Solvable Baumslag-Solitar Groups are Not Almost Convex

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The arguments of Cannon, Floyd, Grayson and Thurston [**CFGT**] showing that solvegeometry groups are not almost convex apply to solvable Baumslag-Solitar groups.

Introduction

The property of almost convexity was first introduced by Cannon in [**C**]. This property has very geometric in flavor, being defined in terms of the geometry of the Cayley graph. If the Cayley graph is almost convex then there are efficient algorithms for calculating in G, or, if you like, constructing the Cayley graph of G.

Given a group G and a finite generating set $C \subset G$, the *Cayley graph* of G with respect to C is the directed, labeled graph whose vertices are the elements of G and whose directed edges are the triples (g, c, g') such that $g, g' \in G$, $c \in C$ and $g' = gc$. Such an edge is directed from g to g' and is labeled by c. We denote this Cayley graph by $\Gamma = \Gamma_C(G)$. We will assume that C is closed under inverses.

A Cayley graph has a natural base point $1 \in G$ and a natural path metric $d = d_C$ which results from identifying each edge with the unit interval. Each element of G has a natural *length* $\ell(G) = \ell_C(g) = d_C(1, g)$. We define the *ball of radius* n to be

$$B(n) = \{x \in \Gamma \mid d_C(1, x) \leq n\}\,.$$

The group G is *almost convex* (k) with respect to C if there is $N = N(k)$ so that if $g, g' \in B(n)$ and $d_C(g, g') \leq k$ then there is a path p from g to g' inside $B(n)$ whose length is at most N. G is *almost convex* with respect to C if it is almost convex (k) with respect to C for each k. It is a theorem of Cannon [**C**] that if G is almost convex (2) with respect to C then G is almost convex with respect to C.

Thiel [**T**] has show that almost convexity is not a group property, i.e, that there are groups which are almost convex with respect to one generating set but not another. Almost convexity is fairly well understood for the fundamental groups of closed 3-manifold groups with uniform geometries [**SS**]. The solvegeometry case was covered in a beautiful paper of Cannon, Floyd, Grayson, and Thurston [**CFGT**]†. They show that any group which

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† Cannon has pointed out to us that there are problems with some of the details in their paper. These concern the relationship between lengths in the given group $G \subset \mathrm{Isom}(\mathbf{Sol})$ and the finite index subgroup $A = G \cap \mathbf{Sol}$. These problems are easily fixed and in our view, their paper remains quite beautiful.

acts co-compactly, discretely by isometries on **Sol** cannot be almost convex with respect to any generating set. In this paper we show that their arguments apply to solvable Baumslag-Solitar groups

$$G = B_{1,p} = \langle a, t \mid t^{-1}at = a^p \rangle$$

with $|p| > 1$.

Theorem. *Let $G = B_{1,p}$ be a solvable Baumslag-Solitar group with $|p| > 1$. Then G is not almost convex with respect to any generating set.*

Proof of the Theorem

Let $G = B_{1,p}$ with $|p| > 1$. Then G has the form

$$G = \mathbb{Z}\,[1/p] \rtimes \mathbb{Z},$$

where the generator of $\mathbb{Z}$ acts via multiplication by p. Thus each element of G has the form (f, c) where f is a fraction of the form

$$f = \frac{m}{p^n},$$

and $m, n, c \in \mathbb{Z}$. For each element $(f, 0) \in \mathbb{Z}[1/p] \subset G$, we will take $|(f, 0)| = |f|$ and if n is minimal such that $f = m/p^n$, we will say that $|p|^n$ is the *denominator* of $(f, 0)$ written $\mathrm{denom}(f, 0)$.

We fix a generating set

$$C = \{(f_i, c_i)\}$$

which we assume is closed under inverses. We take

$$\begin{aligned} c &= \max\{c_i \mid (f_i, c_i) \in C\} \\ f^* &= \max\{f_i \mid (f_i, c_i) \in C\} \\ f^{**} &= \max\{\mathrm{denom}(f_i, 0) \mid (f_i, c_i) \in C\}. \end{aligned}$$

We assume $(f_*, c) \in C$ realizes the first of these maxima. Notice that $c > 0$.

We need the following two lemmas which give information about distance with respect to the generating set C.

Lemma 1. *There is a constant M so that if $(f, 0) \in B(n)$ then either $|f| \leq M|p|^{\frac{nc}{4}}$ or* $\mathrm{denom}(f) \leq M|p|^{\frac{nc}{4}}$. *Further, both $|f| \leq M|p|^{\frac{nc}{2}}$ and* $\mathrm{denom}(f) \leq M|p|^{\frac{nc}{2}}$

Proof. First observe the following product formula:

$$(f_1, c_1) \ldots (f_n, c_n) = \left(\sum_{i=1}^{n} f_i p^{(0-c_1-\cdots-c_{i-1})}, \sum_{i=1}^{n} c_i \right).$$

Since $(f, 0) \in B(n)$, $(f, 0)$ can be written as such a product where each (f_i, c_i) is in C and $\sum_{i=1}^{n} c_i = 0$. For each i, $i = 1, \ldots, n$, we set $e_i = 0 - c_1 - \ldots - c_{i-1}$. Then e_i is positive for at most $n/2$ values of i, or e_i is negative for at most $n/2$ values of i. Suppose that e_i is positive for at most $n/2$ values of i. We then have

$$\begin{aligned}
|f| &= \sum_{i=1}^{n} f_i p^{(0-c_1-\cdots-c_{i-1})} \\
&\leq f^* \sum_{i=1}^{n} |p|^{(0-c_1-\cdots-c_{i-1})} \\
&= f^* \left(\sum_{e_i \leq 0} |p|^{e_i} + \sum_{e_i > 0} |p|^{e_i} \right) \\
&\leq f^* \left(n + \sum_{e_i > 0} |p|^{e_i} \right).
\end{aligned}$$

Let us enumerate the $\{e_i \mid e_i > 0\}$ as i increases, so that these are the $\lfloor n/2 \rfloor$-tuple $(e_{i_1}, e_{i_2}, \ldots, e_{i_{\lfloor \frac{n}{2} \rfloor}})$. If there are less than $\lfloor n/2 \rfloor$ of these, we will consider any final entries in this list to be 0. We now take the m-tuple $(e'_{i_1}, e'_{i_2}, \ldots, e'_{i_m})$ to be $(c, 2c, \ldots, \frac{nc}{4}, \frac{nc}{4}, \ldots, 2c, c)$. Here m is either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$. It is not hard to see that for each j, $e_{i_j} \leq e'_{i_j}$. Consequently,

$$|f| \leq f^* \left(n + 2|p|^c + 2|p|^{2c} + \cdots + 2|p|^{\frac{nc}{4}} \right),$$

and thus for suitable choice of M', $|f| \leq M'|p|^{\frac{nc}{4}}$.

On the other hand if more than $n/2$ of the e_i are positive, then less than $n/2$ of them are negative, and in particular, the most negative any of these can be is $\frac{-nc}{4}$. It immediately follows that

$$\operatorname{denom}(f, 0) \leq f^{**}|p|^{\frac{nc}{4}}.$$

Taking $M = \max\{M', f^{**}\}$ completes the proof of the first part of Lemma 1. After suitably enlarging M, a completely similar proof gives the simultaneous bound on $|f|$ and $\operatorname{denom}(f)$ ∎

We also need the following observation.

Lemma 2. *If $h, h' \in \mathbb{Z}[1/p] \subset G$ with $d_C(h, h') \leq r$ then $||h| - |h'|| \leq M|p|^{\frac{rc}{2}}$ and $|\operatorname{denom}(h) - \operatorname{denom}(h')| \leq M|p|^{\frac{rc}{2}}$.*

Proof. If $d_C(h, h') \leq r$ then (using additive notation in $\mathbb{Z}[1/p]$) $h - h' \in B(r)$. Thus

$$||h| - |h'|| \leq |h - h'| \leq M|p|^{\frac{rc}{2}}.$$

On the other hand $h = h' + (h - h')$ and $h' = h - (h - h')$ so we have

$$\operatorname{denom}(h) \leq \max\{\operatorname{denom}(h'), \operatorname{denom}(h - h')\} \leq \operatorname{denom}(h') + \operatorname{denom}(h - h'),$$

and

$$\operatorname{denom}(h') \leq \max\{\operatorname{denom}(h), \operatorname{denom}(h - h')\} \leq \operatorname{denom}(h) + \operatorname{denom}(h - h').$$

Consequently,

$$|\operatorname{denom}(h) - \operatorname{denom}(h')| \leq \operatorname{denom}(h - h') \leq M|p|^{\frac{rc}{2}}.$$

∎

We now return to the proof of the Theorem. For each $k > 0$ we take

$$T_k = (f_*, c)^{-k}(1,0)(f_*, c)^k = (p^{ck}, 0)$$
$$S_k = (f_*, c)^k(1,0)(f_*, c)^{-k} = (p^{-ck}, 0)$$

We then have $T_k S_k = S_k T_k$. For some j which we will fix later, we take

$$\alpha_k = S_k T_k (f_*, c)^{-j}$$
$$\beta_k = T_k S_k (f_*, c)^j$$

If we take $\ell = \ell_C(1,0)$ and $k > j$, then α_k and β_k both lie in $B(4k + 2\ell - j)$ and within distance $2j$ of each other.

Suppose that, contrary to hypothesis, G is almost convex. Then there is a constant $N = N(2j)$ so that α_k and β_k are joined by a path of length at most N lying entirely within $B(4k+2\ell-j)$. The second coordinates of points along this path vary from $-jc$ to $+jc$ changing by at most $\pm c$ along each edge. In particular, this path must pass through a point P'_k of the form (g_k, i) with $|i| \leq c/2$. We take

$$\epsilon = \max\left\{\ell_C(0,i) \mid |i| \leq c/2\right\}.$$

It follows that the point

$$P_k = (g_k, 0)$$

lies within $B(4k + 2\ell + \epsilon - j)$ and within distance $N/2 + \epsilon + j$ of $S_k T_k$. Notice that the distance from P_k to $S_k T_k$ is bounded by a constant independent of k. It is this fact that we will contradict, thus showing G is not almost convex in the given generating set.

From Lemma 1 above it follows that either

$$|P_k| \leq M|p|^{c(k + \frac{2\ell+\epsilon}{4} - \frac{j}{4})}$$

or

$$\mathrm{denom}(P_k) \leq M|p|^{c(k + \frac{2\ell+\epsilon}{4} - \frac{j}{4})}.$$

Let us fix j so that

$$|p|^{\frac{j}{4}} > M|p|^{\frac{2\ell+\epsilon}{4c}+1}$$

and hence

$$M|p|^{c(k+\frac{2\ell+\epsilon}{4} - \frac{j}{4})} = |p|^{kc}(M|p|^{\frac{2\ell+\epsilon}{4c}}|p|^{-\frac{j}{4}}) \leq |p|^{kc-1}.$$

It then follows that either

$$|P_k| \leq |p|^{kc-1}$$

or

$$\mathrm{denom}(P_k) \leq |p|^{kc-1}.$$

Now $|S_k T_k| = |p^{kc} + p^{-kc}|$ and $\mathrm{denom}(S_k T_k) = |p|^{kc}$. Hence either

$$|S_k T_k| - |P_k| \geq |p^{kc} + p^{-kc}| - |p|^{kc-1} = |p|^{kc-1}(|p + p^{-2kc+1}| - 1) > |p|^{kc-1}$$

or

$$\mathrm{denom}(S_k T_k) - \mathrm{denom}(P_k) \geq |p|^{kc} - |p|^{kc-1} = |p|^{kc-1}(|p| - 1) \geq |p|^{kc-1}.$$

In either case by Lemma 2, as $k \to \infty$ the distance $d_C(P_k, S_k T_k)$ increases without bound. But our assumption of almost convexity implied $d_C(P_k, S_k T_k) \leq N/2 + \epsilon + j$ which is a constant. This is a contradiction. Hence G is not almost convex. ∎

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