- Enough of 1-dimensional systems
- Let's have a look at higher dimensions.
- Our starting point will be discrete dynamical systems, with dynamics governed by:

  $$\mathbf{x}(k+1) = \mathbf{F}(\mathbf{x}(k))$$

  where $\mathbf{x}(k)$ is an n-dimensional vector of variables.

Definition. Our basic definitions carry over:

- A point $\mathbf{x}_*$ is a fixed point of $\mathbf{F}$ if

  $$\mathbf{F}(\mathbf{x}_*) = \mathbf{x}_*$$

- The orbit of a point $\mathbf{x}_0$ is the set:

  $$\{\mathbf{x}_0, \mathbf{x}_1 = \mathbf{F}(\mathbf{x}_0), \mathbf{x}_2 = \mathbf{F}(\mathbf{x}_1), \ldots \mathbf{x}_n = \mathbf{F}(\mathbf{x}_{n-1}), \ldots\}$$

- A point $\mathbf{x}_0$ is periodic with period n if $\mathbf{x}_n = \mathbf{F}^n(\mathbf{x}_0) = \mathbf{x}_0$.
- And similarly for eventually fixed periodic and eventually periodic.

- The stability of fixed and periodic points is a little more complicated in higher dimensions and is governed by the Jacobian of $\mathbf{F}$.

Definition. For a vector function $\mathbf{F}$ given by

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \mathbf{F}(x_1, x_2, \ldots x_n)\\ &= \big(F_1(x_1,\ldots,x_n), F_2(x_1,\ldots,x_n),\ldots,F_n(x_1,\ldots,x_n)\big)\end{aligned}$$

the *Jacobian matrix* is

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

i.e. the x_i change from left to right and the f_i change from top to bottom.

Proposition. *Just as we can approximate a function $f(x)$ close to a point x_* by the linear function:*

$$f(x_* + h) \approx f(x_*) + f'(x_*)h$$

we can approximate a vector function $\mathbf{F}(\mathbf{x})$ close to a point $\mathbf{x}_*$ by the function:

$$\mathbf{F}(\mathbf{x}_* + \mathbf{h}) \approx \mathbf{F}(\mathbf{x}_*) + D\mathbf{F}(\mathbf{x}_*)\,\mathbf{h}$$

Theorem. *Consider a fixed point $\mathbf{x}_*$ of a vector function $\mathbf{F}$ with Jacobian $D\mathbf{F}(\mathbf{x}_*)$.*

- *If the magnitudes of all the eigenvalues of $D\mathbf{F}(\mathbf{x}_*)$ are less than 1, then the fixed point is attracting and the orbits of nearby points converge to it.*
- *If the magnitudes are all greater than 1, then the fixed point is repelling and nearby points are pushed away.*
- *We will do the mixed case later.*

Idea of a proof:

- A point $\mathbf{x_0} = \mathbf{x}_* + \mathbf{h}(0)$ has image:

$$\begin{aligned} \mathbf{x_1} &= \mathbf{F}(\mathbf{x}_* + \mathbf{h}(0)) \\ &\approx \mathbf{F}(\mathbf{x}_*) + D\mathbf{F}(\mathbf{x}_*)\,\mathbf{h}(0) \\ &= \mathbf{x}_* + \mathbf{h}(1) \end{aligned}$$

- So the "distance", $\mathbf{h}(n)$ between the iterate $\mathbf{x}(n)$ and the fixed point is approximately governed by:

$$\mathbf{h}(n+1) = D\mathbf{F}(\mathbf{x}_*)\,\mathbf{h}(n)$$

  Which implies

$$\mathbf{h}(n) = (D\mathbf{F}(\mathbf{x}_*))^n\,\mathbf{h}(0)$$

- Suppose the Jacobian has eigenvalues α_k with corresponding eigenvectors $\mathbf{v}_k$.
- Write $\mathbf{h}(0)$ in terms of the eigenvectors:

$$\mathbf{h}(0) = \sum_{k=1}^{n} c_k \mathbf{v}_k$$

- We can then write

$$D\mathbf{F}(\mathbf{x}_*)\ \mathbf{h}(0) = \sum_{k=1}^{n} c_k D\mathbf{F}(\mathbf{x}_*)\ \mathbf{v}_k = \sum_{k=1}^{n} c_k \alpha_k \mathbf{v}_k$$

  and so

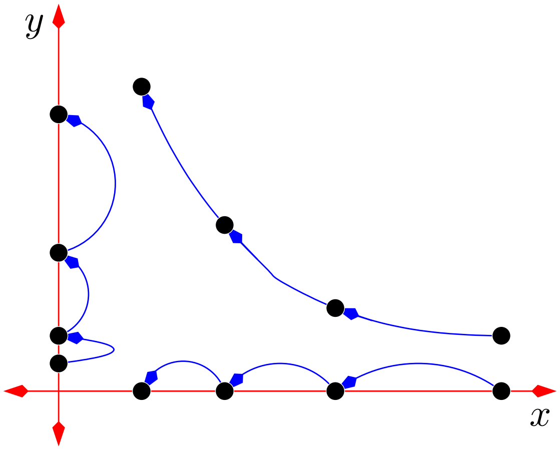
$$\mathbf{h}(n) = (D\mathbf{F}(\mathbf{x}_*))^n\ \ \mathbf{h}(0) = \sum_{k=1}^{n} c_k \alpha_k^n \mathbf{v}_k$$

- So if $|\alpha_k| < 1$ for $k = 1, \ldots, n$ then the distance $\mathbf{h}(n)$ will shrink to zero.
- While if $|\alpha_k| > 1$ for $k = 1, \ldots, n$ then the distance will diverge to ∞.
- If some of the $|\alpha_k| < 1$ and others $|\alpha_k| > 1$ then we find that under certain conditions the distance will converge to zero, and other other conditions it will diverge to ∞.

- An example of the mixed case:

Example. Consider the system defined by

$$(x_{n+1}, y_{n+1}) = (\frac{1}{2}x_n, 3y_n)$$



Solving this gives:

$$(x_n, y_n) = (2^{-n}x_0, 3^n y_0)$$

The Jacobian is:

$$D\mathbf{F}(\mathbf{0}) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 3 \end{pmatrix}$$

and its eigenvalues and eigenvectors are:

$$\begin{aligned} \alpha_1 &= 1/2 & \mathbf{v}_1 &= (1,0) \\ \alpha_2 &= 3 & \mathbf{v}_2 &= (0,1) \end{aligned}$$

- Points along the x-axis converge to $\mathbf{0}$ — this is the *stable-manifold.*
- Points along the y-axis diverge away from $\mathbf{0}$ — this is the *unstable-manifold.*
- The orbits of other points exhibit a mixture of convergent and divergent behaviours.

- More generally:

Definition. For a given fixed point with Jacobian eigenvalues $\{\alpha_k\}$ and eigenvectors $\{\mathbf{v}_k\}$

- The *stable manifold* is the space spanned by:

  $$\{\text{all eigenvectors } \mathbf{v_k} \mid |\alpha_k| < 1\}$$

  All orbits points in this space converge to the fixed point.

- The *unstable manifold* is the space spanned by:

  $$\{\text{all eigenvectors } \mathbf{v_k} \mid |\alpha_k| > 1\}$$

  All orbits in this space move away from the fixed point.

- We also use the following definition:

Definition. A dynamical system $\mathbf{F}(\mathbf{x})$ is

- *conservative* if $|\det D\mathbf{F}(\mathbf{x})| = 1$
- *expansive* if $|\det D\mathbf{F}(\mathbf{x})| > 1$
- *contracting* or *dissipative* if $|\det D\mathbf{F}(\mathbf{x})| < 1$

for all $\mathbf{x}$ in the state space.

- In the previous example $|\det D\mathbf{F}(\mathbf{x})| = 3/2$ for all $\mathbf{x}$ — so it is expansive.

- Consider the system:

$$\mathbf{x}_{n+1} = \begin{pmatrix} 3/2 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{x}_n$$

- It has determinant -1 so it is a conservative system.
- The eigenvalues are the solutions of:

$$\begin{aligned} 0 = \det\left(\begin{pmatrix} 3/2 & -1 \\ -1 & 0 \end{pmatrix} - I\lambda\right) &= \lambda^2 - \frac{3}{2}\lambda - 1 \\ &= (\lambda - 2)(\lambda + 1) \end{aligned}$$

  So they are 2 and $-1/2$.

- The eigenvector corresponding to $\lambda = 2$ is:

$$\mathbf{0} = \begin{pmatrix} -1/2 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

  which gives $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

- Similarly the eigenvector corresponding to $\lambda = -1/2$ is:

$$\mathbf{0} = \begin{pmatrix} 2 & -1 \\ -1 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

  which gives $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

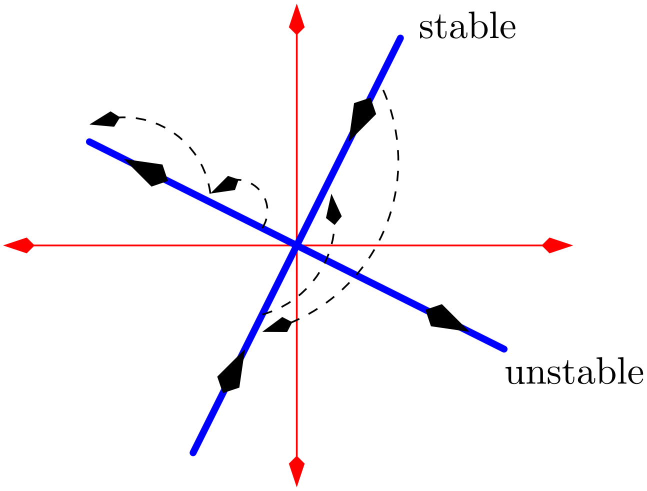
- Hence the stable manifold is:

$$\{(x, y) \mid y = 2x\}$$

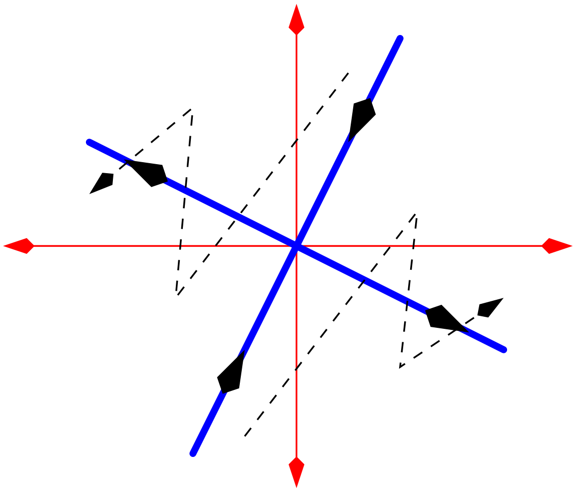
- And the unstable manifold is:

$$\{(x, y) \mid y = -x/2\}$$

- So we have a phase portrait of the form:



- Since $\lambda < 0$, the orbits on the stable manifold alternate on either side of $\mathbf{0}$.
- Orbits zig-zag around the unstable manifold:



- A simple generalisation of the quadratic map in 1 dimension is the Hénon map:

Definition. The two-dimensional Hénon map is given by the equations:

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n \\ y_{n+1} &= bx_n \end{aligned}$$

- This can be rewritten as a single "second order" equation:

$$x_{n+1} = 1 - ax_n^2 + bx_{n-1}$$

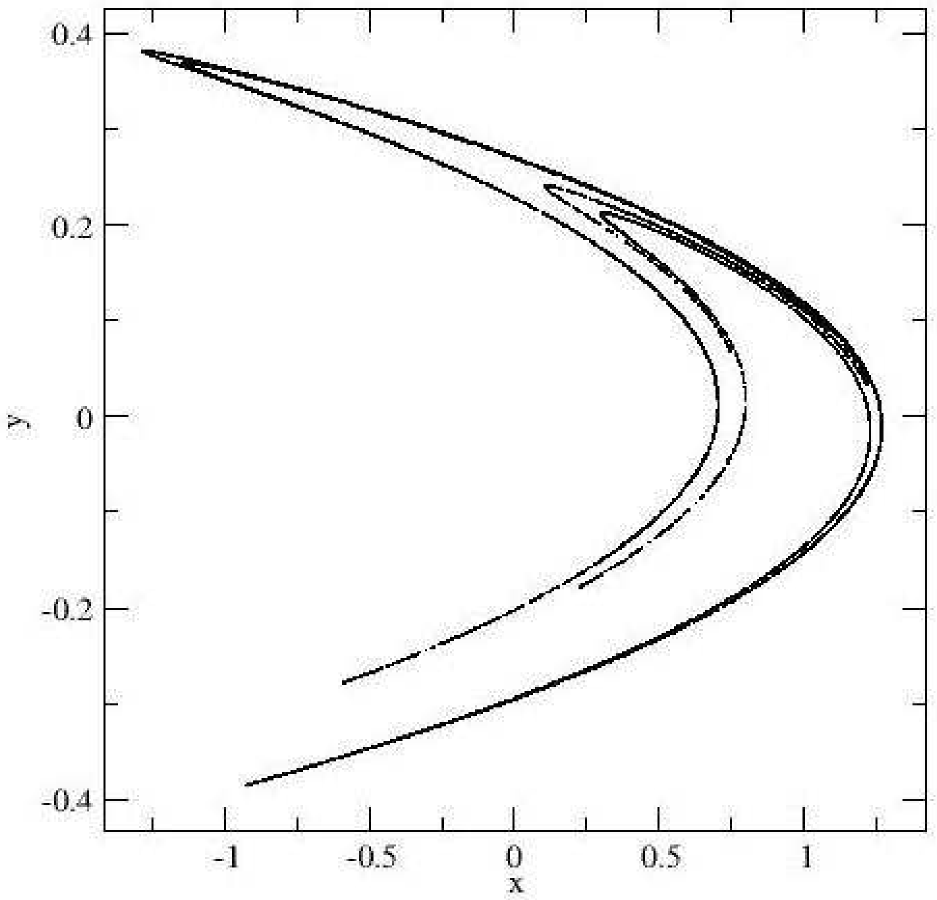
- The y-term introduces some time-feedback.
- The Jacobian of this system is

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

  which has determinant $= b$ — independent of x and y.

- The parameter b determines whether or not the system is dissipative, conservative or expansive.
- If we write the eigenvalues as $\lambda_\pm$, then we must have $\lambda_+\lambda_- = b$.

- In all the 1-d maps we studied, if an orbit converged, then it converged to either a fixed point or a cycle.
- Even this simple generalisation of the quadratic map displays new behaviour.

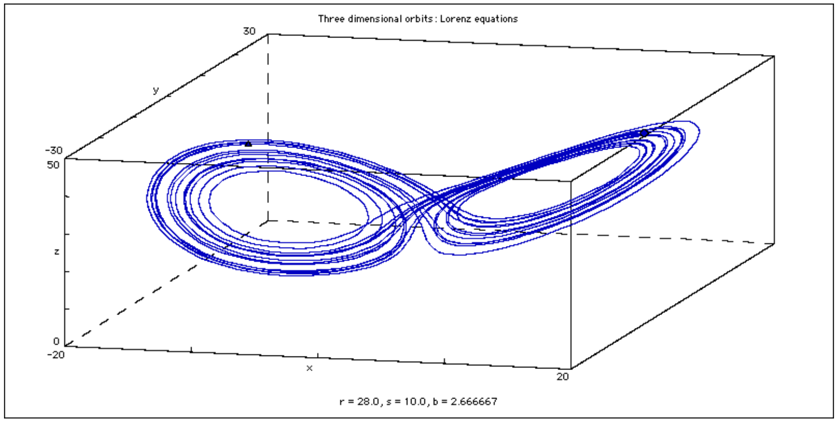


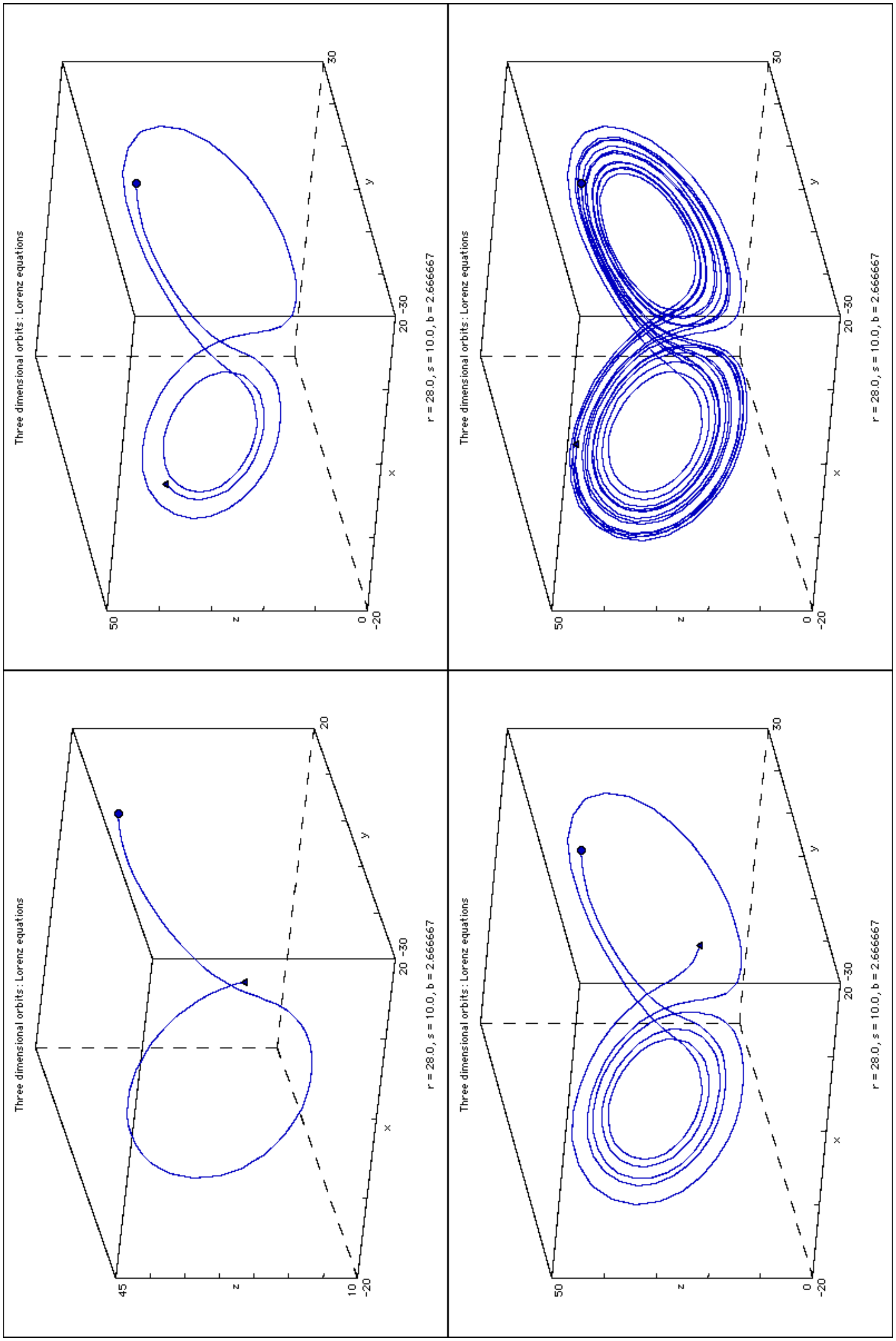
- This is the orbit of a point when $a = 1.4$ and $b = 0.3$.
- See “Iterations-2d” applet.
- It turns out to be a fractal.
- This is an example of a *strange attractor*.

- Perhaps the most famous higher dimensional system is given by Lorenz equations.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y-x)\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= rx-y=xz\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy-bz\end{aligned}$$

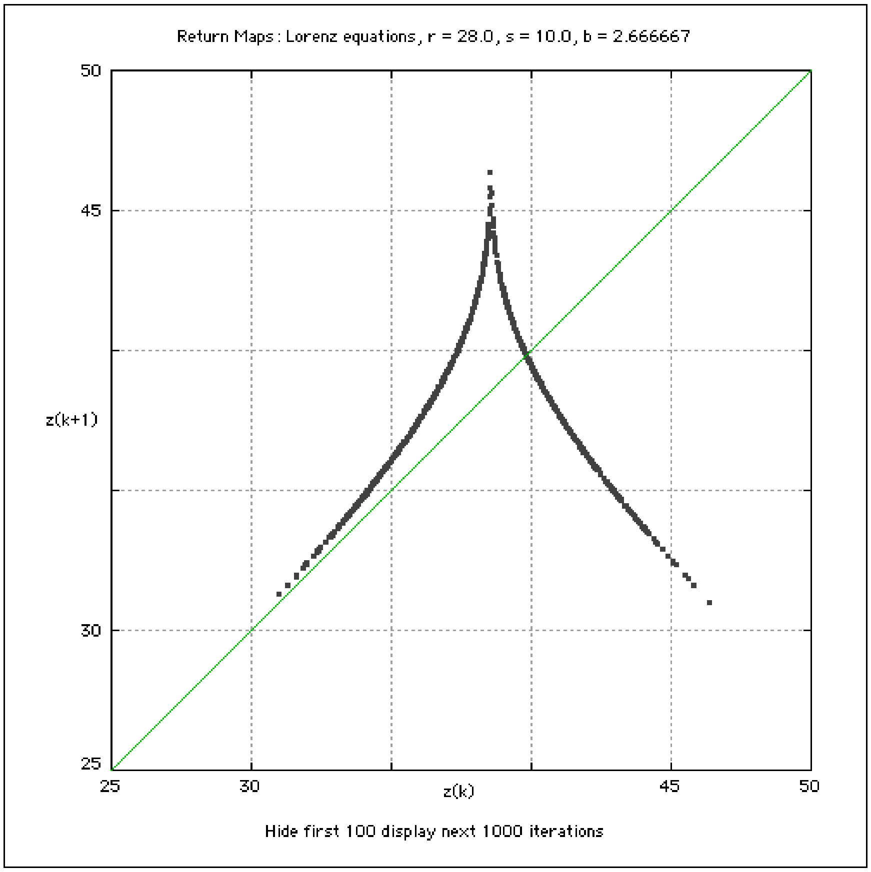
- These model a very simplified fluid in a box heated from below.
- $x(t), y(t)$ and $z(t)$ represent temperature fluctuations and σ, r and b are system parameters.
- Lorenz studied this system in the 60's and found "*the butterfly effect*" or "*sensitive dependence on initial conditions*".
- The system has chaotic orbits:



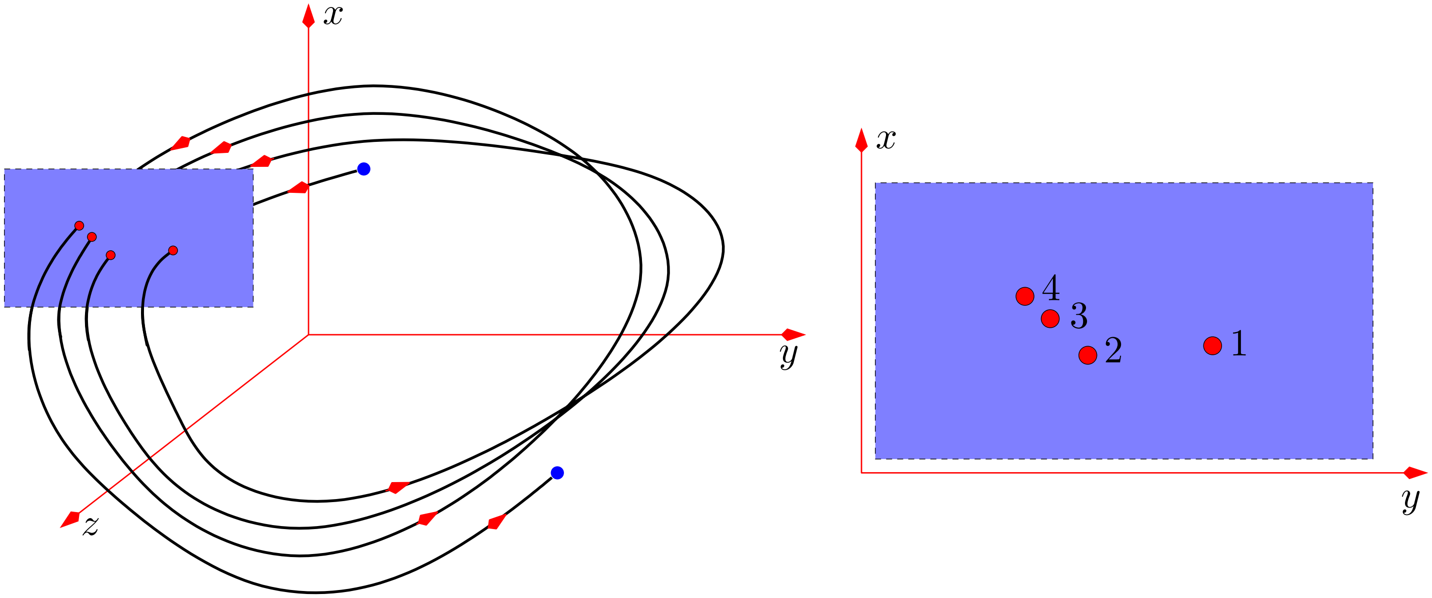

Three dimensional orbits : Lorenz equations
r = 28.0, s = 10.0, b = 2.666667
50
z
0
-20
x
20 -30
y
30
Three dimensional orbits : Lorenz equations
r = 28.0, s = 10.0, b = 2.666667
50
z
0
-20
x
20 -30
y
30
Three dimensional orbits : Lorenz equations
r = 28.0, s = 10.0, b = 2.666667
45
z
10
-20
x
20 -30
y
20
Three dimensional orbits : Lorenz equations
r = 28.0, s = 10.0, b = 2.666667
50
z
0
-20
x
20 -30
y
30

- This can be explored using the “ODE Orbits” applet.
- It is a very difficult system to explore analytically or numerically.
- We can use Poincaré sections and return maps to reduce the dimension of the system:

- The turning point in z is given by $xy = bz$.
- It always occurs when the trajectory intersects the surface $z = xy/b$.
- We can plot z_{tp} against the previous z_{tp}.



- This is an example of a return map.
- See the “Return Maps” applet.
- We see that $z_{n+1} = f(z_n)$ with $f(z)$ being something like the tent-map.
- The idea of reducing a complicated trajectory by plotting its intersections with a surface goes back to Poincaré.



- Such a surface is called a Poicaré section.