W_F (see, for example, [Ta] or [Kud]), we prefer to ignore these representations as they play no crucial role in the sequel. In fact, for the global applications we have in mind to the Reciprocity Conjecture, it is crucial to make explicit only the following *unramified* part of the correspondence.

Example 3.2.3. Recall that if k denotes the residue field of F, then W_F consists of those elements of $\mathrm{Gal}(\bar{F}/F)$ whose image in $\mathrm{Gal}(\bar{k}/k)$ is an integer power of the Frobenius automorphism generator of $\mathrm{Gal}(\bar{k}/k)$. Thus the inertia subgroup I of $\mathrm{Gal}(\bar{F}/F)$ is contained in W_F, and a representation $\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is called *unramified* if it is trivial on I. In this case, since $I \setminus W_F \cong \mathbb{Z}$ (integral powers of the generator of $\mathrm{Gal}(\bar{F}/F)$), σ is completely determined by where it takes the (class of a) Frobenius element Fr of W_F. So suppose (after conjugation, if necessary) that

$$\sigma(\mathrm{Fr}) = \begin{pmatrix} p^{-s_1} & \\ & p^{-s_2} \end{pmatrix} \quad \text{in } \mathrm{GL}_2(\mathbb{C}),$$

with $s_1, s_2 \in \mathbb{C}$. Then the corresponding representation $\pi(\sigma)$ of $\mathrm{GL}_2(F)$ will be the unramified induced representation $\pi(\mu_1, \mu_2)$, with $\mu_1(x) = |x|^{s_1}$, i.e., the Langlands class

$$t_{\pi(\sigma)} = \begin{pmatrix} \mu_1(p) & 0 \\ 0 & \mu_2(p) \end{pmatrix}$$

will be conjugate to $\sigma(\mathrm{Fr})$. (More precisely if $\pi(\mu_1, \mu_2)$ is itself reducible, then $\pi(\sigma)$ will be the unique irreducible *unramified quotient* of $\pi(\mu_1, \mu_2)$, *perforce* one-dimensional....)

§4. The Langlands Reciprocity Conjecture (LRC)

4.1. Reformulation of Theorem 2.6 ("Langlands-Tunnell")

For F a global field, the *Weil group* W_F maps surjectively onto the Galois group $\mathrm{Gal}(\bar{F}/F)$, and there is a canonical isomorphism

$$W_F^{ab} \cong {}_{F^\times}\backslash^{\mathbb{A}_F^\times}.$$

For each place v of F, there is also an injection $W_{F_v} \longrightarrow W_F$, defining a map

$$\sigma \longrightarrow \sigma_v$$

from the two dimensional semisimple representations of W_F to those of W_{F_v} (cf. [Ta] for background). For a given σ, almost all the resulting σ_v's will be *unramified*, and these unramified σ_v's uniquely determine σ. (This is "strong multiplicity one" on the "Galois side.")

Using the *local Langlands correspondence for* GL(2), we can now attach to any nice $\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$ a global representation $\pi(\sigma)$ of $\mathrm{GL}_2(\mathbb{A}_F)$, namely

$$\pi(\sigma) = \otimes\pi(\sigma_v).$$

The thrust of the Conjecture below is that this $\pi(\sigma)$ must be automorphic.

Conjecture (LRC). *Suppose*

$$\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

is irreducible. Then there exists an automorphic cuspidal representation $\pi = \otimes\pi_v$ *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *such that*

$$\pi_v = \pi(\sigma_v) \quad \textit{for all } v.$$

(In particular, the Hecke-Jacquet-Langlands L-function

$$L(s,\pi) = \prod_v L(s,\pi_v)$$

attached to π *— which is known by [JL] to be entire — will equal the Artin L-function* $L(s,\sigma) = \prod_v L(s,\sigma_v)$*.)*

Remarks. (1) This conjecture is actually *equivalent* (via the "converse theorem" for L-functions on GL(2)) to Artin's conjecture for two dimensional irreducible σ; cf. 5.3.1 below. Thus this LRC is sometimes called the "Strong Artin Conjecture."

(2) If σ is *reducible* and the sum of two grossencharacters μ_1 and μ_2, then there is easily seen to be an automorphic (non-cuspidal) representation $\pi = \otimes \pi_v$ of $\mathrm{GL}(2, \mathbb{A}_F)$ with $\pi_v = \pi(\sigma_v)$ for all v, namely the induced "Eisensteinian" representation $\pi(\mu_1, \mu_2)$ (or appropriate irreducible quotient thereof).
(3) When $F = \mathbb{Q}$, and $\sigma : W_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ factors through $G_{\mathbb{Q}}$, the above form of the LRC clearly implies the "almost everywhere" version which we stated in §2.6. According to the Proposition below, these two forms of the LRC are actually equivalent!

Proposition 4.1. *Suppose σ is a two-dimensional representation of W_F, and π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then*

$$\pi_v = \pi(\sigma_v) \quad \textit{for all } v$$

if and only if

$$\mathrm{trace}(t_{\pi_v}) = \mathrm{trace}(\sigma_v(\mathrm{Fr}_v))$$

for almost all v (where both π_v and σ_v are unramified).

Note that this last condition really says that, for the unramified places, $\pi_v = \pi_v(\sigma_v)$ (cf. Example 3.2.3 above). Thus this proposition essentially amounts to "strong multiplicity one" for GL(2); for further discussion, see pages 23–24 of [La].

4.2. Relations with Classical Forms

Proposition. *Suppose $\sigma : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is irreducible, and "odd," and let $\pi(\sigma)$ denote the corresponding automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ (assuming it exists!). Then $\pi(\sigma)$ corresponds (via the correspondence $f \longleftrightarrow \pi_f$ already described) to a normalized new form*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_1(\Gamma_0(N), \psi)$$

with $N =$ conductor(σ), ψ determined by the central character of $\pi(\sigma)$, and

$$a_p = \mathrm{trace}(\sigma(\mathrm{Fr}_p))$$

for almost every p.

Proof By Proposition 2.5, it suffices to check that $\pi_\infty = \pi_\infty(\sigma_\infty)$ is of the form $\pi(\mu_1, \mu_2)$, with $\mu_1 \equiv 1$ and $\mu_2 = \mathrm{sgn}(\ )$. Equivalently, we must check that σ_∞ is a sum of these two characters. But when viewed as a representation of $W_{\mathbb{R}}$, σ_∞ is clearly trivial on $\mathbb{C}^\times$. This means σ_∞ cannot be induced from a *non-trivial* character of $\mathbb{C}^\times$. Thus σ_∞ must be *reducible* (cf. Example 3.1.1), say the sum of two characters μ_i, with $\mu_i \sim (t_i, \varepsilon_i)$.

Since σ_∞ is trivial on $\mathbb{C}^\times$, it follows that each $t_i = 0$. On the other hand, the assumption $\det \sigma(\tau) = -1$ implies that $\sigma(\tau)$ *is not a scalar*; hence

$$\sigma(\tau) \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which means $\pi_\infty = \mathrm{Ind}\, 1 \cdot \mathrm{sgn}$, as claimed.

Concluding Remarks. (1) If $\det \sigma$ is *even*, then by the same reasoning as above, σ_∞ is the sum of two characters, but now either *both* trivial or both the sgn character. Thus one concludes $\pi_\infty = \pi(1,1)$ or $\pi(\mathrm{sgn}, \mathrm{sgn})$, and from Remark 2.5.5 it follows that π_σ corresponds to a cuspidal Maass eigenform of eigenvalue $1/4$ for Δ.

(2) In [DS], Deligne and Serre associated to each normalized new form $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ in $S_1(\Gamma_0(N), \psi)$ an irreducible two-dimensional representation σ of $G_{\mathbb{Q}}$, of conductor N and (odd) determinant ψ, such that

$$\mathrm{trace}\ \sigma(\mathrm{Fr}_p) = a_p$$

for almost all primes p. Taken together with the Langlands reciprocity conjecture for $F = \mathbb{Q}$ and "odd" σ (or equivalently, Artin's conjecture for such σ), their result says that *new forms of weight one are one and the same thing as irreducible, odd two-dimensional representations of $G_{\mathbb{Q}}$ (satisfying Artin's conjecture...).*

(3) One expects an analogue of Deligne-Serre to hold for cuspidal Maass-eigenforms (of eigenvalue $1/4$), i.e., that to each such form there should correspond an irreducible two-dimensional, even representation of $G_{\mathbb{Q}}$ (satisfying Artin's conjecture), with $L(\sigma, s) = L(f, s)$. But this remains an open problem; cf. 4.3 below.

4.3. Representations of W_F vs. "Arithmetic" Automorphic Representations of GL(2)

For further reference, it will be convenient to repeat in a more precise form the classification of two-dimensional "semisimple" representations

$$\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

over a number field F.

Proposition. *Each σ as above is classified according to its image in* $\mathrm{PGL}_2(\mathbb{C})$, *called the "type" of the representation:*

(i) Cyclic type: $\mu \oplus \nu$: *σ is the direct sum of the two one-dimensional representations defined by Hecke characters μ and ν.*

(ii) Dihedral type: *σ is irreducible of the form* $\mathrm{Ind}_{W_E}^{W_F} \theta$, *with θ a character of $E^\times \setminus \mathbb{A}_E^\times$, E a quadratic extension of F, and $\theta \neq \theta^\tau$ for $\tau \neq 1$ in* $\mathrm{Gal}(E/F)$. *(Such representations are also called monomial.)*

(iii) Exceptional type: *The image of σ in* $\mathrm{PGL}_2(\mathbb{C})$ *is A_4, S_4 or A_5.*

Now let's assume that the LRC holds for all irreducible

$$\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C}),$$

and ask which automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ are of the form $\pi(\sigma)$ for some σ? A *necessary* condition is clearly the following.

Definition. Given an irreducible admissible representation $\pi = \otimes\pi_v$, and a real place v, let $\sigma_v : W_{\mathbb{C}/\mathbb{R}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ denote the Langlands parameter of π_v. Then π is called of *type* A_{00} (resp. A_0) if the restriction of σ_v to $\mathbb{C}^\times$ is trivial (resp. the sum of characters of the form $z \longrightarrow z^a \bar{z}^b$ with $a, b, \in \mathbb{Z}$). Alternatively, π of type A_{00} (resp. A_0) is called *of Galois type* (resp. *arithmetic*).

Exmaple 4.3.1. If $\sigma : W_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ actually factors through $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, then the corresponding cuspidal $\pi(\sigma)$ (if it exists) will be of *type* A_{00} (cf. Proposition 4.2 and the Remarks immediately following it). Conjecturally, one expects that all cuspidal π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ of type A_0 are "motivic," i.e., arise in this way; in case $\det(\sigma)$ is odd, this is the result of Deligne-Serre.

On the other hand, cuspidal $\pi(\sigma)$ of *type* A_0 are related to ℓ-adic representations of $G_{\mathbb{Q}}$ or $W_{\mathbb{Q}}$ (or the L-series attached to ℓ-adic cohomology spaces of varieties over $\mathbb{Q}$). This is the subject matter of [Ant] (really a representation theoretic reformulation and strengthening of "Eichler-Shimura" theory). For example, if σ_∞ is induced from the character $z \longrightarrow z^{-n}\bar{z}^{-m}$ of $\mathbb{C}$, with $n > m \geq 0$, let D_k denote the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight $k = n - m + 1$ (and appropriate central character). Then Langlands in [La5] associates to $\pi = \otimes\pi_p$ of type A_0 (with $\pi_\infty = \pi(\sigma_\infty) = D_k$) a two-dimensional ℓ-adic representation σ of $G_{\bar{\mathbb{Q}}}$ whose local L and ε factors are (eventually) shown to agree with those of π_p *for all* p; cf. [Car].

§5. The Langlands Functoriality Principle Theory and Results

All the automorphic results used to prove the Reciprocity Conjecture in the two-dimensional solvable case, as well as the LRC itself, are but special realizations of what Langlands calls "functoriality of automorphic forms with respect to the L-group." Hence it seems worthwhile to review some of the necessary background on "functoriality" in this Section.

5.1. L groups and L-factors

Recall that for GL_2, an unramified representation $\pi_p = \pi_p(\mu_1, \mu_2)$ is parametrized by a semisimple conjugacy class in $\mathrm{GL}_2(\mathbb{C})$, namely the Langlands class

$$t_{\pi_p} = \begin{pmatrix} \mu_1(p) & 0 \\ 0 & \mu_2(p) \end{pmatrix}.$$

More generally, an arbitrary irreducible admissible representation π_p is parametrized by a *Langlands parameter*

$$\sigma_p : W_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\mathbb{C}),$$

and the "local Langlands Conjecture" says that the same should hold for GL_n, over any local field F_v. Namely, each nice representation π of $\mathrm{GL}_n(F_v)$ should be attached to a parameter $\sigma_v : W_{F_v} \longrightarrow \mathrm{GL}_n(\mathbb{C})$.

For an arbitrary reductive group G, over a local field F_v, Langlands introduced the notion of the *L-group* ${}^LG$ to take the place of $\mathrm{GL}_n(\mathbb{C})$ in parametrizing the irreducible admissible representations of $G(F_v)$. Roughly speaking, each nice representation of $G(F_v)$ should be attached to a "semi-simple" homomorphism

$$\varphi : W_{F_v} \longrightarrow {}^LG,$$

and in the case of unramified representations, this should amount to fixing a certain semisimple conjugacy class in ${}^LG$ (again called the *Langlands class* t_{π_v} attached to π_v).

In general, if G is defined over a local or global field F, then ${}^LG$ is a group of the form

$$\hat{G} \rtimes \mathrm{Gal}(\bar{F}/F).$$

Here $\hat{G}$ is the complex Lie group "dual" to the root datum of $G(\mathbb{C})$, and the action of $\mathrm{Gal}(\bar{F}/F)$ on $\hat{G}$ is trivial if and only if G is split over F; cf. §2 of [Bo] for details. It is sometimes convenient to replace $\mathrm{Gal}(\bar{F}/F)$ by $\mathrm{Gal}(E/F)$, where E is any Galois extension of F over which G splits. Indeed, since $\mathrm{Gal}(\bar{F}/E)$ acts trivially on $\hat{G}$, we can take ${}^LG$ to be

$$\hat{G} \rtimes \mathrm{Gal}(E/F)$$

(now a complex reductive Lie group). For example, for $G = \mathrm{GL}(n)$, and F local or global, we can take

$${}^LG = \mathrm{GL}_n(\mathbb{C}) \quad \text{or} \quad \mathrm{GL}_n(\mathbb{C}) \times \mathrm{Gal}(E/F)$$

for any E. It is also convenient to define a semi-direct product

$${}^LG = \hat{G} \rtimes \Sigma$$

for Σ any group endowed with a homomorphism into $\mathrm{Gal}(\bar{F}/F)$, for example, the Weil group W_F. Henceforth, we deal almost exclusively with this "*Weil*" form" of ${}^LG$.

For the moment, let us also assume that G is *unramified over* F, i.e, quasi-split over the local field F, and split over an unramified extension E. For such a group (like $\mathrm{GL}_n(F)$), an irreducible admissible representation is called *unramified* if its restriction to a very special maximal compact subgroup (like $\mathrm{GL}_n(O_F)$) contains the identity representation (and then just once). If F_r denotes a Frobenius generator for $\mathrm{Gal}(E/F)$, then the *unramified* representations π of $G(F)$ are in one-to-one correspondence with the semisimple ${}^LG^0$-conjugacy classes t_π in ${}^LG^0 \rtimes \mathrm{Fr} \subset {}^LG$. The resulting bijection

$$\pi \longleftrightarrow t_\pi$$

attaches to each unramified π its *Langlands class* $t_\pi = g \rtimes \mathrm{Fr}$ in ${}^LG$, and it is in terms of these classes, and their matrix realizations, that the general Langlands L-factors are defined.

Definition. By a *representation* of ${}^LG$ is meant a continuous homomorphism $r : {}^LG \longrightarrow \mathrm{GL}_m(\mathbb{C})$ whose restriction to $\hat{G}$ is a morphism of complex Lie groups. Given such an r, and an unramified representation π, one sets

$$L(s,\pi,r) = \det(1 - r(t_\pi)q^{-s})^{-1}, \tag{5.1.1}$$

where q is the order of the residue class field of F.

Example. If $F = \mathbb{Q}_p$, $\pi = \pi_p$ is the local component of a cuspidal representation of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ associated to a new form in $S_k(SL_2(\mathbb{Z}))$, and $r : \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is the "standard" representation taking g to g, then

$$\begin{aligned} L(s,\pi,r) &= [(1-\mu_1(p)p^{-s})(1-\mu_2(p)p^{-s})]^{-1} \\ &= (1 - a_p p^{-s-\frac{k-1}{2}} + p^{-2s})^{-1}, \end{aligned}$$

with $a_p = p^{(k-1)/2}(\mu_1(p)+\mu_2(p))$.

The question remains: what L-factors can be assigned to irreducible admissible π which are *not* unramified? In the case of GL_n (and some other classical groups as well now), the local representation theory of G may be used to directly define L (and ε-factors $L(s,\pi,r)$ (and $\varepsilon(s,\pi,r)$), at least for r sufficiently close to the standard embedding of ${}^LG$ in some $\mathrm{GL}_d(\mathbb{C})$. For example, for GL_n, and $r : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ the identity, [GoJa] constructs such L and ε factors, and in [Ja] they are related (modulo the local Langlands conjecture for $\mathrm{GL}(n)$) to the L and ε factors of their corresponding Langlands parameter.

In general, a typical *Langlands parameter φ for G* is a continuous homomorphism

$$\varphi : W_F \longrightarrow {}^LG \ (= \hat{G} \times W_F)$$

such that $\varphi(w)$ is "semisimple" for each w in W_F, and such that the composition with projection onto W_F induces the identity map. Then to each representation $r : {}^LG \longrightarrow \mathrm{GL}_d(\mathbb{C})$, and class of parameters φ (modulo conjugation by $\hat{G}$), one can attach the Langlands factor

$$L(s,\varphi,r) = \det(I - q^{-s}r(\varphi(\mathrm{Fr})|_{V^I}))^{-1}. \tag{5.1.2}$$

Here Fr is a Frobenius element in W_F, and V^I is the subspace of $\mathbb{C}^d$ invariant under the action of the inertia subgroup. Note that when φ is *unramified*, i.e., trivial on I, then φ is determined by the semisimple element $\varphi(\mathrm{Fr}) = g \rtimes \mathrm{Fr}$ in ${}^LG$ and $L(s,\varphi,r)$ *reduces to the Langlands L-function* $L(s,\pi,r)$, with π the unramified representation of $G(F)$ such that t_π is conjugate to $\varphi(\mathrm{Fr})$. In general, a form of the "local Langlands conjecture" *for G* asserts that any irreducible admissible π is associated to some parameter $\varphi : W_F \longrightarrow {}^LG$, and then $L(s,\pi,r)$ should be $L(s,\varphi,r)$, for any r.

N.B. If G is not quasi-split, its Langlands parameters must satisfy certain additional "rationality" conditions; cf. 8.2(ii) of [Bo].

5.2. Statement of the Functoriality Principle

A homomorphism between L-groups

$$\rho : {}^LG \longrightarrow {}^LG'$$

is called an $L-$*morphism* if it commutes with the natural projections onto W_F. Such a morphism clearly gives rise to a map

$$\varphi \longrightarrow \rho \circ \varphi$$

between the Langlands parameters of G and G', hence (conjecturally) also between representations of G and G' (by the local Langlands conjecture).

Now suppose F is global, and $\pi = \otimes \pi_v$ is an automorphic cuspidal representation of $G(\mathbb{A}_F)$. For almost every place v of F, G_v and π_v will be *unramified*, and the corresponding Langlands class t_π in ${}^LG_v$ defined. Thus the (partial Langlands) L-function $L^S(s,\pi,r)$ can be defined for any representation $r : {}^LG \to \mathrm{GL}_d(\mathbb{C})$ through the formula

$$L^S(s,\pi,r) = \prod_{\substack{v \\ \text{"unramified"}}} L(s,\pi_v,r_v).$$

(Here each r_v arises through composition of the natural embedding

$${}^LG_v = \hat{G} \rtimes W_{F_v} \hookrightarrow {}^LG = \hat{G} \rtimes W_F.)$$

Langlands has shown that this *Euler product* converges in some half-plane, and conjectured that it admits a meromorphic continuation to $\mathbb{C}$, with only finitely many poles in $\mathrm{Re}(s) \geq 0$.

N.B. (1) If one accepts the local Langlands Conjecture, one can also introduce $L(s,\pi_v,r_v)$ at the remaining places (as in (5.1.2)), and define the "completed" functions $L(s,\pi,r)$ (and $\varepsilon(s,\pi,r)$).

(2) If $G = \mathrm{GL}_n$, and $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ is the standard representation, then $L^S(s,\pi,r)$ is simply denoted $L^S(s,\pi)$.

The Functoriality Principle. *Suppose that G' is quasi-split, and that $\rho : {}^LG \to {}^LG'$ is a morphism of L-groups. For each v, consider the corresponding commutative diagram*

$$\begin{array}{ccc} {}^LG_v & \xrightarrow{\rho_v} & {}^LG'_v \\ \downarrow & & \downarrow \\ {}^LG & \xrightarrow{\rho} & {}^LG'. \end{array}$$

Then to each automorphic cuspidal representation $\pi = \otimes \pi_v$ of $G(\mathbb{A}_F)$ there corresponds an automorphic representation $\pi' = \otimes \pi'_v$ of $G'(\mathbb{A}_F)$ such that for almost all v (where both π_v and π'_v are unramified)

$$t_{\pi'_v} = \rho_v(t_{\pi_v}).$$

In particular, for any representation $r' : {}^L G' \to \mathrm{GL}_d(\mathbb{C})$,

$$L^S(s, \pi', r') = L^S(s, \pi, r' \circ \rho).$$

Moreover, if one accepts the local Langlands Conjecture for G, *then the Langlands parameter of* π'_v *should be the image (under* ρ_v*) of the Langlands parameter of* π_v *for* every v.

Example. Take $G = \{1\}$, and $G' = \mathrm{GL}(n)$. In this case, a morphism

$$\rho : {}^L G \longrightarrow {}^L G' = \mathrm{GL}(n, \mathbb{C}) \times W_F$$

must be of the form $\rho(1, w) = \sigma(w) \times w$, with a continuous representation

$$\sigma : W_F \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

(and conversely, any Artin representation $\sigma : W_F \longrightarrow \mathrm{GL}_n(\mathbb{C})$ determines a morphism ρ_σ through this formula). Since $G = \{1\}$, its only automorphic representation π is the trivial one, with Langlands class $1 \rtimes \mathrm{Fr}_v$ for every (finite) v. Thus the Functoriality Conjecture in this case asserts that (for any given σ) there is an automorphic representation $\pi(\sigma) = \otimes \pi_p$ of $\mathrm{GL}_n(\mathbb{A}_F)$ such that

$$t_{\pi_p} = \sigma(\mathrm{Fr}_r)$$

for almost every p.

This example shows that the general Reciprocity Conjecture is but a special instance of the Functoriality Principle. Hence it is clear that this Principle is more a guiding light than a problem to be solved in the near future!

5.3. Established Examples of Functoriality

We collect here some instances of "functoriality" which are required for the proof of Langlands-Tunnell.

(A) Automorphic Induction

This is a generalization of the classical construction of Hecke and Maass, whereby an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{Q})$ (a modular form, or Maass form, in their language) is attached to each Hecke character of a *quadratic* extension of $\mathbb{Q}$ (which is purely imaginary or real, respectively).

For a general formulation, fix a number field F, and K a *cyclic* Galois extension of F of degree n. Let G denote the group $\mathrm{Res}_{K/F}\,\mathrm{GL}_1$ (defined by "restriction of scalars" from K to F) and let G' denote the group GL_n. Then G is isomorphic to a maximal F-torus of G', and

$${}^L G = (\mathrm{GL}_1(\mathbb{C}) \times \cdots \times \mathrm{GL}_1(\mathbb{C})) \rtimes W_F,$$

with W_F acting on $\hat{G}$ through its projection onto $\mathrm{Gal}(K/F)$, and (the generator of) $\mathrm{Gal}(K/F)$ acting through cyclic permutation of the n $\mathrm{GL}_1(\mathbb{C})$-factors of $\hat{G}$. Let ρ_1 be the natural homomorphism of ${}^LG$ into the normalizer of a maximal torus of $\hat{G}' = \mathrm{GL}(n, \mathbb{C})$, and define a morphism of L-groups

$$\rho : {}^LG \longrightarrow {}^LG'$$

by $\rho(g) = (\rho_1(g), \rho_2(g))$, where $\rho_2 : {}^LG \to W_F$ is the canonical projection. Note that an automorphic form on G is the same thing as a grossencharacter χ of K, since $G(F) = K^\times$; and when v splits (completely) in K, the representation $\pi_v = \pi_v(\chi)$ (induced from the χ_v's above v) satisfies $t_{\pi_v} = \rho(t_{\chi_v})$ if χ is unramified at v. Thus the principle of functoriality *suggests* the following:

Theorem 5.3.1. *For each grossencharacter χ of K there is an automorphic representation $\pi(\chi)$ of $\mathrm{GL}_n(\mathbb{A}_F)$ whose L-function $L^S(s,\pi)$ equals the Hecke-L-function $L^S(s,\chi)$; moreover, $L^S(s,\pi(\chi))$ is entire (and hence $\pi(\chi)$ is* cuspidal *automorphic) if χ does not factor through the norm map $N_{K/F}$ (equivalently χ is not fixed by the natural action of the Galois group* $\mathrm{Gal}(K/F)$*).*

For $n = 2$ and $F = \mathbb{Q}$, this Theorem follows essentially from the classical work of Hecke and Maass. For $n = 2$ and F arbitrary, it is proved in [JL] (using L-functions), [LL] (using the "stable trace formula"), and [ST] (using theta-series); it also follows from Jacquet's "relative trace formula" (cf. §VIII 4 of [Ge2]). For $n = 3$ it is proved in [J-PS-S2] (using L-functions) and for arbitrary n in [AC] (using the trace formula). The only cases needed in the sequel are $n = 2$ or 3, and here it is simplest to (follow [JL] and [J-PS-S2] and) appeal to the so-called "*Converse Theorem to Hecke Theory.*"

For this, suppose that $\pi = \otimes\pi_v$ is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ whose central character is invariant under $F^\times$. If π is actually automorphic, then it is known from "Hecke theory" (cf. [GoJa]) that *π is "nice" relative to any idele class character ω of F*, i.e., $L(s, \pi\otimes\omega)$ and $L(s, \tilde{\pi}\otimes\omega^{-1})$ are absolutely convergent in some half-place, admit analytic continuations to the whole s-plane which are bounded in vertical strips, and have a functorial equation relating s to $1-s$; moreover, if π is cuspidal, then these analytic continuations are also *entire*. For $n = 2$ or 3, the *Converse Theorem* (cf. [JL] and [J-PS-S2]) simply says that the converse to each of these statements is also true.

Remarks 5.3.1. (a) In real life situations, like the application to proving $\pi(\chi)$ automorphic in Theorem 5.3.1, the situation is complicated by the fact that the representation we are trying to prove automorphic may not be easily defined at *every* place, but rather only at almost all places; thus, in fact, a more complicated "almost everywhere" version of the converse theorem is needed; cf. §§13–14 of [J-PS-S2].

(b) In the paper [Co-PS], Cogdell and Piatetski-Shapiro conjecture that the *Converse Theorem* should also hold for any n, with the additional

caveat that for $n \geq 4$, π need only be almost everywhere equivalent to some automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$; cf. [He].

(c) If χ (in Theorem 5.3.1) is not fixed by any non-trivial element of $\mathrm{Gal}(K/F)$, then $\mathrm{Ind}_{W_K}^{W_F} \chi = \sigma$ is an *irreducible* n-dimensional representation of W_F with $L(s,\sigma) = L(s,\chi)$ (a Hecke L-series with grossencharacter χ *over* K). Hence Theorem 5.3.1 may be viewed as an affirmation of *the Langlands Reciprocity Conjecture for monomial representations.*

(d) Note that "on the Galois side," *induction* brings a Langlands parameter for GL_1 (over K), namely $\chi : W_K \longrightarrow \mathbb{C}^\times$, to a Langlands parameter for GL_2 over F, namely $\sigma = \mathrm{Ind}\,\chi : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$. "On the automorphic side," this map is reflected by the correspondence $\chi \longrightarrow \pi(\chi) = \pi(\sigma)$ (hence the aptness of the terminology "automorphic induction").

(e) Finally, we note that in case $n = 2$ or 3, the "converse theorem approach" to Theorem 5.3.1 does *not* depend on K being a normal (Galois) extension of F. This will be crucial in the application to the Reciprocity Conjecture in the Octahedral case; cf. §7.2.

(B) **The Symmetric Square Lifting**

Let A denote the three dimensional representation of $\mathrm{PGL}_2(\mathbb{C})$ determined by the adjoint action of $\mathrm{PGL}_2(\mathbb{C})$ on the Lie algebra of $SL(2,\mathbb{C})$, and denote the resulting (three-dimensional) representation

$$\begin{tikzcd} \mathrm{GL}_2(\mathbb{C}) \arrow[rr, "\mathrm{Ad}"] \arrow[dr] & & \mathrm{GL}_3(\mathbb{C}) \\ & \mathrm{PGL}_2(\mathbb{C}) \arrow[ur, "A"] & \end{tikzcd}$$

of $\mathrm{GL}(2,\mathbb{C})$ by Ad. This representation Ad may be viewed as a natural morphism between the L-groups of $\mathrm{GL}(2)$ and $\mathrm{GL}(3)$.

Theorem 5.3.2. (The "Symmetric Square" Lift from $\mathrm{GL}(2)$ to $\mathrm{GL}(3)$; cf. [GeJa]).

(i) *To each cuspidal automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A}_F)$, *there exists an automorphic representation* Π *of* $\mathrm{GL}_3(\mathbb{A}_F)$ *such that for almost all* v,

$$\Pi_v = \Pi_v(\mathrm{Ad}(\sigma_v))$$

whenever $\pi_v = \pi_v(\sigma_v)$; *equivalently,*

$$t_{\Pi_v} = \mathrm{Ad}(t_{\pi_v}).$$

(ii) *This lift of* π *to* $\mathrm{GL}(3)$ *is* cuspidal *automorphic unless* π *is monomial, i.e., of the form* $\pi(\sigma)$, *with* σ *induced from a Hecke character of some* quadratic *extension* K.

Method of Proof The "converse theorem for GL(3)" says that $L^S(s,\pi,\mathrm{Ad})$ will be the L-function of an automorphic representation Π of $\mathrm{GL}(3)$ (with

$t_{\Pi_v} = \mathrm{Ad}(t_{\pi_v})\cdots)$ as soon as $L^S(s,\pi,\mathrm{Ad})$ is shown to have the expected analytic properties; moreover, this Π will be cuspidal if and only if all $L^S(s,\Pi\otimes\omega)$'s are *entire*. To establish the required analytic properties, it is shown (following [Sh]) that

$$L^S(s,\pi,\mathrm{Ad}) = A_S(s) \int\limits_{SL_2(F)\diagdown SL_2(\mathbb{A})} \varphi_\pi(g)\Theta(g)E(g,s)dg.$$

Hence φ_π belongs to the space of π, $\Theta(g)$ is a theta-function on Weil's metaplectic group, $E(g,s)$ is an Eisenstein series of half-integral weight which is real analytic in g and meromorphic in s, and $A_S(s)$ is a meromorphic function which at the possible poles of $E(g,s)$ can be chosen non-zero.

N.B. The idea of using the integral of an automorphic form to derive analytic properties of its L-function of course goes back to Hecke, and even Riemann. But the idea of mixing automorphic forms in the integral with *Eisenstein series* was first systematically developed by Rankin and Selberg, and is now a flourishing industry; cf. below.

(C) **Rankin-Selberg Products (Especially** $\mathrm{GL}(3)\times\mathrm{GL}(3)$**)**

Underlying this work is the following instance of Langlands functoriality. Viewing $\mathrm{GL}(k,\mathbb{C})$ as the L-group of $\mathrm{GL}(k)$, and $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(m,\mathbb{C})$ as that of $\mathrm{GL}(n)\times\mathrm{GL}(m)$, consider the natural L-group morphism

$$\rho : \mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_m(\mathbb{C}) \xrightarrow{\otimes} \mathrm{GL}_{nm}(\mathbb{C})$$

given by the tensor product map.

So far, there seems no hope of establishing Langlands functoriality in this case, i.e., of proving the existence of an automorphic Π on GL_{nm} such that $t_{\Pi_v} = t_{\pi_v}\otimes t_{\pi'_v}$ for two given cuspidal representations π' on GL_n and π on GL_n. Indeed, this is an important open problem, whose solution would play a crucial role in finding "the" group whose irreducible representations are expected to parametrize *all* the automorphic cuspidal representations of GL_n (not just those "arithmetic" ones coming from representations of W_F); cf. [Ram] for further discussion along these lines. A big first step, however, was taken by Jacquet and Shalika:

Theorem 5.3.3. (cf. [JaSh1,2] and [Mo-Wald]) *Given cuspidal representations π on GL_n and π' on GL_m, let $L^S(s,\pi\times\pi')$ denote the partial L-function*

$$\prod_{v\notin S}[\det(I-(t_{\pi_v}\otimes t_{\pi'_v})q^{-s}]^{-1}.$$

(i) *$L^S(s,\pi\times\pi')$, originally defined only in some right half-plane, extends to a meromorphic function in all of $\mathbb{C}$, with functional equation relating the value at s to the value at $1-s$.*

(ii) *$L^S(s,\pi\times\pi')$ may be "completed" to an Euler product*

$$L(s,\pi\times\pi') = \prod_{\text{all } v} L(s,\pi_v\times\pi'_v)$$

which is holomorphic on $\mathrm{Re}(s)\geq 1$ *if $m\neq n$, and otherwise has a pole at s with* $\mathrm{Re}(s)=1$ *if and only if* $|\det(\ )|^{s-1}\otimes\pi\cong\tilde{\pi}$ *(the* contragredient *of π').*

As already suggested, the proof of this result constitutes a non-trivial representation-theoretic generalization of the classical integral representations of Rankin and Selberg; see [Ja] for the case of GL(2) × GL(2). In the sequel, we need only the case $n=m=3$.

Concluding Remarks. (1) There is one more example of functoriality needed for the proof of Langlands-Tunnell, namely the theory of *base-change* of Saito, Shintani and Langlands. However, since that theory is so intimately tied up with Artin's conjecture, and its proof relies on the trace formula rather than L-functions, it seems convenient to postpone discussion of it until the last lecture.

(2) There are of course large aspects of the Langlands Program which we have not seriously broached here because they have no *immediate* bearing on Wiles' work. Perhaps the most obvious such topic is the (conjectured) relation between Hasse-Weil zeta-functions of algebraic varieties ("motive" L-functions) and automorphic L-functions of type $L(s,\pi,r)$. For example, in [La6] the zeta-functions of certain Shimura varieties are related to automorphic L-functions of degree 2^n. This "program" represents the beginnings of a higher dimensional analogue of the theory of Eichler-Shimura and has greatly influenced much of the work during the last twenty years in representation theory and the theory of automorphic forms. Among other things, it pushed to the forefront the need to refine and generalize the "Selberg trace formula"; more about this in the next lecture. It also brought into representation theory such crucial but different concepts as "L-indistinguishability," "endoscopy," "L-packets," etc., and encouraged the use of new algebro-geometric methods for counting points on these varieties.

(3) Finally, one should say a *few* words about the relation between the Langlands Program and the Shimura-Taniyama-Weil Conjecture. Personally, I do not think that it is so significant that the Langlands Program actually includes the S-T-W conjecture as a special "example" (and that's why I haven't bothered to broach the topic here). After all, Taniyama obviously made his Conjecture — and Shimura and Weil understood its importance — *before* the Langlands Program was conceived. Also, from the other point of view, it is equally clear that including the S-T-W Conjecture inside the Langlands Program is more incidental than crucial to the Program. Rather the crux of the Program is two pronged: its overall *vision* relating motives of all kinds to automorphic representations, and its

methods which push representation theory to the forefront, and infuse the subject with a seemingly endless string of challenging problems. It is these aspects of the Langlands Program which (albeit indirectly) play a role in the proof of Fermat's Last Theorem.

Lecture III
Proof of the Langlands-Tunnell Theorem

Abstract
Our task is to describe the proof of the following:

Theorem. *Suppose F is a number field and the irreducible representation*

$$\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

has a solvable image in $\mathrm{PGL}_2(\mathbb{C})$. *Then there exists a (unique) irreducible automorphic cuspidal representation* $\pi(\sigma) = \otimes \pi_v$ *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *such that*

$$\mathrm{trace}(\sigma(\mathrm{Fr}_v)) = \mathrm{trace}(t_{\pi_v})$$

for almost every v.

The crucial instance of the Functoriality Conjecture required in the proof of this Theorem is the theory of "Base Change" as developed in [La1]. This we describe in §6, along with its proof, which relies heavily on *trace* formula methods. The application of base change to the Langlands Reciprocity theorem is explained in §7, the proof of the actual theorem proceeding in two steps: first the base change (trace formula) methods are exploited to produce the best possible candidate for $\pi(\sigma)$ (which is called $\pi_{ps}(\sigma)$); then the results from the theory of L-functions (recalled in §5) are used to prove that $\pi_{ps}(\sigma)$ actually equals $\pi(\sigma)$.

§6. Base Change Theory

(6.1). Fix E a cyclic extension of the number field F, of prime degree ℓ. Roughly speaking, the theory of "base change" describes the correspondence between automorphic representations of the groups $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_E)$ which reflects the operation of *restriction* of Galois representations of W_F to W_E. The first results on base change for automorphic forms (or representations) used the theory of *L-functions*, and were restricted to the case of *quadratic* E and GL_2. The introduction of the *trace formula* is due to H. Saito, who dealt with GL_2 and arbitrary cyclic E using the classical language of automorphic *forms*; cf. [Sai]. Immediately after that, Shintani reformulated Saito's results using group representations, and gave the correct *local* definition of base change lifting; cf. [Shin]. Finally, Langlands saw the connection with Artin's conjecture, and reshaped the trace formula proof for GL_2 in a form suitable for the later generalization to GL_n developed by Arthur and Clozel; see [La1] and [AC] for a more detailed history. Since only the case $n = 2$ is required here, we restrict ourselves henceforth to this case.

Definition. Suppose $\pi = \otimes \pi_v$ is an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$, and $\Pi = \otimes_w \Pi_w$ is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. Then Π is a *base change lift of* π, denoted $\mathrm{BC}_{E/F}(\pi)$, if for each place v of F, and $w|v$, the Langlands parameter attached to Π_w equals the restriction to W_{E_w} of the Langlands parameter $\sigma_v : W_{F_v} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ of π_v.

Remarks. (i) The above (essentially local) definition of base-change lifting is at the level of Langlands parameters rather than representations. The key idea of [Shin] is to define the lift of π_v on $\mathrm{GL}_2(F_v)$ directly in terms of a character identity between π_v and the extension of Π_w to the group $\mathrm{GL}_2(E_w) \rtimes \mathrm{Gal}(E_w/F_v)$. Implicit here is the fact that Π_w is $\mathrm{Gal}(E_w/F_v)$ invariant and hence this extension, call it $\tilde{\Pi}_w$, exists. If τ is a generator of $\mathrm{Gal}(E_w/F_v)$ the character of this identity reads

$$\chi_{\tilde{\Pi}_w}(g \rtimes \tau) = \chi_{\pi_v}(x)$$

whenever $N_{E/F,\sigma}(g) = g^{\tau^{\ell-1}} \cdots g^{\tau} g$ is conjugate in $\mathrm{GL}_2(E_w)$ to a regular semisimple element x of $G(F_v)$.

(ii) *Functoriality.* Let $G = \mathrm{GL}_2$ and set $G' = \mathrm{Res}_{E/F}(G)$. As recalled in §5.3, $\hat{G}'$ is then a product of ℓ copies of $\mathrm{GL}_2(\mathbb{C})$ indexed and permuted by $\mathrm{Gal}(E/F)$. So let

$$\rho : {}^LG = \mathrm{GL}_2(\mathbb{C}) \times W_F \longrightarrow {}^LG' = \hat{G}' \rtimes W_F$$

be the natural morphism which takes $g \times w$ in ${}^LG$ to $(g, \cdots, g) \rtimes w$ in ${}^LG'$. The Functoriality Principle suggests the existence of a map taking automorphic cuspidal representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ to automorphic representations Π of $\mathrm{GL}_2(\mathbb{A}_E) \approx G'(\mathbb{A}_F)$ such that for π_v and Π_w unramified,

$$(*) \qquad t_{\Pi_w} = \rho_v(t_{\pi_v}).$$

Using either definition of lifting given above, it is easy to check that if $\pi_v = \pi(\mu_1, \mu_2)$ (with μ_i an unramified character of F_v) then

$$\mathrm{BC}_{E/F}(\pi_v) = \Pi(\nu_1, \nu_2) \quad \text{with } \nu_i = \mu_i \circ N_{E_w/F_v}.$$

From this it follows that $(*)$ holds, i.e., Base Change is functorial.

N.B.. In verifying that $(*)$ holds, one must keep in mind that $\pi(\nu_1, \nu_2)$ (viewed as a representation of $\mathrm{GL}_2(E_w)$) corresponds first to the Langlands class $g \times \sigma$ in $\mathrm{GL}_2(\mathbb{C}) \times \sigma$, with

$$g = \begin{pmatrix} \mu_1 \circ N(\tilde{\omega}_w) & 0 \\ 0 & \mu_2 \circ N(\tilde{\omega}_w) \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1(\tilde{\omega}_v)^2 & 0 \\ 0 & \mu_2(\tilde{\omega}_v)^2 \end{pmatrix};$$

but the corresponding class in

$${}^LG' = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \rtimes \sigma$$

(viewing $\pi(\nu_1, \nu_2)$ as a representation of $G'(F_v) \approx \mathrm{GL}_2(E_w)$) is

$$\left(\begin{pmatrix} \mu_1(\tilde{\omega}_v) & 0 \\ 0 & \mu_2(\tilde{\omega}_v) \end{pmatrix}, \begin{pmatrix} \mu_1(\tilde{\omega}_v) & 0 \\ 0 & \mu_2(\tilde{\omega}_v) \end{pmatrix}, \sigma\right),$$

i.e., just $\rho(t_{\pi_v})$. Indeed, the Hecke algebras of $\mathrm{GL}_2(E_w)$ and $G'(F_v)$ are the same, and if f_w and f'_w represent the same element in this algebra $\mathcal{H}_w$, then $(f'_w)^v(g_1, \cdots, g_\ell \times \sigma) = f^v_w(g_e \cdots g_2 g_1)$; see 6.3 below for definitions of $\mathcal{H}_w$ and the Satake isomorphism $(f')^v$.

(iii) Because we are assuming E over F cyclic of *prime* degree, each v of F either remains *inert* or splits completely. In the later case, it is clear that $E_w \approx F_v$ for any $w|v$, and the base change lift of π_v is just $\Pi_w \approx \pi_v$. This case being trivial, we usually assume (as above) that we are dealing with the inert local case.

Theorem. (cf. [La1])
(a) *Every cuspidal representation* π *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *has a unique base change lift to* $\mathrm{GL}_2(\mathbb{A}_E)$; *the lift is itself* cuspidal *(as opposed to "just" automorphic) unless* E *is* quadratic *over* F, *and* π *is monomial (or dihedral) of the form* $\pi(\sigma)$, *with* $\sigma = \mathrm{Ind}_{W_E}^{W_F} \theta$.
(b) *If two cuspidal representations* π *and* π' *have the same base change lift to* E, *then* $\pi' \approx \pi \otimes \omega$ *for some character* ω *of* $F^\times N_{E/F}(\mathbb{A}_E^\times) \setminus \mathbb{A}_F^\times$.
(c) *A cuspidal representation* Π *of* $\mathrm{GL}_2(\mathbb{A}_E)$ *equals* $\mathrm{BC}_{E/F}(\pi)$ *for some cuspidal* π *on* $\mathrm{GL}_2(\mathbb{A}_F)$ *if and only if* Π *is invariant under the natural action of* $\mathrm{Gal}(E/F)$.

In some ways, the *proof* of Base Change is as interesting as the result itself. Since it involves a form of the trace formula which should (and does) generalize, and apply to other instances of functoriality, we devote some time to it below.

(6.2). The Trace Formula of Arthur-Selberg

Recall that the right regular representation R_0 of $G(\mathbb{A}_F)$ in the space of cusp forms $L_0^2(G(F) \setminus G(\mathbb{A}_F), \omega)$ decomposes discretely as

$$R_0 = \sum m_\pi \pi,$$

and it is the cuspidal constituents π which are the building blocks of the theory of automorphic forms on G. What the "trace" in "the trace formula" refers to is the distributional trace of R_0. More precisely, suppose $f(g)$ is any nice compactly supported "test function" on $G(\mathbb{A})$, and define the operator $R_0(f)$ on $L_0^2(G(F) \setminus G(\mathbb{A}), \omega)$ through the formula

$$R_0(f) = \int_{Z(\mathbb{A}) \setminus G(\mathbb{A})} f(g) R_0(g) dg.$$

(For simplicity, assume that the central character ω of R_0 is trivial.) Then clearly

$$\operatorname{trace} R_0(f) = \sum m_\pi \operatorname{trace}(\pi(f));$$

but as we know next to nothing about the π's which occur in R_0, we also know next to nothing about $\operatorname{trace}(R_0(f))$. The original idea of the trace formula was to give an *alternative* formula for trace $R_0(f)$, which ultimately gives some of the sought after information about R_0 and its constituents π.

The original trace formula was introduced by Selberg, in the context of a semisimple Lie group G and discrete subgroup Γ (in place of our $G(\mathbb{A})$ and $G(F)$). In his famous 1956 paper [Sel], Selberg first of all described a general formula for the case of *compact* $\Gamma \setminus G$ (equivalently $G(F) \setminus G(\mathbb{A})$); it took the form

$$\operatorname{trace} R_0(f) = \sum_{\pi} m_\pi \operatorname{trace} \pi(f) = \sum_{\{\gamma\}} m_\gamma \Phi_f(\gamma) \tag{6.2.1}$$

with $\{\gamma\}$ running over the conjugacy classes in $G(F)$, and each $\Phi_f(\gamma)$ an "orbital integral"

$$\int_{G_\gamma(\mathbb{A}) \setminus G(\mathbb{A})} f(g^{-1}\gamma g)\, dg.$$

Secondly, Selberg treated in detail certain *non*-compact quotient cases such as $SL_2(\mathbb{Z}) \setminus SL_2(\mathbb{R})$, which already required the analytic continuation of Eisenstein series to handle the *continuous* spectrum of L^2 outside L_0^2.

Subsequently, in the 1960's and 70's, Langlands developed a general theory of Eisenstein series valid for any reductive group G, and Arthur used it to develop a general trace formula in the context of not necessarily compact quotients $G(F) \setminus G(\mathbb{A})$. The resulting *trace formula of Arthur* takes the form

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f) = \sum_{\chi} J_\chi(f). \tag{6.2.2}$$

Here the left (or *geometric*) side of Arthur's formula is a sum of special types of equivalence classes in $G(F)$ (generalizing the ordinary notion of equivalence in the case of compact quotient), and the sum on the right (or *spectral*) side is over certain classes of automorphic cuspidal representations of "Levi subgroups" of G (as opposed to just the cuspidal representations of $G(\mathbb{A})$ itself in the case of compact quotient). Although it *looks* like the "trace" has been lost in Arthur's trace formula, this is not really so; certain of the spectral terms $J_\chi(f)$ add up to exactly trace $R_0(f)$, and so making (6.2.2) explicit still (ultimately) gives us the information we seek about $\operatorname{trace}(R_0(f))$.

Now instead of focusing efforts on finding an explicit, new formula for $\operatorname{trace}(R_0(f))$, it was the idea of Langlands to *compare* the trace formulas for *two different* groups G and G', in order to find *relations* (presumably "functorial") between their automorphic representations. For example, this is exactly the strategy exploited in §16 of [JL] to establish the correspondence between automorphic cuspidal representations of $G = D^\times$ and $G' = \mathrm{GL}_2$, D a division quaternion algebra over F. In the case of $G = \mathrm{GL}_2$ and $G' = \operatorname{Res}_{E/F} G$ this strategy brings us back to the proof of Theorem 6.1 which we now explain.

(6.3) The Proof of Base Change.
If τ denotes a generator of $\mathrm{Gal}(E/F)$, then τ acts naturally on $G'(\mathbb{A}) \approx \mathrm{GL}_2(\mathbb{A}_E)$, and hence on $L^2(G'(F)\backslash G'(\mathbb{A}_F))$ through the rule

$$(\tau \cdot \varphi)(g) = \varphi(g^\tau).$$

We can also define a *twisted* regular representation R^τ through the composition of R with τ. Then for any nice f' on $G'(\mathbb{A}_F)$, there is a "twisted" version of the trace formula of the form

$$\sum_{\mathfrak{o}'} J^\tau_{\mathfrak{o}'}(f') = \sum_{\chi'} J^\tau_{\chi'}(f'), \tag{6.3.1}$$

with the "twisted" $\operatorname{trace}(R_0^\tau(f'))$ hidden inside the right side of (6.3.1), and "twisted" orbital integrals $\Phi^\tau_{f'}(\gamma') = \int f'(g^{-\tau}\gamma g)dg$ on the left. The significance of working with a *twisted* formula for G' is that then only Galois invariant cuspidal representations Π will contribute (to $\operatorname{trace}(R_0^\tau(f'))$)†; hence we might indeed establish the desired base change map $\pi \longrightarrow \Pi$ between G and G' by relating (6.3.1) to (6.2.2), and ultimately $\operatorname{trace}(R_0(f))$ to $\operatorname{trace}(R_0^\tau(f'))$.

The first step is to prove that the left-hand (i.e., *geometric*) sides of the trace formulas for G and G' coincide, at least for certain "matching" f and f'. This matching is a non-trivial *local* step, which first of all requires that the orbital integrals $\Phi_{f_v}(N\gamma')$ on G_v match the *twisted* orbital integrals $\Phi^\tau_{f'_w}(\gamma')$ on G'_w. Moreover, it must be shown that this matching $f' \longrightarrow f$ is compatible with "base change at the unramified places," in the following sense:

If $\mathcal{H}_v$ denotes the Hecke algebra of bi-K_v-invariant (compactly supported smooth) functions on G_v, each f_v in $\mathcal{H}_v$ may be viewed as a function on ${}^LG_v$ through the formula

$$f_v^V(t) = \operatorname{trace} \pi_v(f_v)$$

whenever $t_{\pi_v} = t$. (This is the Satake *isomorphism* $f_v \longrightarrow f_v^V$, defined analogously for the Hecke algebra $\mathcal{H}'_w$ of G'_w.) Then the base change map

† This is because τ permutes the constituents Π of R_0, and a permutation matrix without fixed points has zero trace...

of Hecke algebras, dual to the base change morphism $\rho : {}^LG_v \longrightarrow {}^LG'_w$, is defined by

$$\rho^V : (f'_w)^V \longrightarrow f'_v(g) = (f'_w)^V(\rho(g)),$$

and the compatibility condition mentioned above is that f'_w will match its image $\rho^V(f'_w) = f'_v$ in the above sense, for any f'_w in $\mathcal{H}'_w$. (This is what is known as the *fundamental lemma*, in the context of "base change.")

The next step, which takes a great deal more work on the *spectral* sides of (6.2.2) and (6.3.1), is to conclude from the equality of the geometric trace formulas that

$$\operatorname{trace} R_0(f) \qquad \text{essentially equals} \qquad \operatorname{trace} R_0^\tau(f')$$

for such matching f and f'. Equivalently,

$$\sum_{\substack{\pi \\ \text{cuspidal}}} \operatorname{trace} \pi(f) = \sum_{\substack{\pi' \text{ cuspidal} \\ (\pi')^\tau \approx \pi}} \operatorname{trace}(\tau \circ \pi')(f') \tag{6.3.2}$$

with the sum on the right only over Galois fixed π'.

Now for π_v, π'_w, f_v, f'_w all "unramified," we will have π'_w equal to the base change lift of π_v if and only if

$$\operatorname{trace}(\tau \circ \pi'_w)(f'_w) = \operatorname{trace} \pi_v(f_v)$$

for any f_v the base change image of f'_w as above. (Indeed, in terms of Satake transforms, this last identity reads $(f'_w)^V(t_{\Pi_w}) = f_v^V(t_{\pi_v}) = (f'_w)^V(\rho(t_{\pi_v}))$, i.e., $t_{\Pi_w} = \rho(t_{\pi_v})$, as required.)

In this way, with a "linear independence of characters" argument, (6.3.2) ultimately implies that for a given π occurring on the left-hand side, there must be a π' on the right-hand side which is Galois invariant, and almost everywhere the base change lift of π (and, conversely, all such Galois invariant π' thus arise). Thus the required correspondence is established.

Remark. Whenever the trace formula can be used to establish an instance of functoriality (like base change above), it offers the additional bonus of *characterizing the image* of the automorphic representations in question. This is *not* so for the method of L-functions (witness the example of the lifting from GL(2) to GL(3), where the image is left uncharacterized, or Proposition 7.2 below giving base change for non-Galois cubic E).

§7. Application to Artin's Conjecture

The idea of applying "base change" to attack Artin's Conjecture arises from the following observation. Suppose that for any $\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$ there really is a corresponding cuspidal representation $\pi(\sigma)$ of $\mathrm{GL}_2(\mathbb{A}_F)$. Then it follows from the original definition of base change lifting that

$$\mathrm{BC}_{E/F}(\pi(\sigma)) = \pi(\operatorname{Res} \sigma|_{W_E})$$

for any cyclic extension E of F. This means that if we *start* with σ, and want to find *candidates* for $\pi(\sigma)$, then the thing to do is to pick an E such that $\pi(\operatorname{Res}\sigma|_{W_E})$ is already known to exist, and look among the cuspidal π's such that $\mathrm{BC}_{E/F}(\pi) = \pi(\operatorname{Res}\sigma|_{W_E})$. In this way the following "obvious" strategy unfolds: Among the possible candidates for $\pi(\sigma)$, pick a "best possible" one, call it $\pi_{\mathrm{ps}}(\sigma)$ (for $\pi_{\mathrm{pseudo}}(\sigma)$), and then prove that $\pi_{\mathrm{ps}}(\sigma)$ must equal $\pi(\sigma)$. Roughly speaking, the first step uses the trace formula (via base change), while the second uses L-functions.
Convention. Henceforth, if we are given $\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$, and any field E over F, then by σ_E we denote the restriction of σ to W_E.

(7.1). The Tetrahedral Case

(a) *Choosing* $\pi_{\mathrm{ps}}(\sigma)$

We are given an irreducible representation

$$\sigma : W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

whose image in $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to A_4. This group is solvable, with composition series

$$A_4 \rhd D_2 \rhd \{e\}.$$

(In general, D_n will denote the dihedral group of $2n$ elements; in this case, D_2 is the Klein 4-group). Since $A_4/D_2 \cong A_3 \cong \mathbb{Z}_3$, the inverse image of D_2 in W_F under the map

$$W_F \longrightarrow A_4 \subset \mathrm{PGL}_2(\mathbb{C})$$

is a (normal) subgroup of index 3, hence the Weil group of a *cubic* extension of F, call it E. Pictorially:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & W_E & \longrightarrow & W_F & \longrightarrow & \mathrm{Gal}(E/F) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
1 & \longrightarrow & D_2 & \longrightarrow & A_4 & \longrightarrow & \mathbb{Z}_3 & \longrightarrow & 1
\end{array}$$

Thus the resulting representation $\sigma_E : W_E \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is "monomial" in the sense of Proposition 4.3.

Let $\pi(\sigma_E)$ denote the automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ attached to this monomial representation by Theorem 5.3.1. This representation of $\mathrm{GL}_2(\mathbb{A}_E)$ is clearly invariant under the action of $\mathrm{Gal}(E/F)$; indeed, $\pi(\sigma_E)^\tau = \pi(\sigma_E^\tau) = \pi(\sigma_E)$. So by (the Base Change) Theorem 6.1, $\pi(\sigma_E)$ will be the base change lift of exactly *three* classes of irreducible cuspidal representations π_i of $\mathrm{GL}_2(\mathbb{A}_F)$, each one related to the other by a twist $\omega \circ \det$ for some character ω of $F^\times N_{E/F}(\mathbb{A}_E^\times) \setminus \mathbb{A}_F^\times$, i.e.,

$$\pi_i = \pi_j \otimes \omega \circ \det .$$

These π_i's are our natural candidates for $\pi(\sigma)$.

Recall that the *central character* of $\pi(\sigma)$ is to be $\det \sigma$. On the other hand, the central character ω_i of each π_i above "base change lifts" to the central character of $\pi(\sigma_E)$, which is $\det \sigma_E = (\det \sigma) \circ N_{E/F}$. Since each $\omega_i = \omega_j \omega^2$ if $\pi_i = \pi_j \otimes \omega \circ \det$, it is clear that *exactly one of these* π_i*'s has central character* $\det \sigma$, and this is the one we choose to be $\pi_{\mathrm{ps}}(\sigma)$.

(b) *Proving* $\pi_{\mathrm{ps}}(\sigma) = \pi(\sigma)$

Write $\pi_{\mathrm{ps}}(\sigma) = \otimes \pi_v$. Then for each v, $\pi_v = \pi_v(\sigma'_v)$ for some

$$\sigma'_v : W_{F_v} \longrightarrow \mathrm{GL}_2(\mathbb{C}),$$

and what we must prove is that

$$\sigma'_v = \sigma_v \tag{7.1.1}$$

for almost every v.

Note that the *restriction* of σ'_v to W_{E_w} (for $w|v$) is by construction the same as the *restriction* of σ_v to W_{E_w}. Thus there is nothing to prove in case v splits (completely) in E, and we henceforth assume E_w *cubic and unramified* over F_v.

If Fr_v denotes a Frobenius element of $\mathrm{Gal}(E_w/F_v)$ we can suppose

$$\sigma_v(\mathrm{Fr}_v) = \begin{pmatrix} a_v & 0 \\ 0 & v_v \end{pmatrix} \qquad \text{and} \qquad \sigma'_v(\mathrm{Fr}_v) = \begin{pmatrix} c_v & 0 \\ 0 & d_v \end{pmatrix}$$

for some a_v, b_v, c_v, d_v in $\mathbb{C}^\times$. Then to prove (7.1.1) it will suffice to prove that $\begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix}$ is conjugate to $\begin{pmatrix} c_v & 0 \\ 0 & d_v \end{pmatrix}$. But the fact that σ_v and σ'_v have the same restrictions to W_{E_w} means that $\sigma_v(\mathrm{Fr}_v)^3$ is conjugate to $\sigma'_v(\mathrm{Fr}_v)^3$ (since Fr_v^3 belongs to W_{E_w}). Thus

$$\begin{pmatrix} a_v^3 & 0 \\ 0 & b_v^3 \end{pmatrix} \qquad \text{is conjugate to} \qquad \begin{pmatrix} c_v^3 & 0 \\ 0 & d_v^3 \end{pmatrix}.$$

In particular, for some pair of cube roots of 1, say ξ and ξ', either

$$c_v = \xi a_v \quad \text{and} \quad d_v = \xi' b_v,$$

$$\text{or else}$$

$$c_v = \xi b_v \quad \text{and} \quad d_v = \xi' a_v.$$

We claim now that $\xi' = \xi^2$. Indeed $\pi_{\mathrm{ps}}(\sigma)$ was chosen so that

$$\omega_{\pi_{\mathrm{ps}}(\sigma)} = \det(\sigma).$$

Since this implies $\det \sigma'_v = \det \sigma_v$, we must have $\xi\xi' = 1$, i.e., $\xi' = \xi^2$. So to prove (7.1.1) it will suffice to prove

$$\xi = 1. \tag{7.1.2}$$

To continue, let us assume (for the moment) that

$$\mathrm{Ad} \circ \sigma'_v = \mathrm{Ad} \circ \sigma_v. \tag{7.1.3}$$

Since the kernel of $\mathrm{Ad} : \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_3(\mathbb{C})$ is precisely the group of scalar matrices $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\}$, it follows from (7.1.3) that $\sigma_v(\mathrm{Fr}_v)$ and $\sigma'_v(\mathrm{Fr}_v)$ must differ by some scalar $\lambda \neq 0$. Thus

$$\begin{pmatrix} \xi a_v & 0 \\ 0 & \xi^2 b_v \end{pmatrix} \quad \text{is conjugate to} \quad \begin{pmatrix} \lambda a_v & 0 \\ 0 & \lambda b_v \end{pmatrix},$$

and it suffices to prove

$$\lambda = 1.$$

If $\lambda a_v = \xi a_v$ and $\lambda b_v = \xi^2 b_v$ then $\lambda = \xi = \xi^2 = 1$ for the trivial reason that ξ is a cube root of 1. On the other hand, if $\lambda a_v = \xi^2 b_v$ and $\lambda b_v = \xi a_v$, then $\lambda^2 = 1$ (since $a_v = \xi^2/\lambda b_v = (\lambda/\xi)\, b_v$). If $\lambda = -1$, this means that the image of

$$\sigma_v(\mathrm{Fr}_v) = \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} = \begin{pmatrix} a_v & 0 \\ 0 & a_v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \xi\lambda \end{pmatrix}$$

in $\mathrm{PGL}_2(\mathbb{C})$ is *of order 6* (since $\xi\lambda$ will then have order 6). But as A_4 has no elements of order 6, this means we are done.

It remains to prove (7.1.3). For this, we note (following Serre) that $\mathrm{Ad} \circ \sigma : W_F \longrightarrow \mathrm{GL}_3(\mathbb{C})$ is a *monomial* representation. In particular, there is a character θ of W_E (not invariant by $\mathrm{Gal}(E/F)$) such that

$$\mathrm{Ad} \circ \sigma = \mathrm{Ind}_{W_E}^{W_F} \theta.$$

This means (again by Theorem 5.3.1, this time with $n = 3$) that there is associated to this irreducible representation $\mathrm{Ad} \circ \sigma$ a *cuspidal* automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$, call it Π_1. On the other hand, by the "symmetric square lift" (Theorem 5.3.2) $\pi_{\mathrm{ps}}(\sigma)$ has a lift to $\mathrm{GL}_3(\mathbb{A}_F)$, call it Π_1^*, which is almost everywhere associated to the Langlands parameter $\mathrm{Ad} \circ \sigma'_v$. Thus to prove (7.1.3), it clearly suffices to prove that

$$\Pi_1 \approx \Pi_1^*. \tag{7.1.4}$$

N.B. The automorphic representation Π_1^* will be *cuspidal* automorphic (by Theorem 5.3.2) if and only if $\pi_{\mathrm{ps}}(\sigma)$ is not monomial. But if $\pi_{\mathrm{ps}}(\sigma)$ were equal to $\pi(\sigma')$ for any irreducible two dimensional (let alone monomial) representation of W_F, we would have to conclude that $\sigma' = \sigma$ (which is impossible, since σ is tetrahedral, not monomial). Therefore Π_1^* is also cuspidal, and the proof of 7.1.4 reduces to the following:

Lemma. *The Rankin-Selberg L-function* $L(s, \Pi_1^* \times \tilde{\Pi}_1)$ *on* $\mathrm{GL}(3) \times \mathrm{GL}(3)$ *has a pole at* $s = 1$ *(and so, by Theorem 5.3.3,* Π_1^* *is indeed isomorphic to* Π_1*).*

Proof By definition

$$L(s, \Pi_1^* \times \tilde{\Pi}_1) = \prod_v L(s, (\Pi_1^*)_v \times (\tilde{\Pi}_1)_v),$$

where for almost every v (namely the "unramified" v),

$$L(s, (\Pi_1^*)_v \times (\tilde{\Pi}_1)_v) = L(s, (\mathrm{Ad} \circ \sigma_v') \otimes (\mathrm{Ad} \circ \tilde{\sigma}_v)).$$

Keeping in mind that $\mathrm{Ad} \circ \sigma$ is monomial, it is possible to check that we also have

$$L(s, (\Pi_1^*)_v \times (\tilde{\Pi}_1)_v) = L(s, (\Pi_1)_v \times (\tilde{\Pi}_1)_v) \tag{7.1.5}$$

(again for almost every v). Indeed, since $\mathrm{Ad} \circ \sigma$ is induced from θ on E, we have

$$\mathrm{Ad} \circ (\tilde{\sigma}_v) = \bigoplus_{w|v} \mathrm{Ind}_{W_{E_w}}^{W_{F_v}} \theta_w^{-1}.$$

Hence

$$\mathrm{Ad}(\sigma_v') \otimes \mathrm{Ad}(\tilde{\sigma}_v) = \bigoplus_{w|v} \mathrm{Ind}_{W_{E_w}}^{W_{F_v}} (\theta_w^{-1} \otimes \Sigma_w')$$

if Σ_w (resp. Σ_w') denotes the restriction of $\mathrm{Ad}(\sigma_v)$ (resp. $\mathrm{Ad}(\sigma_v')$) to W_{E_w}. (Here we are using the fact that for σ (resp. Σ) a representation of some group G (resp. a subgroup H),

$$\sigma \otimes \mathrm{Ind}_H^G \Sigma \cong \mathrm{Ind}_H^G (\Sigma \otimes \mathrm{Res}\ \sigma|_H).)$$

Similarly we have

$$\mathrm{Ad}(\sigma_v) \otimes \mathrm{Ad}(\tilde{\sigma}_v) = \bigoplus_{w|v} \mathrm{Ind}_{W_{E_w}}^{W_{F_v}} (\theta_w^{-1} \otimes \Sigma_w).$$

So since $\Sigma_w \cong \Sigma_w'$ almost everywhere (by construction), we indeed have

$$\begin{aligned} L(s, (\Pi_1^*)_v \times (\tilde{\Pi}_1)_v) &= L(s, \mathrm{Ad}(\sigma_v') \otimes \mathrm{Ad}(\tilde{\sigma}_v)) \\ &= L(s, \mathrm{Ad}(\sigma_v) \otimes \mathrm{Ad}(\tilde{\sigma}_v)) \\ &= L(s, (\Pi_1)_v \times (\tilde{\Pi}_1)_v) \end{aligned}$$

for almost every v.

Using (7.1.5), it remains to show that $\Pi_1^* = \Pi_1$. So suppose (7.1.5) holds for all v outside the finite set S. Then

$$L(s,\Pi_1^* \times \tilde{\Pi}_1) = \left(\prod_{v\in S} \frac{L(s,(\Pi_1^*)_v \times (\tilde{\Pi}_1)_v)}{L(s,(\Pi_1)_v \times (\tilde{\Pi}_1)_v)}\right) \cdot L(s,\Pi_1 \times \tilde{\Pi}_1).$$

But by Theorem 5.3.3, $L(s,\Pi_1 \times \tilde{\Pi}_1)$ has a pole at $s = 1$; moreover, the quotient expression in parentheses above is non-zero at $s = 1$. Therefore $L(s,\Pi_1^* \times \tilde{\Pi}_1)$ also has a pole at $s = 1$, as asserted, and this in turn implies (by the same Theorem 5.3.3) that $\Pi_1^* \cong \Pi_1$.

(7.2). The Octahedral Case

(a) *Choosing* $\pi_{\mathrm{ps}}(\sigma)$

In this case, the image of $\sigma(W_F)$ in $\mathrm{PGL}_2(\mathbb{C})$ is S_4, and the pull-back of the normal subgroup $A_4 \subset S_4$ is the Weil group W_E of a quadratic extension E of F.

Pictorially:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W_E & \longrightarrow & W_F & \longrightarrow & \mathrm{Gal}(E/F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow\wr & & \\ 1 & \longrightarrow & A_4 & \longrightarrow & S_4 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \end{array}$$

Since $\sigma_E = \mathrm{Res}\,\sigma\,|_{W_E}$ is now of *tetrahedral* type, we know $\pi(\sigma_E)$ exists as an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ (by the results of the last paragraph). Moreover, we again have $\pi(\sigma_E)$ invariant under the action of $\mathrm{Gal}(E/F)$. So again by Theorem 6.1, we conclude that $\pi(\sigma_E)$ must equal $\mathrm{BC}_{E/F}(\pi_i)$ for (this time) *two* irreducible cuspidal representations π_i of $\mathrm{GL}_2(\mathbb{A}_F)$. The problem now is that we can no longer distinguish these π_i's by their central characters. Indeed, π_1 now equals $\pi_2 \otimes \omega$ for a *quadratic* character of $F^\times \setminus \mathbb{A}^\times$; hence $\omega_{\pi_1} = \omega_{\pi_2}\omega^2 = \omega_{\pi_2}$!

Tunnell's contribution to the "Langlands-Tunnell Theorem" was to get around this problem by appealing to a new kind of base-change which appeared only after the publication of [La1], namely the following result:

Proposition. (cf. [J-PS-S3]) *If L is a cubic* not necessarily *Galois extension of F, then each automorphic cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ has a base change lift Π on $\mathrm{GL}_2(\mathbb{A}_L)$, i.e., $\Pi = \mathrm{BC}_{L/F}(\pi)$ is automorphic, and for almost every place v of F, and place w of L dividing v, $\pi_v = \pi_v(\sigma_v)$ implies $\Pi_w = \pi(Res_{L_w/F_v}(\sigma_v))$.*

The proof of [J-PS-S3] uses the theory of L-functions for the groups GL(3) and GL(2) $\times$ GL(3) (and is entirely analogous to Jacquet's original proof of base change for GL_2 over a *quadratic* extension in [Ja]). The idea is to introduce the representation Π on $\mathrm{GL}_2(\mathbb{A}_L)$ through the formula

$$L(s,\Pi \times \chi) = L(s,\pi \times \pi(\chi));$$

here χ is any Hecke character of L, $\pi(\chi)$ is the corresponding automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ (whose existence is assured by Theorem 5.3.1 in the *non-Galois case* — recall Remark 5.3.1 (e)), and $L(s, \pi \times \pi(\chi))$ is the Rankin-Selberg L-function on $\mathrm{GL}(2) \times \mathrm{GL}(3)$. Then one shows that $L(s, \Pi \times \chi)$ has the analytic properties required by the Converse Theorem to ensure that Π is automorphic. (The fact that each Π_w is the base change lift of π_v is relatively easy to check, from the definitions.)

N.B. The trace formula methods of [La1] fail in this context precisely because there may not be any Galois group attached to L over F (hence no way to define the twisted trace $R_0^\tau \cdots$). On the other hand, because L-function methods are used, there is no way to *characterize* the image of this base change map; fortunately, as we shall now see, there is also no need for this in the application Tunnell found for this result.

What Tunnell did in [Tu] is introduce L/F as the cubic (*non-normal*) subextension of K/F fixed by a 2-Sylow subgroup (of order 8) of S_4. (More precisely, L is the cubic subextension fixed by all elements of $\mathrm{Gal}(K/F)$ mapping to this chosen Sylow subgroup.) Then if M is the composition in K of L and E (the quadratic Galois extension chosen above), we have the diagram shown in Figure 1,

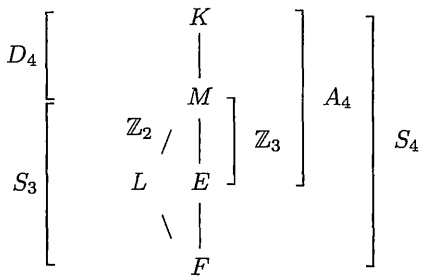

Figure 1

and the crucial:

Lemma. (cf. [Tu], page 174) *There is a unique $i = 1, 2$ such that*

$$\mathrm{BC}_{L/F}(\pi_i) = \pi(\sigma_L)$$

(and this is the π to be designated as $\pi_{\mathrm{ps}}(\sigma)$).

Proof Note first that $\pi(\sigma_L)$ actually exists, since the 2-Sylow subgroup used to define L is just D_4, and therefore σ_L is *monomial*; similarly, $\mathrm{BC}_{L/F}(\pi_i)$ exists for $i = 1, 2$ by the Base Change Theorem quoted above. To prove the Lemma, one appeals to the identity

$$\mathrm{BC}_{M/L}(\mathrm{BC}_{L/F}(\pi_i)) = \pi(\sigma_M) \quad \text{for } i = 1, 2.$$

(This is "transitivity of base change"; it follows immediately from the definition of base change.) Since $\mathrm{BC}_{L/F}(\pi_2)$ and $\mathrm{BC}_{L/F}(\pi_1)$ have the same (quadratic) base change to M, it follows that

$$\mathrm{BC}_{L/F}(\pi_2) \approx \mathrm{BC}_{L/F}(\pi_1) \otimes \omega_{M/L}.$$

Now we claim that the representations $\mathrm{BC}_{L/F}(\pi_i)$ are distinct for $i = 1, 2$. Indeed, if they were not, we would have

$$\mathrm{BC}_{L/F}(\pi_1) \approx \mathrm{BC}_{L/F}(\pi_1) \otimes \omega_{M/L},$$

which by Lemma 11.7 of [La1] implies π_1 is "monomial." By part (b) of Theorem 6.1, this would then imply $\mathrm{BC}_{M/L}(\mathrm{BC}_{L/F}(\pi_1)) = \pi(\sigma_M)$ is *not* cuspidal. But the image of σ_M in $\mathrm{PGL}_2(\mathbb{C})$ is $S_3 \approx D_3$, which means that σ_M itself is monomial and irreducible, i.e., $\pi(\sigma_M)$ *is* cuspidal. This contradiction establishes that $\mathrm{BC}_{L/F}(\pi_1)$ and $\mathrm{BC}_{L/F}(\pi_2)$ are *the* two (distinct) cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_L)$ yielding $\pi(\sigma_M)$ upon base change to M. Since we also have $B_{M/L}(\sigma_L) = \pi(\sigma_M)$, it must be that $\pi(\sigma_L) = \mathrm{BC}_{L/F}(\pi_i)$ for (exactly) one i, as required.

(b) *Proving* $\pi_{\mathrm{ps}}(\sigma) = \pi(\sigma)$.

Write $\pi_{\mathrm{ps}}(\sigma) = \otimes\pi_v(\sigma'_v)$ as before. Then one proves exactly as in the tetrahedral case (but *without* having to take a lift to $\mathrm{GL}_3(\mathbb{C})$) that the non-existence of an element of order 6 in S_4 implies $\sigma_v \cong \sigma'_v$ for almost all v. Since no new ideas are involved, we simply refer the reader to [Tu] for details.

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