MATHS COMPETITION 2000
SOLUTIONS

Problem 1
Denote by $b_1, b_2, \ldots$ non-negative and by $-c_1, -c_2, \ldots$ negative terms in the sequence $\{a_n\}$ (in the order they appear in the sequence). Clearly,

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = 0, \quad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n = \infty. \quad (1)$$

For an $A \in (-\infty, \infty)$, let

$$n_1 = \min\{n > 0 : \sum_{k=1}^{n} b_k > A\},$$
$$m_1 = \min\{m > 0 : \sum_{k=1}^{n_1} b_k - \sum_{k=1}^{m} c_k \leq A\},$$
$$n_2 = \min\{n > n_1 : \sum_{k=1}^{n} b_k - \sum_{k=1}^{m_1} c_k > A\},$$
$$m_2 = \min\{m > m_1 : \sum_{k=1}^{n_2} b_k - \sum_{k=1}^{m} c_k \leq A\}$$

and so on. Due to (1), all such $n_i$ and $m_i$, $i \geq 1$, exist, and this algorithm produces the desired permutation: in the permuted series, first we have the first $n_1$ non-negative terms in $\{a_n\}$, then the first $m_1$ negative ones, then the next $n_2 - n_1$ non-negative terms, and so on. The partial sums they form will converge to $A$. When $A = \infty$, one can take

$$n_1 = \min\{n > 0 : \sum_{k=1}^{n} b_k > 3^1\},$$
$$m_1 = \min\{m > 0 : \sum_{k=1}^{m} c_k > 2^1\},$$
$$n_2 = \min\{n > n_1 : \sum_{k=1}^{n} b_k > 3^2\},$$
$$m_2 = \min\{m > m_1 : \sum_{k=1}^{m} c_k > 2^2\}$$

and so on, Similarly when $A = -\infty$.  

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Problem 2
The identity can be proved using induction. First note that for $n = 1$, both sides are equal to 1/2. Next, assuming that the relation is true for $n > 1$, show that it will still be true for $n + 1$. Indeed, when the value of $n$ increases by one, the right-hand side increases by
\[
\frac{1}{2n + 1} + \frac{1}{2n + 2} - \frac{1}{n + 1} = \frac{1}{2n + 1} - \frac{1}{2n + 2}.
\]
But exactly the same value is added to the left-hand side under the same transition from $n$ to $n + 1$.

As for the limiting value, note that clearly
\[
\frac{1}{n + 1} < \int_n^{n+1} \frac{dx}{x} < \frac{1}{n},
\]
and hence
\[
\sum_{k=n+1}^{2n} \frac{1}{k} < \int_n^{2n} \frac{dx}{x} < \sum_{k=n}^{2n-1} \frac{1}{k},
\]
so that obviously
\[
\lim_{n \to \infty} \sum_{k=n+1}^{2n} \frac{1}{k} = \lim_{n \to \infty} \int_n^{2n} \frac{dx}{x} = \lim_{n \to \infty} (\ln(2n) - \ln n) = \ln 2.
\]

Problem 3
Note that
\[
(X_1 + X_2)(X_1 - X_2) = X_1^2 - X_2^2 + X_2X_1 - X_1X_2
= -A(X_1 - X_2) + [X_2X_1 - X_1X_2].
\]
Hence
\[
(X_1 + X_2) = -A + [X_2X_1 - X_1X_2][X_1 - X_2]^{-1}
\]
and
\[
\text{tr}X_1 + \text{tr}X_2 = \text{tr}(X_1 + X_2) = -\text{tr}A
\]
since $\text{tr}(CDE) = \text{tr}(DCE)$ for any matrices $D$, $C$ and $E$.

To show the second equality, note that
\[
(X_i + A)X_i = -B \quad i = 1, 2,
\]
implying that
\[
\det(X_i + A) \det X_i = \det[(X_i + A)X_i] = (-1)^n \det B. \tag{2}
\]
Since
\[
(X_2 + A)(X_1 - X_2) = X_2X_1 - X_2^2 + AX_1 - AX_2 = X_2X_1 + AX_1 + B
= X_2X_1 - X_1^2 = -(X_1 - X_2)X_1,
\]
it follows that
\[ (X_1 - X_2)^{-1}(X_2 + A)(X_1 - X_2) = -X_1 \]
and
\[ \det(X_2 + A) = (-1)^n \det X_1. \]  
(3)
Combining (2) and (3) we get \( \det X_1 \det X_2 = \det B \).

**Problem 4**
The intersection points \( A, B \) and \( C \) lie on the \( x-, y- \) and \( z- \)axes, respectively. Let \( \gamma \) be the angle between the \( xy \) coordinate plane and our section plane (i.e. between the planes containing triangles \( OAB \) and \( ABC \), resp.), \( \beta \) the angle between the \( xz \)-plane and \( ABC \)-plane, and \( \alpha \) the angle between the \( yz \)-plane and \( ABC \)-plane.

Denote \( O' \) the projection of the point \( O \) onto our plane. Clearly,
\begin{align*}
\text{Area}(O'AB) &= \text{Area}(OAB) \cos \gamma, \\
\text{Area}(O'BC) &= \text{Area}(OBC) \cos \alpha, \\
\text{Area}(O'AC) &= \text{Area}(OAC) \cos \beta. 
\end{align*}

(4)
On the other hand, viewing \( OAB \) as a projection of the triangle \( ABC \) onto the \( xy \)-plane etc., we similarly have
\begin{align*}
\text{Area}(OAB) &= \text{Area}(ABC) \cos \gamma, \\
\text{Area}(OBC) &= \text{Area}(ABC) \cos \alpha, \\
\text{Area}(OAC) &= \text{Area}(ABC) \cos \beta. 
\end{align*}

(5)
Combining (4) and (5), we get
\[ \text{Area}(ABC) = \text{Area}(O'AB) + \text{Area}(O'BC) + \text{Area}(O'AC) \]
\[ = \frac{\text{Area}(OAB)^2}{\text{Area}(ABC)} + \frac{\text{Area}(OBC)^2}{\text{Area}(ABC)} + \frac{\text{Area}(OAC)^2}{\text{Area}(ABC)}, \]
which completes the proof. The multidimensional case can be dealt with in exactly the same way.

**Problem 5**
Denote by \( Z_p \) a random variable (r.v.) obtained as a sum \( \sum_{n=1}^{\infty} 2^{-n} X_n \), where \( X_n \) are independent Bernoulli r.v.’s with
\[ \mathbf{P}(X_n = 1) = 1 - \mathbf{P}(X_n = 0) = p \in (0, 1), \quad n = 1, 2, \ldots, \]
and by \( F_{Z_p} \) its distribution function. Denote by \( x_n(u) \) the \( n \)th digit in the binary expansion of the number \( u \in (0, 1) \):
\[ x_1(u) = \begin{cases} 1 & \text{if } u > 1/2, \\ 0 & \text{otherwise}, \end{cases} \quad x_2(u) = \begin{cases} 1 & \text{if } 2u - x_1(u) > 1/2, \\ 0 & \text{otherwise}, \end{cases} \]
etc. Clearly, $x_n(Z_p) = X_n$ (with probability one).

Next note that all $F_{Z_p}$ are continuous. Indeed, any point $u$ which is not binary rational (i.e. of the form $m/2^n$ for integer $m$ and $n$), has a unique binary decomposition, and hence by independence

$$P(Z_p = u) = P(x_k(Z_p) = x_k(u), \ k = 1, 2, \ldots) = \prod_{k=1}^{\infty} P(X_k = x_k(u))$$

$$= \prod_{k=1}^{\infty} p^{x_k(u)}(1-p)^{1-x_k(u)} = p^{\sum x_k(u)} (1-p)^{\sum (1-x_k(u))} = 0$$

due to the fact that $\sum_{k=1}^{\infty} x_k(u) = \infty$. For binary rational points, we have only two possible binary representations, each having probability zero for a similar reason. Setting

$$A_p = \left\{ u \in (0, 1) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k(u) = p \right\}, \ p \in (0, 1),$$

we see that

$$A_p \cap A_q = \emptyset \quad \text{for} \quad p \neq q$$

and, since $x_n(Z_p) = X_n$,

$$P(Z_p \in A_p) = 1, \quad p \in (0, 1),$$

by the strong law of large numbers. Hence all the points of increase of the distribution function $F_{Z_p}$ of $Z_p$ (i.e. such points $x$ that $F_{Z_p}(x + \varepsilon) - F_{Z_p}(x - \varepsilon) > 0$ for any $\varepsilon > 0$) belong to $A_p$. So $F_{Z_p}$ can have a positive density at the points of $A_p$ only.

It is easy to verify that, for a uniformly distributed on $(0,1)$ r.v. $U$, $X_n = x_n(U)$ are independent Bernoulli r.v. with “success” probability $p = 1/2$, so that $Z_{1/2}$ is uniform on $(0,1)$. From (6) we see that for any $p \neq 1/2$,

$$P(U \in A_p) = 0,$$

i.e. the total length of such an $A_p$ is zero. Which implies that all $F_{Z_p}$, $p \neq 1/2$, are singular.

**Problem 6**

**Proof by contradiction.** Denote by

$$f(x) = (T(x), P(x)) : S^2 \mapsto \mathbb{R}^2$$

the mapping of the unit sphere $S^2 \subset \mathbb{R}^3$ (representing the Earth surface) whose components are the temperature and pressure at the respective point $x = (x_1, x_2, x_3)$. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be
the North and South poles, respectively, \( \Gamma = S^2 \cap \{ x_3 = 0 \} \) the equator. Assume that there are no such points \( a \neq b \) on \( S^2 \) that \( f(a) = f(b) \).

The image \( \gamma = f(\Gamma) \) of the equator under the mapping \( f \) is a closed continuous contour. It has no self-crossings (by our assumption above). Denote by \( C_i \) and \( C_e \) the interior and exterior of \( \gamma \), respectively. Assume that one of the poles is in \( C_i \); we can always assume that it is \( N \):

\[
\begin{align*}
\text{Show that the image of northern hemisphere will fill the whole } C_i: \\
f(S^2 \cap \{ x_3 > 0 \}) = C_i.
\end{align*}
\]

Let us continuously deform \( \gamma \) “pulling” it towards \( N \) so that it “shrinks” to the single point \( N \); we can do that by setting \( \Gamma_t^+ = S^2 \cap \{ x_3 = t \} \), \( t \in [0, 1] \). The image \( f(\Gamma_t^+) \) of the “deformed contour” will then continuously shrink to the point \( f(N) \), which would be impossible if there were points in \( C_i \) not belonging to \( f(S^2 \cap \{ x_3 > 0 \}) \).

Therefore \( f(S) \in C_e \) (otherwise we would have \( f(S) = f(x) \) for some \( x \in S^2 \cap \{ x_3 > 0 \} \)). Now letting \( \Gamma_t^- = S^2 \cap \{ x_3 = -t \} \), \( t \in [0, 1] \), we see that all \( \Gamma_t^- \) must be in \( C_e \) (otherwise it would contradict our assumption). But \( f(\Gamma_t^-) \) shrinks to the single point \( f(S) \) as \( t \) runs from 0 to 1! And yet all the time \( \gamma \) is inside \( f(\Gamma_t^-) \). Contradiction.

If both \( N \) and \( S \) belong to \( C_e \), the argument is even simpler (the last contradiction is seen immediately).

**Problem 7**

Assume that \( |z| > 1 \) and \( z \) is a solution of the equation \( z^n + z^{n-1} + \cdots + z + 1 = (n+1)z^n \). Then \( (n+1)|z|^n \leq |z|^n + \cdots + |z| + 1 \) implying that

\[
(n+1) \leq 1 + \frac{1}{|z|} + \cdots \left( \frac{1}{|z|} \right)^n.
\]

But since \( 1/|z| < 1 \), the right side of the above inequality is less than \( n = 1 \). Contradiction.

**Problem 8**

You may notice that \( A \) is a *doubly stochastic irreducible matrix*, hence the statement. To prove it anyway, set

\[
V = \{ \mathbf{v} = (v_1, v_2, v_3) : v_j \geq 0, \; j = 1, 2, 3; \; v_1 + v_2 + v_3 = 1 \}.
\]

It is obvious that the mapping

\[
\mathbf{v} \mapsto \mathbf{v}A \quad \text{maps } V \text{ into } V.
\]
Next assume that $a \leq b \leq c$ (for any other order, our argument works with obvious amendments). For any $v, v' \in V$, using $v_3 = 1 - v_1 - v_2$ and $v'_3 = 1 - v'_1 - v'_2$ etc, we get

$$
(v - v')A = (a(v_1 - v'_1) + c(v_2 - v'_2) + b(v_3 - v'_3),
\quad b(v_1 - v'_1) + a(v_2 - v'_2) + c(v_3 - v'_3),
\quad c(v_1 - v'_1) + b(v_2 - v'_2) + a(v_3 - v'_3)),
$$

$$
= ((a - b)(v_1 - v'_1) + (c - b)(v_2 - v'_2),
\quad (a - b)(v_2 - v'_2) + (c - b)(v_3 - v'_3),
\quad (c - b)(v_1 - v'_1) + (a - b)(v_3 - v'_3)).
$$

Therefore, for the norm $\|v\| = \max_j |v_j|$, 

$$
\|(v - v').A\| \leq (|a - b| + |c - b|)\|v - v'\| = \rho\|v - v'\|
$$

for $\rho = c - a < 1$. That is, the mapping (7) is a contraction on $V$. [What follows is a proof of the so-called contraction principle.]

So if we put $v^{(n+1)} = v^{(n)}.A$ for some initial value $v^{(0)} \in V$, 

$$
\|v^{(n+1)} - v^{(n)}\| \leq \rho\|v^{(n)} - v^{(n-1)}\| \leq \cdots \leq \rho^n,
$$

i.e. $\{v^{(n)}\}$ is a Cauchy sequence and hence has a limit $v^*$ which will clearly satisfy 

$$
v^*A = v^*, \quad v^* \in V.
$$

This is a system of linear equations which is easily seen to have a unique solution 

$$
k = (1/3, 1/3, 1/3) \in V.
$$

So whatever the initial point $v^{(0)} \in V$ is, we get $v^{(n)} \to k$ as $n \to \infty$. It remains to notice that each of the rows of $A^{n+1}$ has the form $vA^n$, so that all the rows converge to $k$.

**Problem 9**

Assume that $f : \mathbb{R} \to \mathbb{R}$ has only one critical point, say $x_0$, at which $f$ attains its local minimum. If $x_0$ is not a global minimum, then there exists a point $x_1$ such that $f(x_1) < f(x_0)$. We may assume that $x_0 < x_1$. Then $f$ attains a local maximum at some point $x_2 \in (x_0, x_1)$ at which $f'(x_2) = 0$. Contradiction. In higher dimensions this does not hold in general. Consider 

$$
f(x, y) = x^2(1 + y)^3 + y^2.
$$

Then $f$ has exactly one critical point at $(0, 0)$. Application of the second derivatives test shows that $f$ has a local minimum at $(0, 0)$. Consider now values of $f$ along the line $y = x$, $f(x, x) = x^2[(1 + x)^3 + 1]$ which
becomes negative for sufficiently large negative values of \( x \). For \( n > 2 \), just take
\[
f(x_1, x_2, \ldots, x_n) = x_1^2(1 + x_2)^3 + x_2^2.
\]

**Problem 10**
Let \( x_0 \) be the unique fixed point of the \( n \)th iterate of \( f \), i.e., \( f^{(n)}(x_0) = x_0 \). Then \( f(f^{(n)}(x_0)) = f(x_0) \) which implies that \( f^{(n)}(f(x_0)) = f(x_0) \).

Hence \( f(x_0) \) is a fixed point of \( f^{(n)} \) and since \( f^{(n)} \) has a unique fixed point \( x_0 \), \( f(x_0) = x_0 \) as required.

**Problem 11**
Since \( x_n = x_0 + \sum_{k=1}^{n} [x_{k+1} - x_k] \), it suffices to show that \( \sum_{k=1}^{\infty} |x_{k+1} - x_k| \) converges. For any \( n \)
\[
x_{k+1} - x_k = \frac{1}{2}[f(x_k) - x_k],
\]
and
\[
|f(x_{k+1}) - x_{k+1}| = \left| f(x_{k+1}) - f(x_k) + f(x_k) - \frac{1}{2}[f(x_k) + x_k] \right|
\leq |f(x_{k+1}) - f(x_k)| + \frac{1}{2}|f(x_k) - x_k|
\leq |x_{k+1} - x_k| + \frac{1}{2}|f(x_k) - x_k|
= |f(x_k) - x_k|.
\]

Thus
\[
|x_{k+1} - x_k| = \frac{1}{2}|f(x_k) - x_k| \leq \frac{1}{2}|f(x_{k-1}) - x_{k-1}| = \frac{1}{2}|x_k - x_{k-1}|
\leq \frac{1}{2^2}|x_{k-1} - x_{k-2}| \leq \cdots \leq \frac{1}{2^{k-1}}|x_2 - x_1|
\]
This shows that \( \sum_{k=1}^{\infty} |x_{k+1} - x_k| < \infty \). Hence there exists an \( x^* \in [a, b] \) such that \( x_n \to x^* \) as \( n \to \infty \), and, since \( f \) is continuous, \( x^* = \frac{1}{2}[x^* + f(x^*)] \). So \( f(x^*) = x^* \) as required.

**Problem 12**
Denote by \( P_n(x) = x^n + x^{n-1} + \cdots + x - 1 \). Since \( P_n(x) > (n-1) \) for \( x \geq 1 \), and \( P_n(0) = -1 \), \( P_n(x) \) have zeros in \((0, 1)\). It has exactly one zero \( a_n \in (0, 1) \) since \( P_n'(x) > 0 \) for \( x > 0 \). Let \( f(x) = P_{n+1} - P_n(x) \).
Since \( f(x) = x^{n+1} \), it follows that \( a_{n+1} < a_n \). This implies that \( a_n \to \)
\( a \geq 0. \) Since \( P_n(x) = \frac{x^{n+1} - 1}{x - 1} - 2, \) we conclude that \( a_n \) satisfies
\[
 a_n^{n+1} = 2a_n - 1.
\]
The left hand side is \( 0 < a_1^{n+1} \to 0 \) as \( n \to \infty \) since \( a_1 \in (0, 1). \) The right hand side converges to \( 2a - 1. \) So \( a = 1/2. \)

**Problem 13**

The solution follows from the following lemma:

**Lemma** (Dirichlet)
If \( \alpha \) is an irrational number, then the set
\[
 A = \{ m + n\alpha \mid m, n \in \mathbb{Z} \}
\]
is dense in \( \mathbb{R}. \)

From the conditions of the problem, \( f(x) = f(0) \) for any \( x \in A. \) So if \( A \) is everywhere dense, \( f(x) = f(0) \) for any \( x \in \mathbb{R} \) since \( f \) is continuous.

To prove the lemma, it suffices to show that \( A \) contains arbitrary small positive numbers. Indeed, if this is the case, then an arbitrary interval \( I \subset \mathbb{R} \) will contain points of the form \( nx, x \in A, \) and then clearly \( x \in A. \)

So take an arbitrary natural \( r \) and consider the intervals
\[
 I_1 = (0, 1/r), I_2 = (1/r, 2/r), \ldots, I_2 = ((r - 1)/r, 1)
\]
and the points
\[
 x_j = j\alpha - \lfloor j\alpha \rfloor, \quad j = 1, \ldots, r + 1,
\]
\( \lfloor x \rfloor \) being the integer part of \( x. \) All the \( r + 1 \) points \( x_j \in (0, 1) \) are irrational and different, hence at least one of the \( r \) intervals \( I_m \) will contain two points: say,
\[
 x_j, x_k \in I_m, \quad x_j < x_k.
\]
But then
\[
 0 < x_k - x_j = (k - j)\alpha + \lfloor j\alpha \rfloor - \lfloor k\alpha \rfloor < 1/r,
\]
which proves the lemma, since clearly \( x_k - x_j \in A. \)