Semester 2, 2003 620-231 Exam Answers

Q1 (a) $f_x = \dfrac{-2x}{(1+x^2+y^2)^2}$, $f_{xy} = \dfrac{8xy}{(1+x^2+y^2)^3}$

$f_y = \dfrac{-2y}{(1+x^2+y^2)^2}$, $f_{xx} = \dfrac{-2(1+y^2-3x^2)}{(1+x^2+y^2)^3}$

(b) $f(x,y)$ is continuous, C^1 and differentiable for all $(x,y) \in R^2$.

(c) $f(x,y) = \sum_{n=0}^{\infty} (-1)^n (x^2+y^2)^n$ which converges for $x^2+y^2<1$

Q2 (a) solve $(1,2) = \lambda(4x, 2y)$ where λ is the Lagrange Multiplier to give

* maximum of $f = \frac{9}{\sqrt{6}}$ at $\left(\frac{1}{\sqrt6}, \frac{4}{\sqrt6}\right)$
* minimum of $f = \frac{-9}{\sqrt{6}}$ at $\left(\frac{-1}{\sqrt6}, \frac{-4}{\sqrt6}\right)$

(b) $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/2 & y \\ 1/2 & -y \end{vmatrix} = -y$, $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ \frac{1}{2\sqrt{u-v}} & \frac{-1}{2\sqrt{u-v}} \end{vmatrix} = -\frac{1}{\sqrt{u-v}} = -\frac{1}{y}$

Q3 (a) * $\underline{c}'(t) = (1+t^2, 1-t^2, 2t)$

* $\underline{T}(t) = \frac{1}{\sqrt2}\left(1, \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$

* $k = \dfrac{|\underline{T}'(t)|}{|\underline{c}'(t)|} = \dfrac{1}{(1+t^2)^2}$

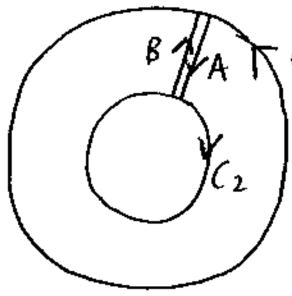
(b) $\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\nabla\left(\frac1r\right) = \nabla\cdot\left(\frac{-\underline{r}}{r^3}\right) = \underbrace{-\nabla\left(\frac{1}{r^3}\right)}_{=\frac{3}{r^5}\underline{r}}\cdot\underline{r} - \frac{1}{r^3}\underbrace{\nabla\cdot\underline{r}}_{=3} = 0$

Q4 (a) Prove $\nabla\times\underline{F} = \underline{0}$

(b) $\phi = 3x^2y^2 + y^3 + xz + C$

(c) $\displaystyle\int_{(0,0,0)}^{(1,1,2)} \underline{F}\cdot d\underline{r} = \left[3x^2y^2+y^3+xz\right]_{(0,0,0)}^{(1,1,2)} = 6$

$\boxed{Q5}$ (a)



$\circledast$ Boundary $= C_1 + A + C_2 + B$

Now $\int_A M\,dx + N\,dy = -\int_B M\,dx + N\,dy$

as width of cut shrinks to zero.

On C_1 $\quad x = 2\cos t,\ y = 2\sin t, \quad 0 \le t \le 2\pi$

$$\int_{C_1} M\,dx + N\,dy = \int_0^{2\pi} 8\sin^2 t + 4\cos^2 t\ dt = 10\pi$$

On C_2. $\quad x = \cos t, \ y = -\sin t, \quad 0 \le t \le 2\pi$

$$\int_{C_2} M\,dx + N\,dy = \int_0^{2\pi} -2\sin^2 t - \cos^2 t\ dt = -\pi$$

$$\Rightarrow \int_{\text{Boundary}} M\,dx + N\,dy = 10\pi - \pi = 9\pi.$$

$\circledast$ $\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\ dx\,dy = 3\iint_R dx\,dy = 9\pi$ (area of anulus)

(b) $\iint_S A \cdot d\underline{S} = \int_0^1 \int_0^3 \frac{xy}{\sqrt{1-y^2}} + 1\ dx\,dy = \frac{15}{2}$

$\boxed{Q6}$ (a) $h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1$

$\underline{e_\rho} = \cos\phi\,\underline{i} + \sin\phi\,\underline{j}, \quad \underline{e_\phi} = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}, \quad \underline{e_z} = \underline{k}$

(b) $\nabla f = \cos\phi\,\underline{e_\rho} - \sin\phi\,\underline{e_\phi} - 2z\,\underline{e_z}$

(c) $dV = \rho\,d\rho\,d\phi\,dz$

$$\iiint_V xyz\ dx\,dy\,dz = \int_0^1 \int_0^{\pi/2} \int_0^1 \rho^3 z \sin\phi \cos\phi\ dz\,d\phi\,d\rho$$
$$= \frac{1}{16}$$