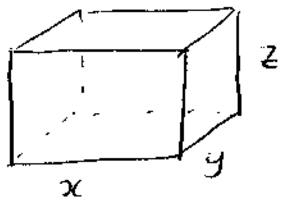
620-231 Semester 1, 2002 Exam Solution

Q1 (a) $f_x = \dfrac{2}{1+2x-y}$, $f_y = \dfrac{-1}{1+2x-y}$, $f_{xx} = \dfrac{-4}{(1+2x-y)^2}$

$f_{yy} = \dfrac{-1}{(1+2x-y)^2}$, $f_{xy} = f_{yx} = \dfrac{2}{(1+2x-y)^2}$

(b) $P_2(x,y) = 2x - y - 2x^2 + 2xy - \frac{1}{2}y^2$

Q2



* Cost of making box

$f(x,y,z) = 5(xy + xz) + 1\ (xz + 2yz)$

* Volume $= 60\ m^3$

$g(x,y,z) = xyz - 60 = 0$

* minimize f subject to constraint g by solving

$(5y + 6z,\ 5x + 2z,\ 6x + 2y) = \lambda(yz,\ xz,\ xy)$

* minimum cost of 180 occurs at $x = 2m$, $y = 6m$, $z = 5m$.

Q3 (a) $\underline{c}$ is not a flow line of $\underline{F}$

(b) Flow lines are circles centred at $(0,0)$, arbitrary radius, traversed anticlockwise. ie $x^2 + y^2 = r^2$.

(c) If $\underline{F}$ is velocity of fluid. A flow line is the path a particle suspended in the fluid follows as the fluid moves. The flow line is tangent to the vector field $\underline{F}$.

Q4 (a)
$$\nabla \cdot (f\nabla g \times \nabla h) = f\nabla \cdot (\nabla g \times \nabla h) + (\nabla g \times \nabla h) \cdot \nabla f$$
(identity 7)
$$= f\left[\nabla h \cdot (\nabla \times \nabla g) - \nabla g \cdot (\nabla \times \nabla h)\right] + (\nabla g \times \nabla h) \cdot \nabla f$$
(identity 8)
$$= (\nabla g \times \nabla h) \cdot \nabla f \quad \text{(identity 11)}.$$
$$= \nabla f \cdot (\nabla g \times \nabla h)$$

(b) (i) $\nabla^2(r^3) = \nabla\cdot(\nabla r^3) = 12r$

(ii) $\nabla \cdot r$ is not defined

(iii) $\nabla \times \left(\frac{\underline{r}}{r^5}\right) = \underline{0}$ use identity 10.

$\boxed{Q5}$ (a) $\int_0^1\int_0^1\int_0^1 yz^2 e^{xyz}\,dx\,dy\,dz = e - \frac{5}{2}$

(b) change order of integration

$$\int_0^1\int_x^1 \frac{y^3}{x^2+y^2}\,dy\,dx = \int_0^1\int_0^y \frac{y^3}{x^2+y^2}\,dx\,dy = \frac{\pi}{12}$$

$\boxed{Q6}$ $I_z = \iiint_R (x^2+y^2)\rho\,dV$

* for problem written in problem sheet booklet which is different from real exam.

$$I_z = \int_0^{2\pi}\int_0^5\int_1^3 \rho^4\,d\rho\,dz\,d\phi = 2\pi(3^5-1) = 484\pi$$

Using cylindrical coordinates.

$\boxed{Q7}$ (a) normal $= \underline{T_u} \times \underline{T_v} = (-e^u\cos v, -e^u\sin v, e^{2u})$

(b) total charge density $= \iint_S \sqrt{1+e^{2u}}\,dS$

$$= \int_0^\pi\int_0^1 e^u(1+e^{2u})\,du\,dv$$

$$= \pi\left(e + \frac{e^3}{3} - \frac{4}{3}\right)$$

$\boxed{Q8}$ Let $x = 4\sin\theta\cos\phi$, $y = 4\sin\theta\sin\phi$, $z = 4\cos\theta$

$0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$.

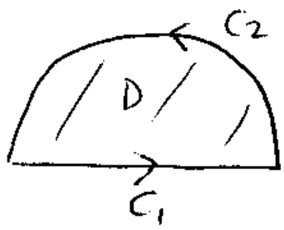
$$\text{Flux} = \iint_S \underline{F}\cdot d\underline{S} = \iint_D \underline{F}\cdot(\underline{T_\theta}\times\underline{T_\phi})\,d\theta\,d\phi$$

$$= \iint_D (-4\sin\theta\sin\phi, 4\sin\theta\cos\phi, -1)\cdot(16\sin^2\theta\cos\phi, 16\sin^2\theta\sin\phi, 16\sin\theta\cos\theta)\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} -8\sin 2\theta\,d\theta\,d\phi$$

$$= -16\pi$$

$\boxed{Q9}$



$*$ $\int_C P\,dx + Q\,dy = \int_{C_1} + \int_{C_2}$

on C_1 put $x = t$, $y = 0$, $-3 \le t \le 3$ $\quad \int_{C_1} = 0$

On C_2 put $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le \pi$

$$\int_{C_2} = \int_0^{\pi} -162 \sin^2 t \cos^2 t \; dt = \frac{-81\pi}{4}$$

$*$ $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \; dx\,dy = \iint_D -y^2 - x^2 \; dx\,dy$

$$= -\int_0^{\pi}\int_0^3 r^3 \; dr\,d\theta \quad \text{using polar coordinates.}$$

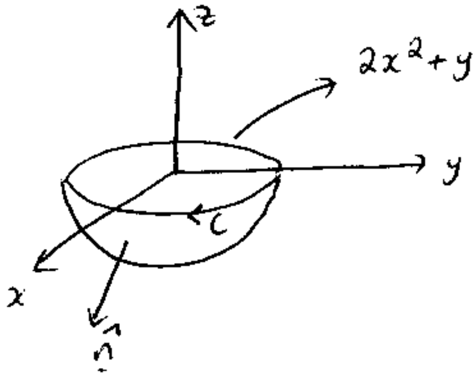
$$= -\frac{81\pi}{4}$$

$\boxed{Q10}$ (a) see lecture notes:

$$\iint_S \underline{F} \cdot d\underline{S} = \iiint_V \nabla \cdot \underline{F} \, dV$$

(b) $\iint_S \underline{F} \cdot d\underline{S} = \int_0^3 \int_0^{3-x} \int_0^{3-x-y} 2y \; dz\,dy\,dx = \frac{27}{4}.$

$\boxed{Q11}$



$2x^2 + y^2 = 4 \quad$ or $\quad x^2 + \frac{y^2}{4} = 1.$

(b) $\iint_S \nabla \times \underline{F} \cdot d\underline{S} = \iint_{S_1} \nabla \times \underline{F} \cdot d\underline{S}$ where S_1 is elliptical region $2x^2 + y^2 \le 4$, $z = 0$ by Stoke's Theorem. Here $\hat{\underline{n}} = -\hat{\underline{k}}$, $\nabla \times \underline{F} = -3\hat{\underline{k}}$

$$= \iint_{S_1} 3 \; dS$$

$$= 3 \cdot \text{area of ellipse} = \pi ab \qquad a = \sqrt{2},\; b = 2.$$

$$= 3 \cdot 2\sqrt{2}\pi = 6\sqrt{2}\pi.$$