

- At a few times so far we have come across complicated, self-similar objects called “*Fractals*”.
- The word was coined by Mandelbrot
- How do we define them more rigorously?
- Alligood, Sauer and York:
“*Scientists know a fractal when they see one but there is no universally accepted definition.*”

- Mandelbrot gave the following:

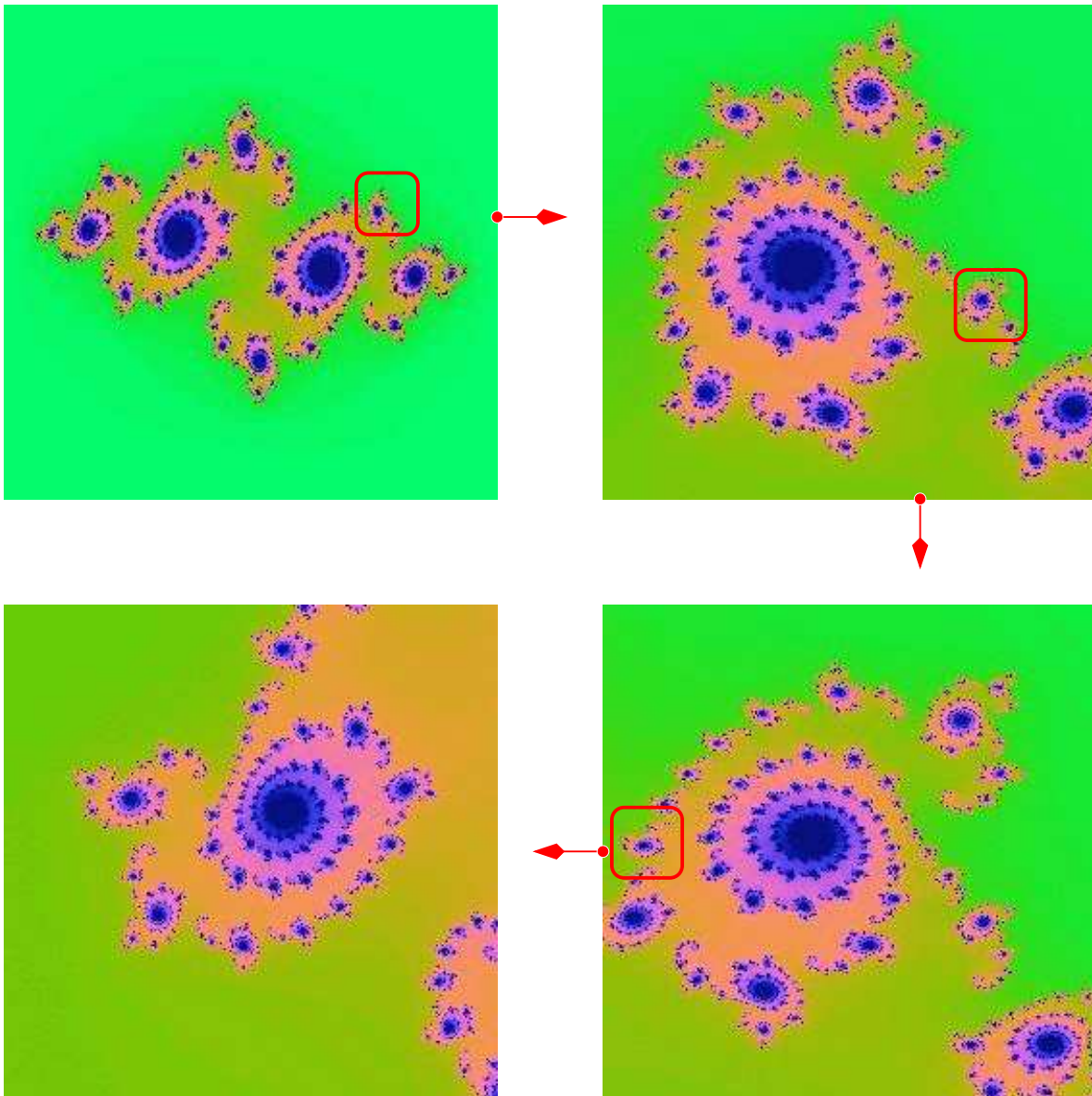
Definition (Mandelbrot — simplified a little). An object is a *fractal* if its fractal dimension exceeds its topological dimension.

- Falconer gives a more qualitative definition:

Definition (Falconer). An object is a *fractal* if

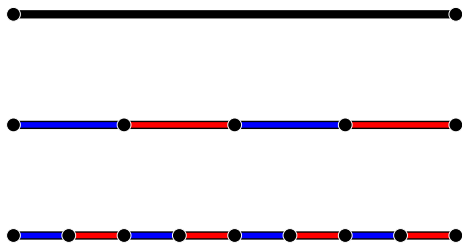
1. It has fine structure at all scales — no matter how much you zoom in, it never becomes smooth or simple.
2. It is too irregular to be described using traditional geometry — it is not a finite collection of smooth geometric objects,
3. It has some form of exact or approximate self-similarity — if we zoom in we see exact or approximate replicas of the whole object.
4. The fractal dimension is not the natural geometric dimension.

- Consider the Julia set of $c = -0.75 - 0.2i$:



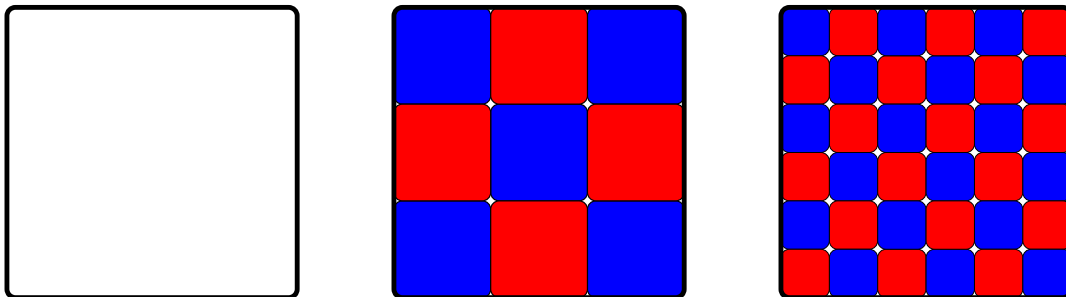
- It certainly satisfies the first 3 points of Falconer's definition:
 1. Structure at all scales
 2. Irregular
 3. Self-similar

- How do we define dimension?
- We are used to the idea that:
 - isolated points have dimension 0
 - curves have dimension 1
 - surfaces have dimension 2
- I don't want to do the more precise definition.
- We will define the fractal dimension as the *capacity dimension* (there are other ways to define it).
- Consider a straight line of length 1.
- We can cover it by smaller straight lines.



- We need $N(\varepsilon) = \varepsilon^{-1}$ lines of length ε to cover the line.
- The number of lines required *scales* with 1 over their size.

- Similarly for a unit square:



- We need $N_\varepsilon = \varepsilon^{-2}$ squares of size ε to cover it.
- The number of squares required *scales* with 1 over their size \times size.

Definition. The capacity (fractal) dimension, d , of a set A is defined as the exponent in the following relation:

$$N(\varepsilon) \approx A\varepsilon^{-d} \quad \text{as } \varepsilon \rightarrow \infty$$

or

$$\log N(\varepsilon) \approx \log A - d \log \varepsilon \quad \text{as } \varepsilon \rightarrow \infty$$

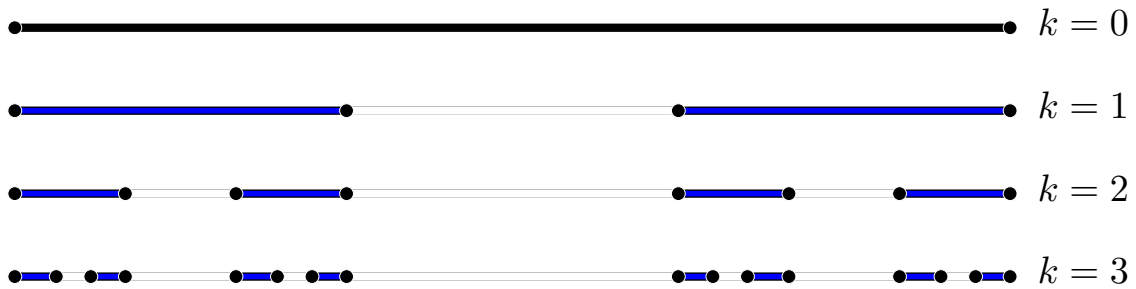
- The fractal dimension of usual sets / objects agrees with the geometric dimension.

Theorem. *The geometric dimension of the Cantor middle thirds set is 0, while its capacity (fractal) dimension is*

$$d = \frac{\log 2}{\log 3} \approx 0.630929754\dots$$

Proof:

- Since the Cantor set is made up of isolated points, and contains no intervals, it follows that its geometric dimension is 0.
- To find the capacity dimension we cover the middle thirds set in intervals of length $1/3^k$.
- We need 2^k such intervals.

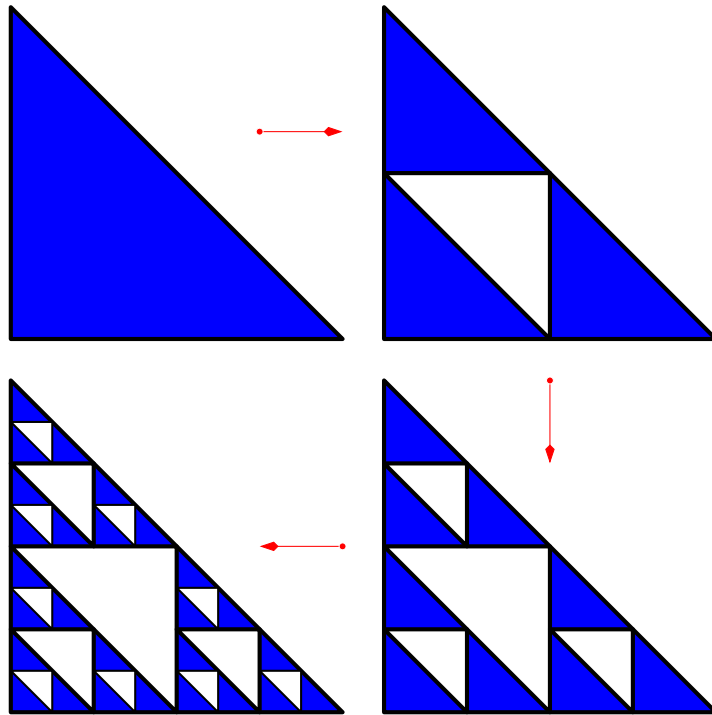


- *i.e.* dividing the length of the intervals by 3 means we need twice as many.
- Plugging this into the equation for the fractal dimension gives:

$$\begin{aligned}k \log 2 &= \log A + kd \log 3 \\d &= \frac{\log 2}{\log 3} - \frac{\log A}{k \log 3}\end{aligned}$$

- Taking the limit $k \rightarrow \infty$ gives the result.

- Another classic fractal object not unlike the Cantor middle thirds set is the Sierpinski triangle:



- At each iteration we split each triangle into 4 equal triangles and remove the central one.

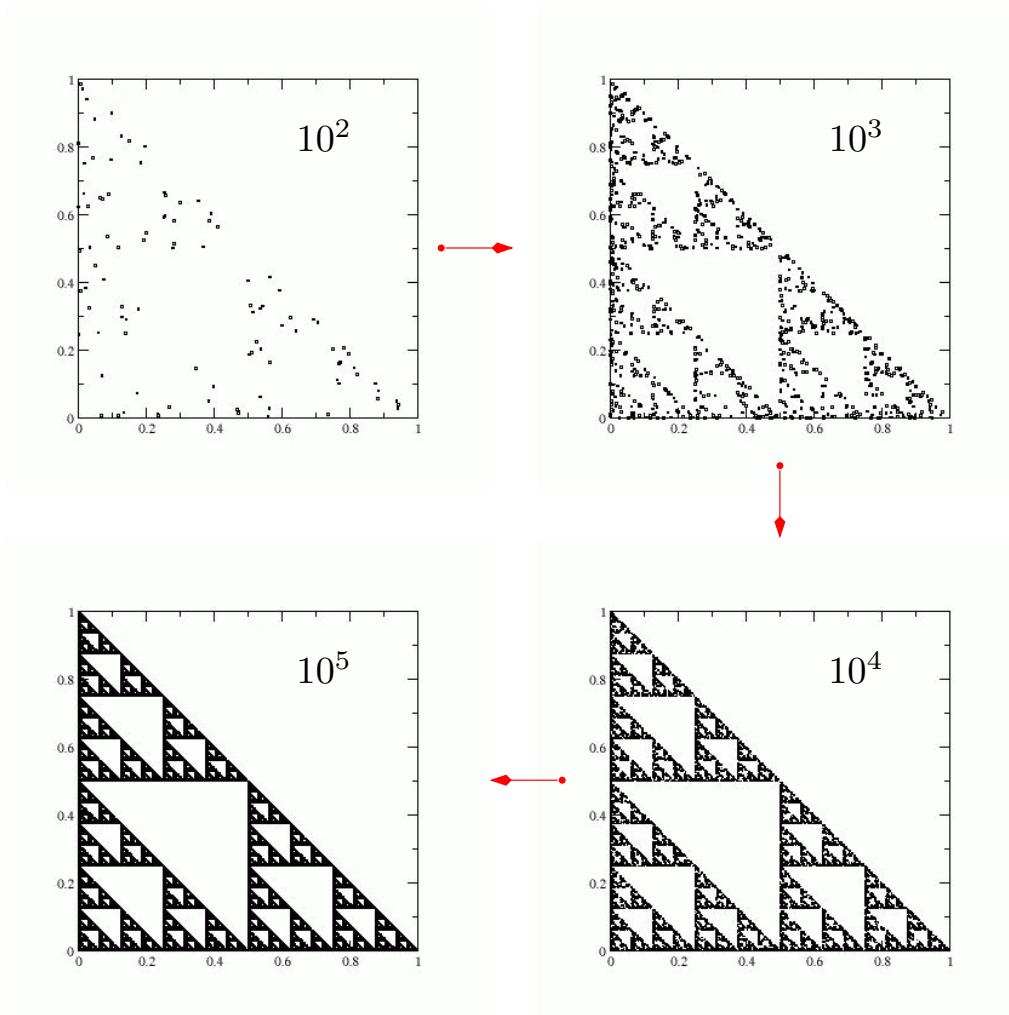
Theorem. *The Sierpinski triangle has geometric dimension 1 and fractal dimension*

$$d = \frac{\log 3}{\log 2} = 1.584962501 \dots$$

Proof:

- The Sierpinski triangle contains no area, but does contain lines and so has geometric dimension 1.
- Much like the previous proof — we cover the set in 3^k triangles of size 2^{-k} .
- *i.e.* if we halve the side-length of the triangles, we need 3 times as many.

- There are other ways of generating fractals.
- One important method is that of “iterated function systems” — much of the work in this area as done by Michael Barnsley.
- Consider a small set of maps that take the whole plane \mathbb{R}^2 to a smaller subset of itself.
- If you *randomly* apply these maps to a given point, it can be shown that the orbit converges to a unique subset of the plane.
- Such a subset is called an *attractor*.
- The Sierpinski triangle is an example of such a subset.
- The attractors are often fractals.
- This technique was used as the basis of an image compression technique — made Barnsley a very rich mathematician.



- The above image results from plotting the orbit of a point under random iteration of the following functions:

$$\mathbf{x}' = \frac{1}{2}(\mathbf{x} - \mathbf{p}_i) + \mathbf{p}_i \quad i = 1, 2, 3$$

where $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (1, 0)$ and $\mathbf{p}_3 = (1, 1)$.

- *i.e.* randomly pick one of the corners of the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ and move the current point halfway to it.