SUBGROUPS OF DIRECT PRODUCTS WITH A FREE GROUP

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Abstract. A result of Baumslag and Roseblade says that a finitely presented subgroup of a direct product of two free groups is either free or virtually a direct product of free groups. We give a particularly straightforward proof of this result using an argument of a more general character. Additional applications of the method are given.

It is well known that the finitely generated subgroups of the direct product of two finitely generated free groups can have unsolvable membership problem and unsolvable conjugacy problem (see for instance [8]). The particular subgroups in question are fibre products of maps from a free group onto a group with unsolvable word problem. Grunewald [6] showed such groups are not finitely presented. More remarkably, Baumslag and Roseblade [2] showed that the only finitely presented subgroups of a direct product of two free groups are either free or are virtually a direct product of two free groups. In this note we present a particularly straightforward proof of their result based on the following theorem which has wider applications.

Theorem 1. *Let $A \times F$ be the direct product of a group A with a free group F. Suppose that $G \leq A \times F$ is a subgroup which intersects F non-trivially.*

(1) *If G is finitely generated, then G has a subgroup G_0 of finite index which is an HNN-extension of the form*

$$G_0 = \langle D, t \mid t^{-1}bt = b, \ \forall b \in L \rangle$$

where D is finitely generated and $L = G \cap A \leq D$.

(2) *If G is finitely presented, then $L = G \cap A$ is finitely generated.*

Proof. Since subgroups of free groups are free, we may assume that both of the projection induced maps $p_A : G \to A$ and $p_F : G \to F$ are surjective. Since G intersects F non-trivially, there is some $1 \neq t \in G \cap F$. By a theorem of Marshall Hall [7] (see also [10]), F contains a subgroup M of finite index which has the cyclic group $< t >$ as a

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free factor. Thus M has a basis of the form $\{t, s_1, s_2, \ldots, s_n\}$. For each $i = 1, \ldots, n$ pick a lift $\hat{s}_i \in p_F^{-1}(s_i)$.

If we put $L = A \cap G$, then $G_0 = p_F^{-1}(M)$ is a subgroup of finite index in G which has the structure of an HNN-extension

$$G_0 = \langle L, t, \hat{s}_1, \ldots, \hat{s}_n \mid t^{-1}bt = b, \hat{s}_i^{-1} b \hat{s}_i = \phi_i(b), i = 1, \ldots, n, \forall b \in L\rangle$$

where ϕ_i is the automorphism of L induced by conjugation by $\hat{s}_i$.

Suppose now that G is finitely generated (so that $n < \infty$ in the above). Then G_0 is also finitely generated and hence is generated by a finite set of elements $a_1, \ldots, a_k$ in L together with $t, \hat{s}_1, \ldots, \hat{s}_n$. Notice that since t acts trivially by conjugation, L is generated by the a_i's together with their conjugates by words in the $\hat{s}_i$. Let D be the subgroup generated by the a_i and $\hat{s}_i$, that is $D = \langle a_1, \ldots, a_k, \hat{s}_1, \ldots, \hat{s}_n\rangle$. Of course L is a subgroup of D. Now G_0 also has the structure of an HNN-extension of D, namely

$$G_0 = \langle D, t \mid t^{-1}bt = b, \ \forall b \in L\rangle.$$

Assume now that G is finitely presented. Then G_0 is also finitely presented since it has finite index in G. The following lemma, originally proved by Baumslag [1] for amalgamated free products, implies that L must also be finitely generated and completes the proof of Theorem 1. □

Lemma 2 (HNN analog of Baumslag[1]). *Let*

$$H = \langle D, t \mid t^{-1}bt = \phi(b), \ \forall b \in L\rangle$$

be the HNN-extension of the finitely generated group D with associated subgroups L and $\phi(L)$. If H is finitely presented, then L is finitely generated.

Sketch: Since H is finitely presented it can be defined by finitely many relations of D together with finitely many $t^{-1}b_it = \phi(b_i), i = 1, \ldots, m$. Then the normal form theorem for HNN-extensions implies $b_1, \ldots, b_m$ generate L. □

The theorem of Baumslag and Roseblade is an easy consequence of Theorem 1.

Corollary 3 (Baumslag and Roseblade [2]). *Let $F_1 \times F_2$ be the direct product of two free groups F_1 and F_2. Suppose that $G \leq F_1 \times F_2$ is a subgroup and define $L_i = G \cap F_i$.*

1. *If either $L_i = 1$ then G is free.*
2. *If both L_i are non-trivial and one of them is finitely generated, then $L_1 \times L_2$ has finite index in G.*
3. *Otherwise, G is not finitely presented.*

Proof. Since subgroups of free groups are free, we can assume both of the induced projections $p_i : G \to F_i$ are surjective. The subgroups L_i

are normal in G and, since the p_i are surjective, each L_i is also normal in F_i. If one of the L_i is trivial, say $L_1 = 1$, then the projection p_2 is an isomorphism and hence G is free.

Suppose both L_i are non-trivial and one of them, say L_1, is finitely generated. Then L_1 is a non-trivial, finitely generated normal subgroup of F_1 and hence has finite index. Thus $p_1^{-1}(L_1) = L_1 \times L_2$ has finite index in G.

The final assertion now follows immediately from Theorem 1. □

Two recent proofs of the Baumslag-Roseblade Theorem which use geometric methods can be found in [9] and [4].

In [2] a spectral sequence argument is used to show that if A is free then $H_1(L, \mathbb{Z})$ is a section (quotient of a subgroup) of $H_2(G_0, \mathbb{Z})$. If L is not finitely generated, it follows that $H_2(G_0, \mathbb{Z})$ is not finitely generated and so G_0 is not finitely presented.

In the context of the above Lemma 2, the exact Mayer-Vietoris sequence for the HNN-extension (see [5]) is

$$\cdots \to H_{n+1}(G_0, \mathbb{Z}) \to H_n(L, \mathbb{Z}) \to H_n(D, \mathbb{Z}) \to \cdots$$

where the map $H_n(L, \mathbb{Z}) \to H_n(D, \mathbb{Z})$ is the difference of the induced maps on associated subgroups. But this is the zero map since t commutes with L. This proves the following:

Theorem 4. *Let $A \times F$ be the direct product of a group A with a free group F. Suppose that $G \leq A \times F$ is a subgroup which intersects F non-trivially. Let $L = G \cap A$. Then G has a subgroup G_0 of finite index with $L \leq G_0$ such that $H_n(L, \mathbb{Z})$ is a quotient of $H_{n+1}(G_0, \mathbb{Z})$ for $n \geq 0$.* □

By a *surface group* we mean the fundamental group of a connected 2-manifold. Such a group is either free (of finite or countably infinite rank) or else has a subgroup of index at most two which is trivial or has a presentation of the form $\mathcal{P}_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g] \rangle$. Recently Bridson, Howie, Miller and Short [3] have given a generalization of Baumslag-Roseblade to subgroups of a direct product of several surface groups, concentrating on homological aspects.

Like free groups, the groups of closed, orientable surfaces of genus $g > 1$ have the property that a non-trivial finitely generated normal subgroup has finite index. So the above arguments carry over to the product of such a surface group and a free group. This proves the following for genus $g > 1$:

Theorem 5. *Suppose that A is the group of a closed, orientable surface and that F is free. If $G \leq A \times F$ is a finitely presented subgroup of their direct product, then G is either a surface group or virtually a direct product of surface groups.*

In the case genus $g = 1$, $A = \mathbb{Z} \times \mathbb{Z}$ is the group of the torus and so every subgroup is finitely generated and abelian. Thus every finitely generated subgroup G of $(\mathbb{Z} \times \mathbb{Z}) \times F$ is finitely presented. Also any subgroup $G \leq (\mathbb{Z} \times \mathbb{Z}) \times F$ is isomorphic to the direct product of $G \cap (\mathbb{Z} \times \mathbb{Z})$ with the projection of G into F. So the Theorem holds for genus $g = 1$ as well, albeit for a much simpler reason.

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