Theorem. *The structure of K_c is governed by the orbit of $z = 0$:*

- *If $Q_c^n(0) \not\to \infty$ then K_c is connected.*
- *If $Q_c^n(0) \to \infty$ then K_c is a Cantor set and is completely disconnected.*

In this case $K_c = J_c$ and the dynamics of Q_c on this set is conjugate to the shift map on two symbols.

- This theorem suggests another set worth plotting:

Definition. The *Mandelbrot set* $\mathcal{M}$ is the set:

$$\mathcal{M} = \{c \in \mathbb{C} \mid K_c \text{ is connected } \}$$

By the above, this is equivalent to:

$$\mathcal{M} = \{c \in \mathbb{C} \mid |Q_c^n(0)| \not\to \infty\}$$

Note:

- While K_c is a set of z-values — it lives in state-space.
- M is a set of c-values — it lives in parameter space.

- We can use the above, together with the escape criterion to build an algorithm to find $\mathcal{M}$

- Reminder...

Corollary (Escape criterion). *Suppose* $\exists k \geq 0$ *such that* $|Q_c^k(z)| > \max\{|c|, 2\}$ *then* $|Q_c^n(z)| \to \infty$.

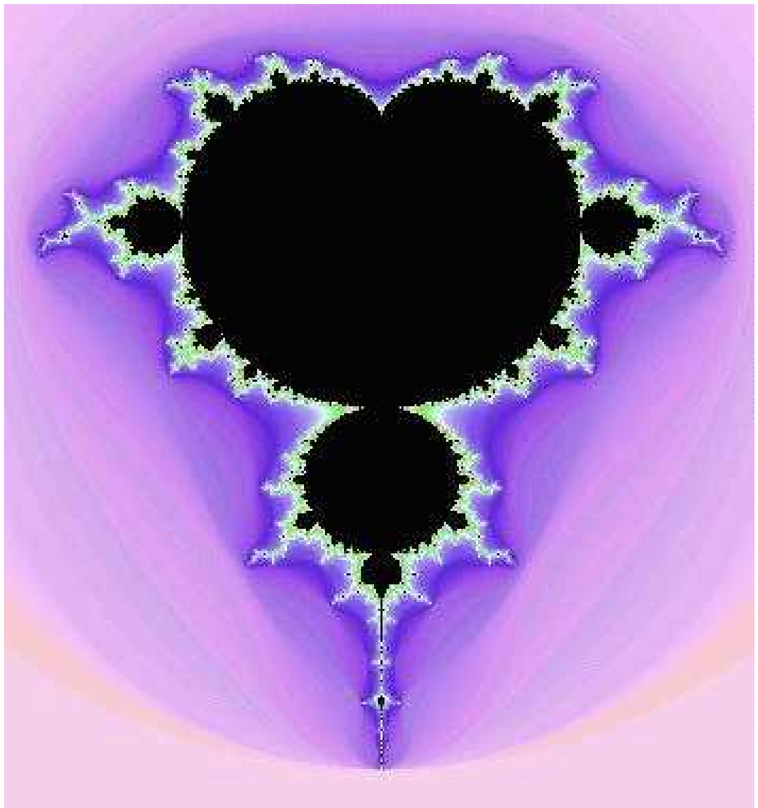
- Since $Q_c(0) = c$ it follows that:

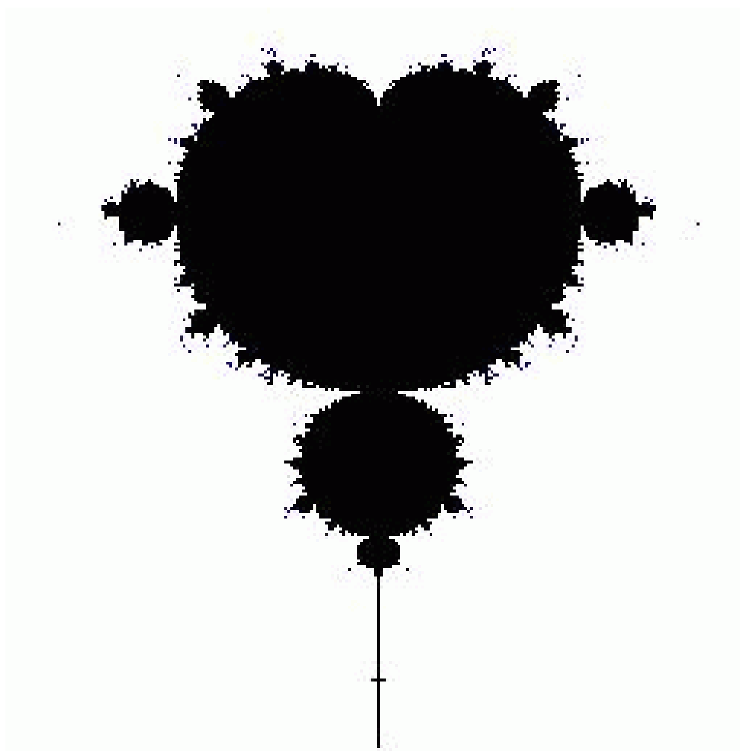
Corollary. *To which points are not in* $\mathcal{M}$*:*

- *If* $|c| > 2$ *or* $|Q^k(c)| > 2$ *(for some* $k > 0$*) then* $|Q_c^n(0)| \to \infty$
- *Hence* $c \notin \mathcal{M}$

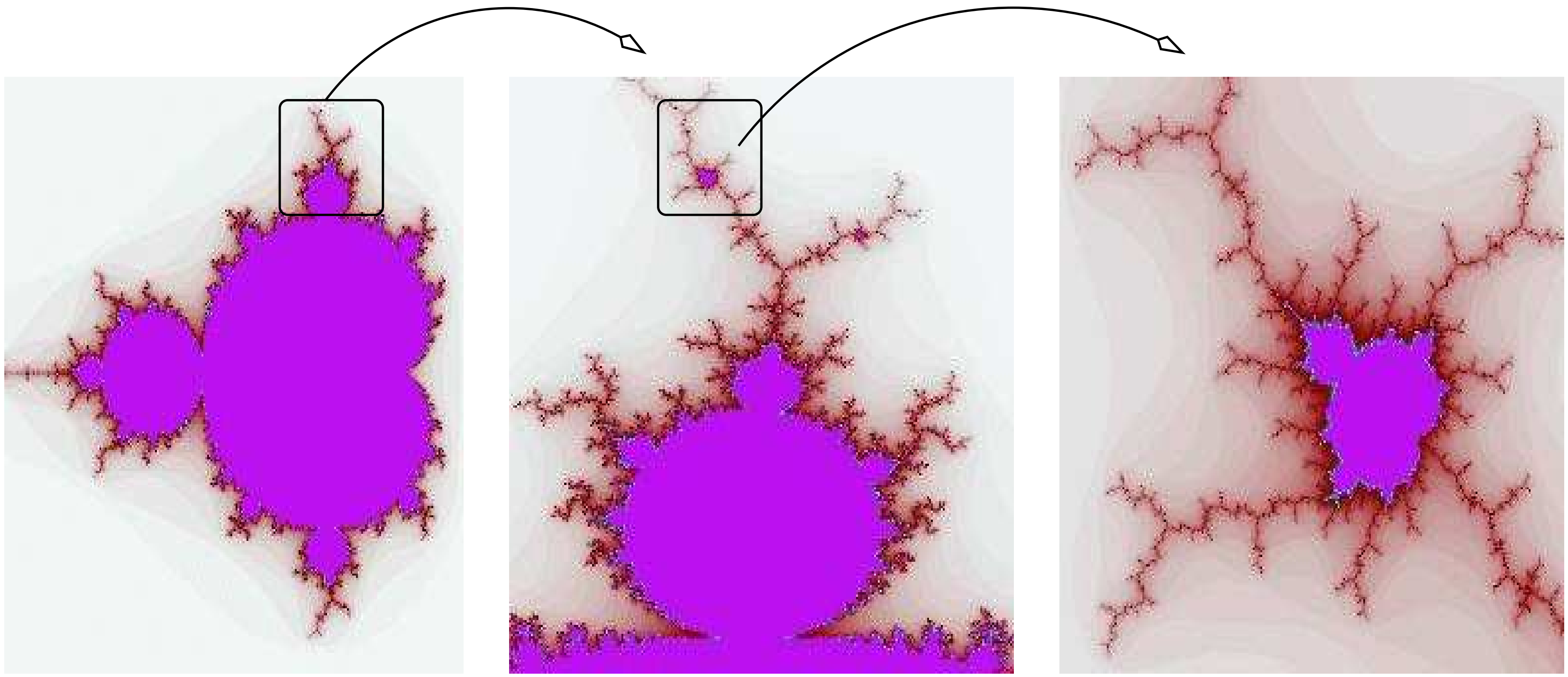
Algorithm for the Mandelbrot set:

- Pick a grid of points in $\mathbb{C}$ around $z = 0$ and some maximum number of iterations, N.
- For each point c in the grid, compute the first N points of the orbit of $z = 0$ under Q_c.
- If $Q_c^k(0)$ "escapes" for some $k \leq N$ then that c value is not in $\mathcal{M}$.
- Otherwise the c value is *probably* in $\mathcal{M}$.
- Colour points in $\mathcal{M}$ black, and other points white.
- Alternatively we can make prettier pictures by colouring those points that escape according to how fast they escape.





- Like K_c, the Mandelbrot set is incredibly complex.
- It is self-similar and is a fractal.



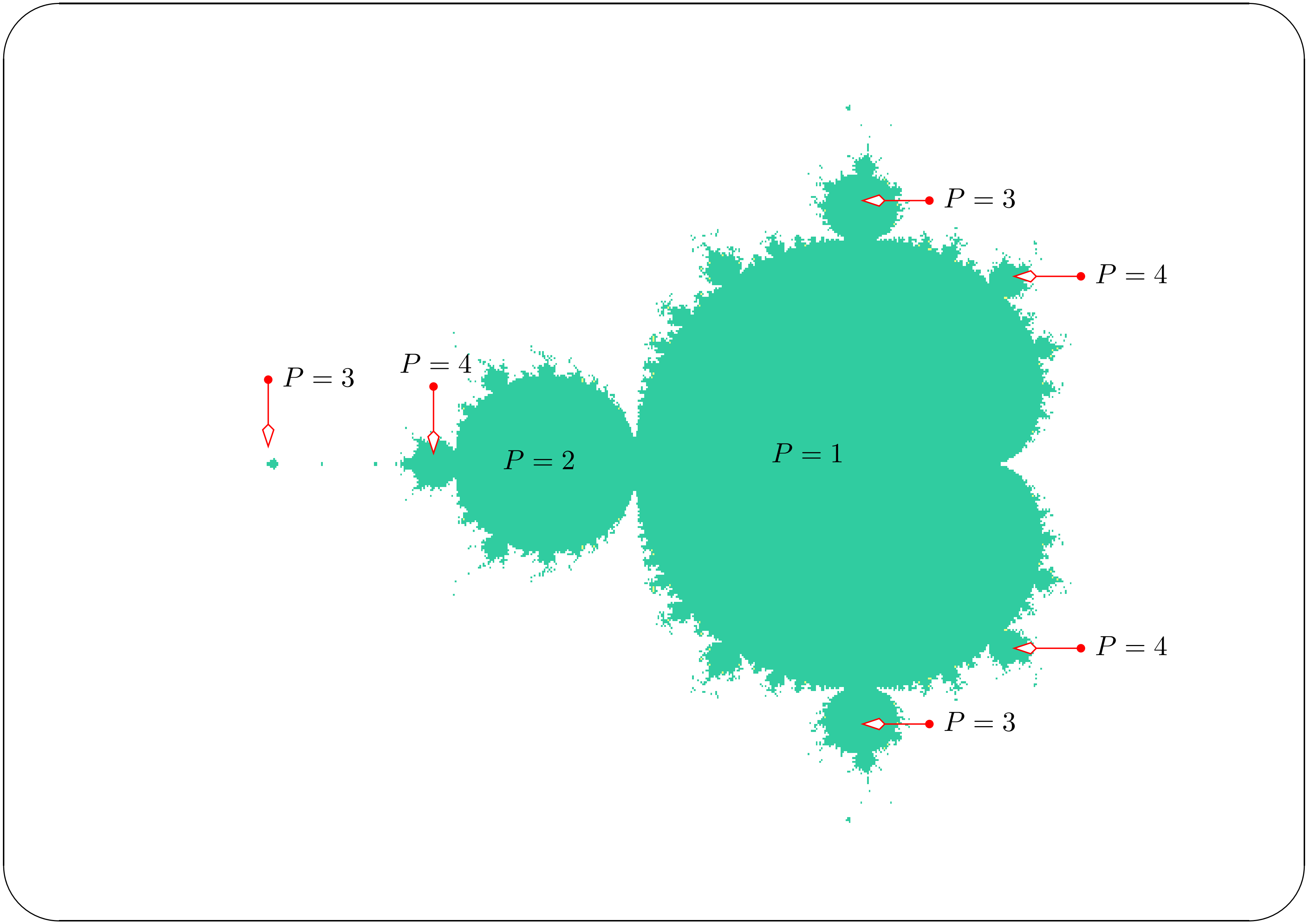
- So what does this image mean?

Definition. An alternative definition of $\mathcal{M}$ is:

$$\mathcal{M} = \{c \in \mathbb{C} \mid Q_c(z) \text{ has an attracting fp or pp}\}$$

- Say $Q_c(z)$ has an attracting fixed or periodic point.
- Then around this point there is a neighbourhood of points whose orbits converge to this point.
- All these points lie in K_c.
- Hence K_c is *not* a Cantor set, and so (by the above theorem) must be simply connected.
- So c is in $\mathcal{M}$.

- Indeed each "blob" or "bulb" of $\mathcal{M}$ corresponds to a region of c in which Q_c has attracting periodic points of a given period.



$P = 3$
$P = 4$
$P = 3$
$P = 4$
$P = 2$
$P = 1$
$P = 4$
$P = 3$

- While this picture doesn't look like it, $\mathcal{M}$ is acutally connected.

- We can compute some of these bulbs exactly.

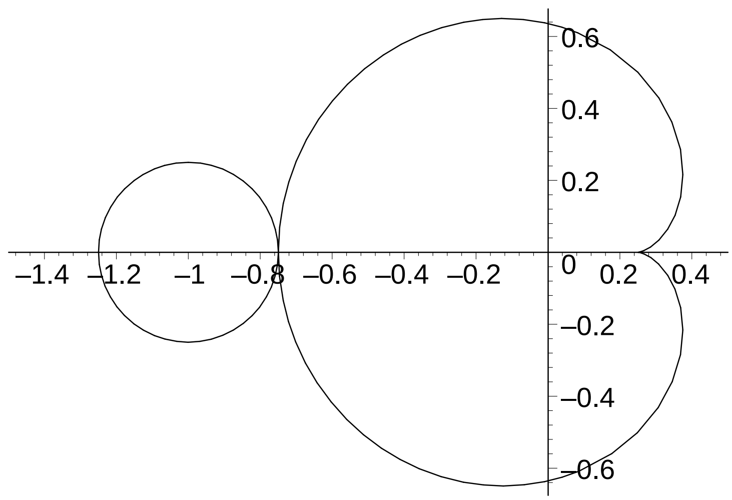
Proposition. *Period 1 and 2 bulbs:*

- $Q_c(z)$ *has an attracting fixed point inside the region defined by*

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$$

- $Q_c(z)$ *has an attracting 2-cycle point inside the region defined by*

$$c = \frac{1}{4}e^{i\theta} - 1$$



- Other bulbs we have to check numerically.
- Though there is a general theorem for which bulbs are which period (we aren't going to do it).

Proof: (fixed points)

- Denote the fixed points of Q_c by $p_\pm$.
- We require $|Q_c'(p_\pm)| = 2|p_\pm| < 1$.
- So we have an attracting fp if $p_\pm = \rho e^{i\theta}$ with $\rho < 1/2$.
- Substituting this into $Q(z) = z$ gives

$$\begin{aligned} \rho^2 e^{2i\theta} + c &= \rho e^{i\theta} \qquad \text{or} \\ c &= \rho e^{i\theta} - \rho^2 e^{2i\theta} \end{aligned}$$

- Which corresponds to c inside the region defined by

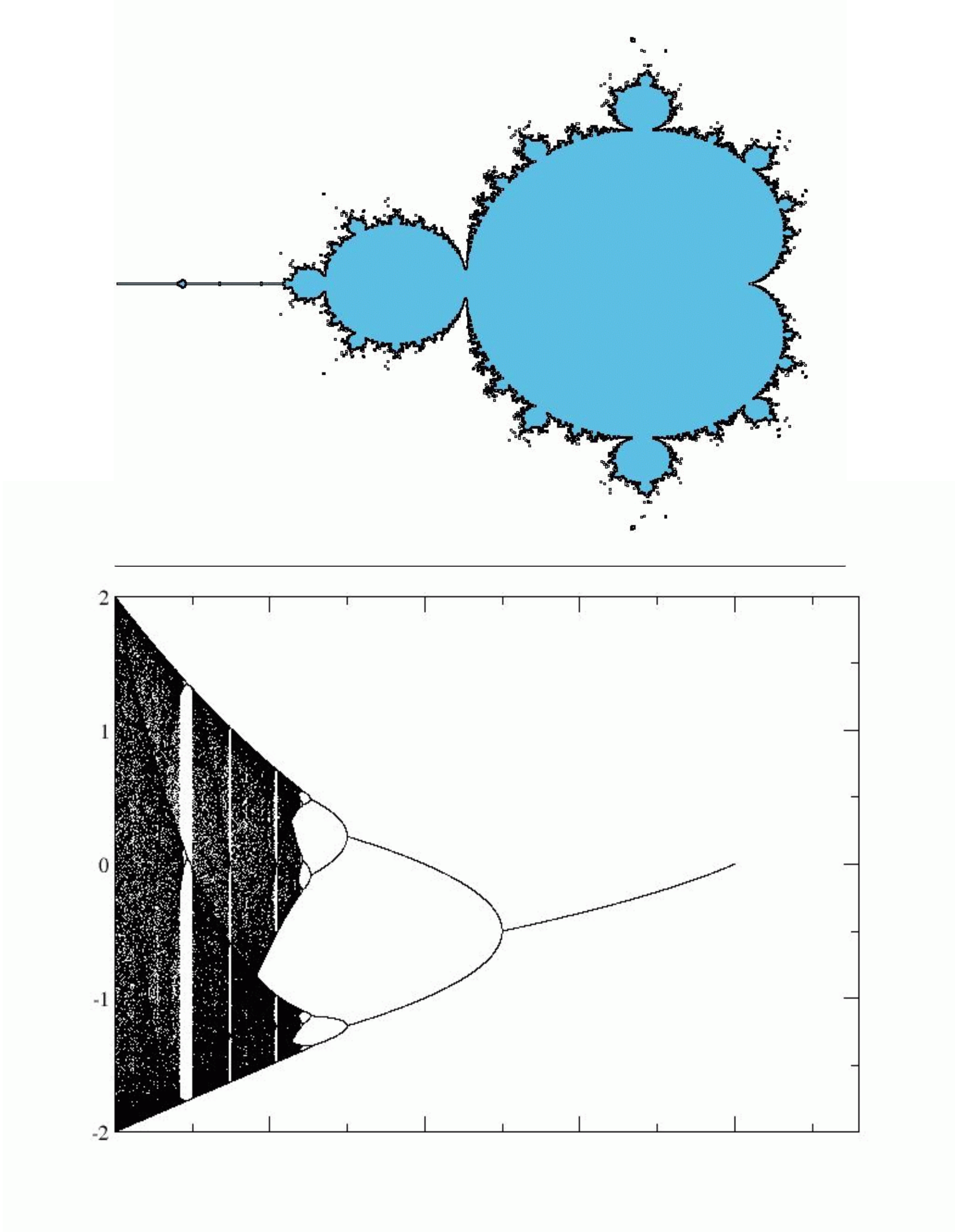
$$c = \frac{1}{2} e^{i\theta} - \frac{1}{4} e^{2i\theta}$$

Proof: (2-cycle)

- The 2-cycle of Q_c is given by $q_\pm = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3-4c}$.
- We have $|(Q_c^2)'(z_0)| = |Q_c'(z_0)||Q_c'(z_1)|$.
- So for the 2-cycle we need $4|q_+||q_-| = 4|q_+ q_-| < 1$
- Now $q_+ q_- = c + 1$.
- Hence we need $|c+1| < 1/4$, which corresponds to c inside the region

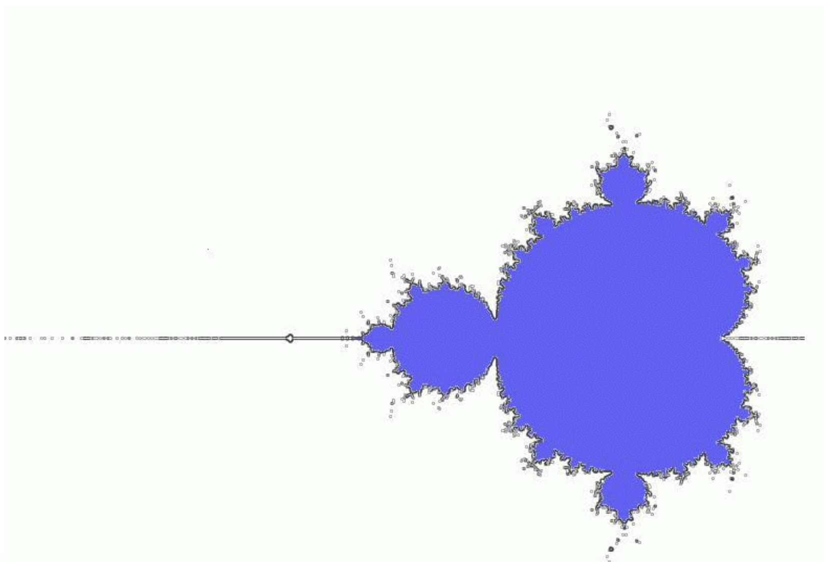
$$c = \frac{1}{4} e^{i\theta} - 1$$

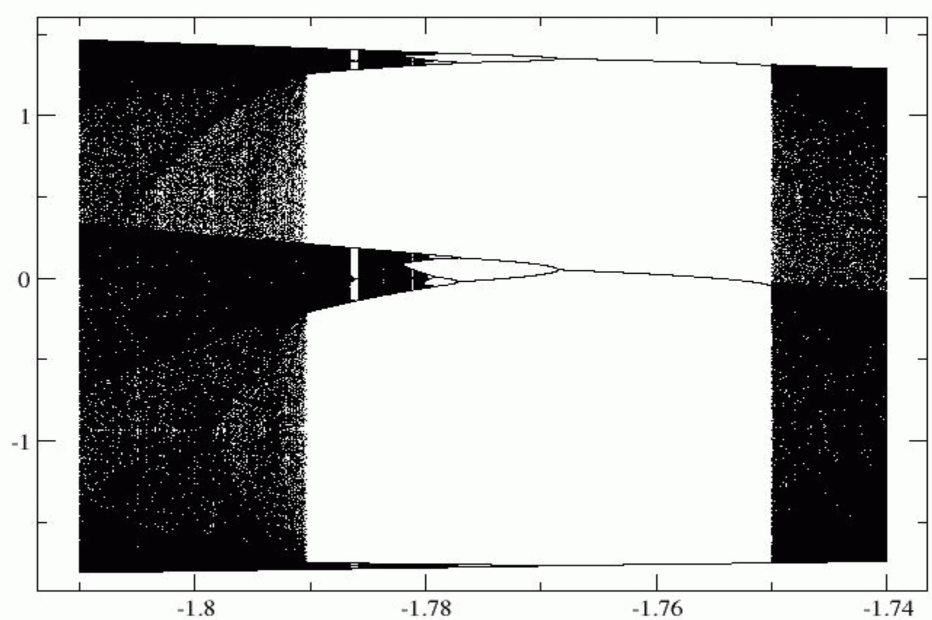
- The bifurcation diagrams we studied earlier correspond to the slice of $\mathcal{M}$ along the real axis.



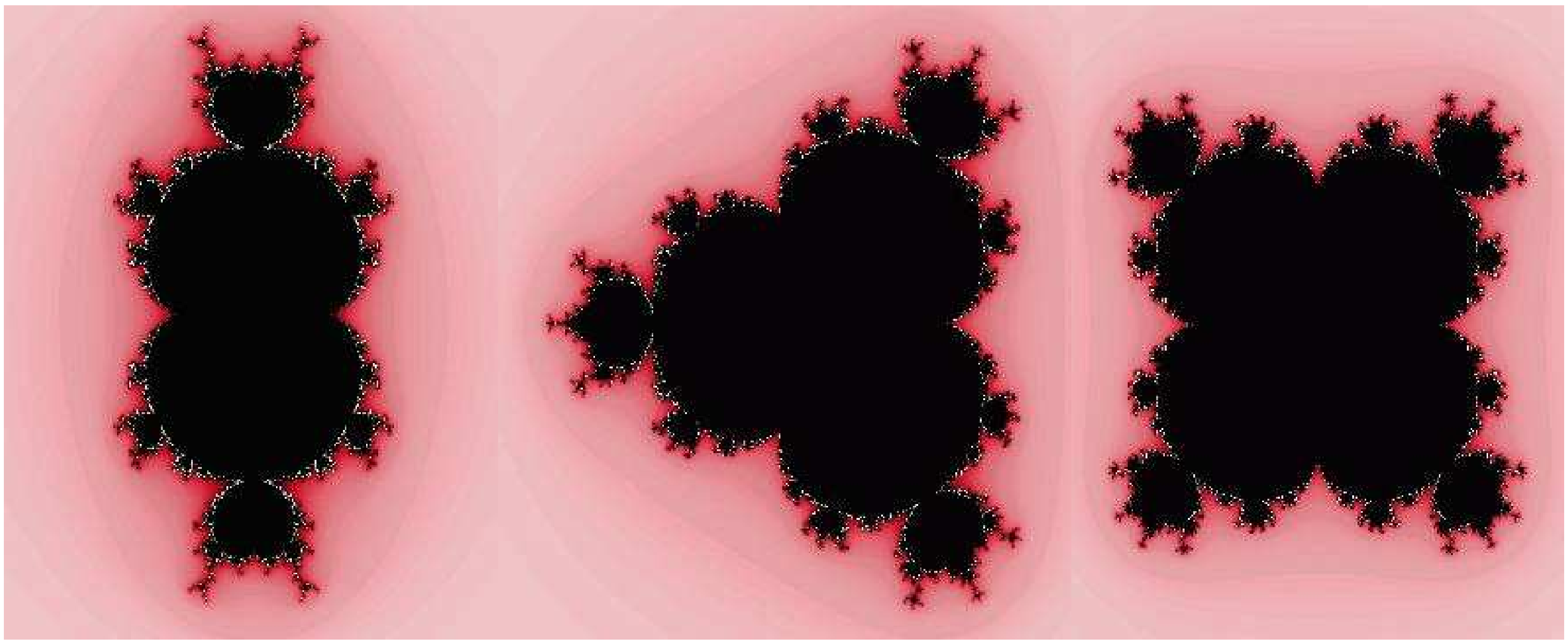
- The period 3 bulb in $\mathcal{M}$ corresponds to the period 3 window in the bifurcation diagram

- Here is the period 3 bulb and corresponding window of the bifurcation diagram:





- We can also generate Mandelbrot sets for other functions:



- These are the Mandelbrot sets of $z^3 + c$, $z^4 + c$ and $z^5 + c$.

- We can find the period-1 bulbs of the Mandelbrot sets of $F(z) = z^n + c$:

Theorem. *The period-1 bulb of the Mandelbrot set of $z^n + c$ is given by the curve:*

$$c = n^{-1/(n-1)} e^{i\theta} - n^{-n/(n-1)} e^{ni\theta}$$

- We do this in the same way we did for $z^2 + c$.
- For a fixed point, we require that $|F'(z_0)| < 1$
- This gives the equation

$$|F'(z_0)| = n|z_0|^{n-1} < 1 \qquad \text{or} \qquad |z_0| < n^{-1/(n-1)}.$$

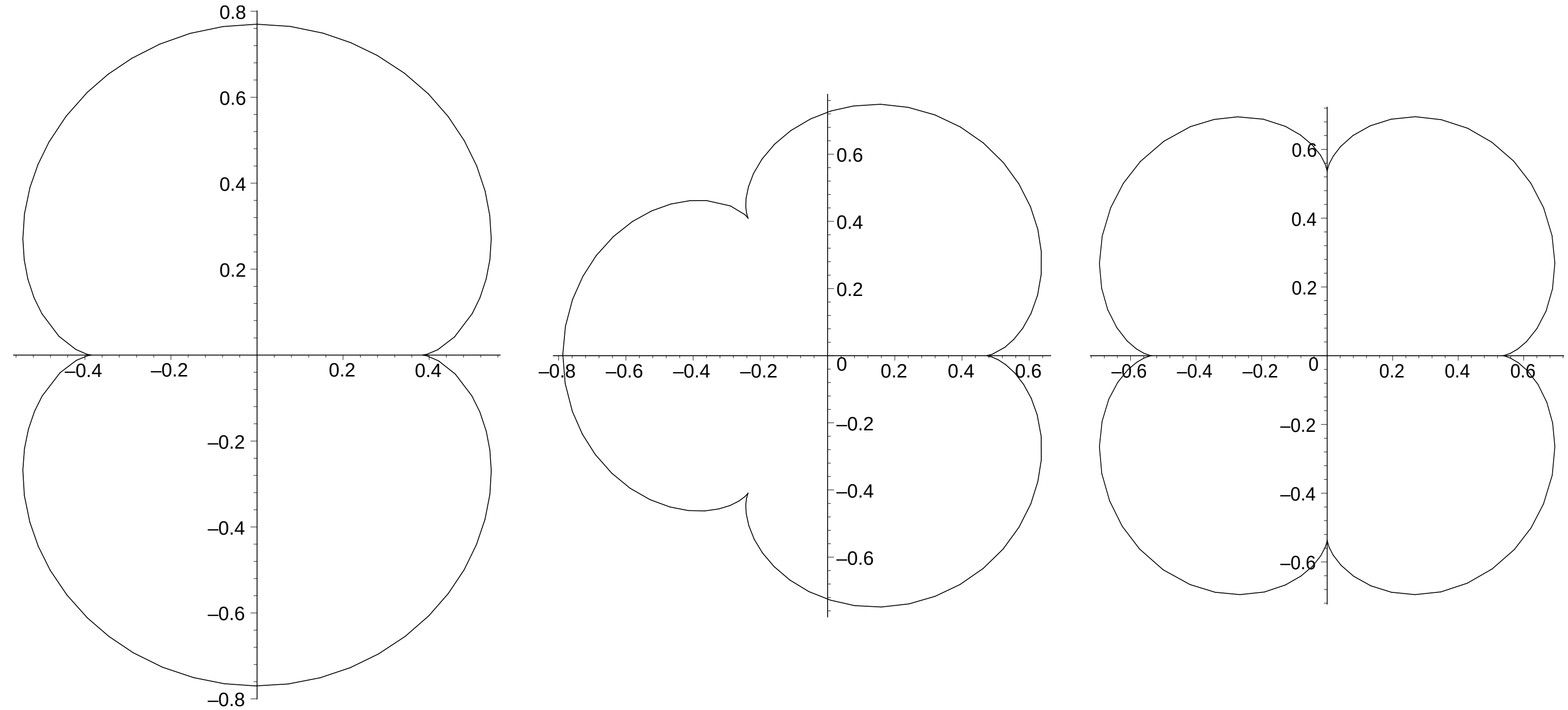
- Hence if we have a fixed point of the form $z_0 = \rho e^{i\theta}$ with $\rho < n^{-1/(n-1)}$, then it is attracting.
- Substituting this into $z^n + c = z$ to obtain c we get:

$$c = \rho e^{i\theta} - \rho^n e^{ni\theta}$$

- As so the required c values lie inside the curve:

$$c = n^{-1/(n-1)} e^{i\theta} - n^{-n/(n-1)} e^{ni\theta}$$

□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