THE WORD PROBLEM IN GROUPS OF COHOMOLOGICAL DIMENSION 2

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Abstract. We show that the finitely presented groups with unsolvable word problem given by the Boone-Britton construction have cohomological dimension 2. More precisely we show these groups can be obtained from a free group by successively forming HNN-extensions where the associated subgroups are finitely generated free groups. Also the presentations obtained for these groups are aspherical. Using this we show there is no algorithm to determine whether a presentation is aspherical. There is no algorithm to determine whether a finite 2-complex is aspherical.

1. Introduction

Fundamental algorithms in combinatorial group theory (the original due to Nielsen [6]) enable one to decide membership in a finitely generated subgroup of a free group. It follows that the free product of two free groups with finitely generated amalgamation has a solvable word problem. Similarly, an HNN-extension of a free group with finitely generated associated subgroups has a solvable word problem. (However, such groups can have unsolvable conjugacy problem and the problem of deciding whether an arbitrary pair of them are isomorphic is unsolvable - see [7].) More generally, the fundamental group of a finite graph of groups whose edge and vertex groups are all finitely generated and free is finitely presented and has a solvable word problem.

Using the Mayer-Vietoris sequence for the (co)homology of a graph of groups [3], it is easy to check that such groups have cohomological dimension ≤ 2. In fact, if the vertex groups of a graph of groups have cohomological dimension ≤ 2 and the edge groups are all free, then its fundamental group will have cohomological dimension ≤ 2.

In particular, groups obtained from free groups by repeatedly forming HNN-extensions and amalgamated free products where the associated or amalgamated subgroups are always free will have cohomological dimension ≤ 2.

The purpose of this article is to establish the following:

Theorem 1.1. *There exists a finitely presented group $\mathcal{B}$ of cohomological dimension 2 having unsolvable word problem. Indeed, $\mathcal{B}$ can be obtained from a free group by applying three successive HNN-extensions where the associated subgroups are finitely generated free groups.*

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The group in question is one constructed by Boone as later modified by Boone, Britton and the authors. The construction is described in Rotman's textbook [9]. The verification that the group has the required properties is somewhat technical and requires detailed understanding of the proof in [9]. In fact we discovered this result some years ago and have even mentioned it in print (see for example [8] p.29). As some interesting applications have recently been found [1], it seems timely to publish a proof.

We will write down a presentation $\mathcal{P}_{\mathcal{B}}$ of $\mathcal{B}$ which exhibits its structure as a successive HNN-extension with free associated subgroups. Each of the relators of $\mathcal{P}_{\mathcal{B}}$ is non-empty, reduced and not conjugate to another relator or its inverse; that is, $\mathcal{P}_{\mathcal{B}}$ is *concise* in the sense of [2]. No relator of $\mathcal{P}_{\mathcal{B}}$ is a proper power. In [2] such presentations are shown to be aspherical in a number of senses. In particular, a presentation is *aspherical* means the standard 2-complex associated with it is topologically aspherical. By applying [2] we conclude the following:

Corollary 1.2. *The finite presentation $\mathcal{P}_{\mathcal{B}}$ for the group $\mathcal{B}$ having unsolvable word problem is aspherical, combinatorially aspherical, diagramatically aspherical and Cohen-Lyndon aspherical.*

Applying one of the constructions used in proving a theorem of Adian and Rabin showing one cannot recognise the trivial group (as for instance in [8] Theorem 3.3 and Lemma 3.6), we obtain a collection of presentations Π_w indexed by words w in the generators of $\mathcal{B}$ such that:

1. each Π_w is concise, has more relators than generators, and no relator is a proper power;
2. if $w \neq_{\mathcal{B}} 1$, then the group presented by Π_w is infinite and the presentation Π_w is aspherical, combinatorially aspherical, diagramatically aspherical and Cohen-Lyndon aspherical;
3. if $w =_{\mathcal{B}} 1$, then the group presented by Π_w is the trivial group and Π_w is not aspherical in any of these senses.

Most of this follows easily from [2] and the construction [8] which applies ordinary free products with free groups and amalgamated free products with free amalgamated subgroups in the case $w \neq_{\mathcal{B}} 1$. In case $w =_{\mathcal{B}} 1$, the standard finite 2-dimensional CW-complex K_w associated with Π_w is simply connected, has a single 0-cell, and has more 2-cells than 1-cells and so is not aspherical. Hence if $w =_{\mathcal{B}} 1$, then Π_w is not aspherical in any of the above senses. Since $\mathcal{B}$ has unsolvable word problem, we conclude the following:

Corollary 1.3. *There is no algorithm to determine of an arbitrary finite presentation Π of a group whether or not Π is any of aspherical, combinatorially aspherical, diagramatically aspherical or Cohen-Lyndon aspherical.*

Corollary 1.4. *There is no algorithm to determine of an arbitrary finite 2-dimensional CW-complex K whether or not K is aspherical.*

The principal difficulty in the proof of the main theorem is showing that a certain subgroup is free. Our proof of this relies on an argument due to Post [4] which involves the deterministic nature of a Turing machine T. Computations in a Turing machine can be viewed as a directed graph with instantaneous descriptions as vertices and operations in the machine as edges. Since T is deterministic, the component of the halting state of T is actually a tree. It is this fact which underlies Post's argument.

2. The structure of the group $\mathcal{B} = \mathcal{B}(T)$

The construction of Boone's group begins with a Turing machine T having an unsolvable halting problem. A construction of Markov and Post is then applied to obtain a finitely presented semigroup $\Gamma(T)$ of the form

$$\Gamma(T) = \langle q, q_0, \ldots, q_N, s_0, \ldots, s_M \mid F_i q_{i_1} G_i = H_i q_{i_2} K_i, i \in I\rangle$$

where the F_i, G_i, H_i, K_i are positive s-words and $q_{i_j} \in \{q, q_0, \ldots, q_N\}$. By an *$s$-word* we mean a word on the symbols $s_0, \ldots, s_M$ and their inverses. A word is *positive* if it contains no s_i^{-1} symbols. Of course the above presentation for $\Gamma(T)$ is a semigroup presentation and the symbols s_i^{-1} are not present in $\Gamma(T)$. We will give more details concerning $\Gamma(T)$ later.

We use $X \equiv Y$ to mean the words X and Y are identical (letter by letter). If $X \equiv s_{b_1}^{e_1} \cdots s_{b_m}^{e_m}$ is an s-word, we define $X^\# \equiv s_{b_1}^{-e_1} \cdots s_{b_m}^{-e_m}$. Note that $X^\#$ is not the same as X^{-1}. Also, if X and Y are s-words, then $(X^\#)^\# \equiv X$ and $(XY)^\# = X^\# Y^\#$.

The group $\mathcal{B} = \mathcal{B}(T)$ constructed (essentially) by Boone is then presented as follows:

generators: $q, q_0, \ldots, q_N, s_0, \ldots, s_M, r_i, i \in I, x, t, k$;

relations: for all $i \in I$ and all $b = 0, \ldots, M$,

$$\left.\begin{array}{l} \left.\begin{array}{l} \left.\begin{array}{l} xs_b = s_b x^2 \end{array}\right] \Delta_1 \\ r_i s_b = s_b x r_i x \\ r_i^{-1} F_i^\# q_{i_1} G_i r_i = H_i^\# q_{i_2} K_i \end{array}\right] \Delta_2 \\ tr_i = r_i t \\ tx = xt \end{array}\right] \Delta_3$$

$$\begin{array}{l} kr_i = r_i k \\ kx = xk \\ k(q^{-1}tq) = (q^{-1}tq)k \end{array}$$

The subsets $\Delta_1 \subset \Delta_2 \subset \Delta_3$ of the relations each define a presentation of a group $\mathcal{B}_i$ generated by the symbols appearing in the Δ_i. Also let $\mathcal{B}_0$ denote the infinite cyclic group generated by x, and let Q denote the free group with basis $\{q, q_0, \ldots, q_N\}$. The following is Lemma 12.11 in [9]:

Lemma 2.1. *In the chain*

$$\mathcal{B}_0 \leq \mathcal{B}_1 \leq \mathcal{B}_1 * Q \leq \mathcal{B}_2 \leq \mathcal{B}_3 \leq \mathcal{B}$$

each group is an HNN-extension of its predecessor; moreover, the free product $\mathcal{B}_1 * Q$ *is an HNN-extension of* $\mathcal{B}_0$. □

We want to look at $\mathcal{B}_3$ somewhat differently. Let $\mathcal{A}$ be the group with presentation

$$\mathcal{A} = \langle x, s_0, \ldots, s_M, q, q_0, \ldots, q_N, t \mid t^{-1}xt = x, s_b^{-1}xs_b = x^2, b = 0, \ldots, M\rangle.$$

Then $\mathcal{A}$ is an HNN-extension of $\mathcal{B}_0 = \langle x \mid \rangle$ with all the listed generators other than x as stable letters. In particular, the associated subgroups are either cyclic or trivial and hence are finitely generated free groups. We now have the following easy fact.

Lemma 2.2. *In the chain*

$$\mathcal{B}_0 \leq \mathcal{A} \leq \mathcal{B}_3$$

each group is an HNN-extension of its predecessor; moreover, the associated subgroups are finitely generated free groups.

Proof. Let $\mathcal{F}$ denote the free group on the stable letters of $\mathcal{A}$ and $\phi : \mathcal{A} \to \mathcal{F}$ the retraction sending stable letters to themselves and x to 1. One of the associated subgroups for r_i is generated by the $M+3$ elements $\{F_i^{\#}q_{i_1}G_i, t, s_0x, \ldots, s_Mx\}$. The image of this subgroup under ϕ is easily seen to be the (free) subgroup of $\mathcal{F}$ generated by $\{q_{i_1}, t, s_0, \ldots, s_M\}$ which has rank $M+3$. Hence the associated subgroup is free on the given generators. Similar considerations show the other associated subgroup for r_i is free. This completes the proof □

Lemma 2.3. *The elements* $\{x, r_i, i \in I\}$ *freely generate a free subgroup of* $\mathcal{B}_3$.

Proof. Consider $\mathcal{B}_3$ as an HNN-extension of $\mathcal{A}$ and adopt the notation in the previous proof. If w is a (non-empty) freely reduced word in x and the r_i and if $w = 1$ in $\mathcal{B}_3$ then Britton's Lemma implies w contains a subword of the form $r_i^{-e}x^nr_i^e$ and x^n belongs to one of the associated subgroups of r_i depending on the sign of e. But $x \in \ker\phi$ while $\ker\phi$ intersects the associated subgroups in the identity. Thus $n = 0$ and w is not freely reduced which is a contradiction. □

What is needed to finish the proof of the main theorem is the following improvement of the previous lemma.

Lemma 2.4 (Main Lemma)**.** *The elements* $\{q^{-1}tq, x, r_i, i \in I\}$ *freely generate a free subgroup of* $\mathcal{B}_3$. *Hence in the chain*

$$\mathcal{B}_0 \leq \mathcal{A} \leq \mathcal{B}_3 \leq \mathcal{B}$$

each group is an HNN-extension of its predecessor; moreover, the associated subgroups are finitely generated free groups.

The main theorem is an immediate consequence of this result. We are going to prove this "Main Lemma" by contradiction. We begin with the following.

Lemma 2.5. *Assume there is some non-empty freely reduced word W in the generators $\{q^{-1}tq, x, r_i, i \in I\}$ such that $W = 1$ in $\mathcal{B}_3$. Then there are non-empty freely reduced words L_1 and L_2 on $\{x, r_i, i \in I\}$ such that $L_1 q L_2 = q$ in $\mathcal{B}_3$.*

Proof. By Lemma 2.3, W must contain $q^{-1}tq$ or $q^{-1}t^{-1}q$ and hence must involve t. Now $\mathcal{B}_3$ is the HNN-extension of $\mathcal{B}_2$ by the stable letter t having associated subgroup freely generated by $\{x, r_i, i \in I\}$ and t commutes with these elements. So by Britton's Lemma, W must contain a subword $t^{-e} q L_2 q^{-1} t^{e}$ such that $q L_2 q^{-1} = L_1^{-1}$ in $\mathcal{B}_2$, where L_1 and L_2 are words in the associated subgroup. Hence $L_1 q L_2 = q$ in $\mathcal{B}_2$ as desired. ☐

3. Reduction to proofs in $\Gamma(T)$

A word Σ in $\mathcal{B}_3$ is *special* if $\Sigma \equiv X^{\#} q_j Y$ where X and Y are positive s-words and $q_j \in \{q, q_0, \dots, q_N\}$. If $\Sigma \equiv X^{\#} q_j Y$ is special, then we define $\Sigma^* \equiv XqY$, which is a word in the semigroup $\Gamma(T)$.

By a *proof* in $\Gamma(T)$ is meant a sequence of words

$$w_1 \to w_2 \to \dots \to w_n$$

on the generators where $w_j \to w_{j+1}$ means that w_j yields w_{j+1} by a single application of a relation of $\Gamma(T)$. We say this is a *forward* application if the left-hand side of the relation is replaced by the right-hand side and a *backward* application if the right-hand side is replaced by the left-hand side. A proof is *reversal-free* if it does not contain an application $w_{j-1} \to w_j$ of a relation immediately followed by an application $w_j \to w_{j+1}$ of the same relation in the opposite direction. (Note that when w_j contains only a single letter from $\{q, q_0, \dots, q_N\}$ such a reversal would mean $w_{j-1} \equiv w_{j+1}$.)

The inductive proof of Lemma 12.15 in [9] actually shows the following more detailed result:

Lemma 3.1. *Suppose that Σ_1 and Σ_2 are special words and that L_1 and L_2 are freely reduced words on $\{x, r_i, i \in I\}$. If the equation*

$$L_1 \Sigma_1 L_2 = \Sigma_2$$

holds in $\mathcal{B}_2$, then for some choice of $e_i = \pm 1$ and integers m_i, n_i,

$$L_1 \equiv x^{m_p} r_{i_p}^{-e_p} \cdots x^{m_1} r_{i_1}^{-e_1} x^{m_0},$$

$$L_2 \equiv x^{n_0} r_{i_1}^{e_1} x^{n_1} \cdots r_{i_p}^{e_p} x^{n_p},$$

and the sequence

$$(i_1, e_1), \dots, (i_p, e_p)$$

determines a reversal-free proof that $\Sigma_1^* = \Sigma_2^*$ in $\Gamma(T)$. Under this correspondence (i_j, e_j) corresponds to a forward application of the j-th relation if $e_j = +1$ and a backward application of the j-th relation if $e_j = -1$.

Proof. The only point not covered by the argument given in [9] is the fact that the resulting proof in $\Gamma(T)$ is reversal-free. We need to show that if $i_j = i_{j+1}$, then $e_j = e_{j+1}$. To obtain a contradiction we may without loss of generality assume that $p = 2$ and $i = i_1 = i_2$ and $e_1 = -e_2$. We consider the case $e_1 = 1$ (the case $e_1 = -1$ is similar). The left hand side of the equation $L_1\Sigma_1 L_2 = \Sigma_2$ is transformed to the right by two successive pinches,

$$\begin{aligned}
L_1\Sigma_1 L_2 &\equiv x^{m_2} r_i x^{m_1} r_i^{-1} x^{m_0} X_1^{\#} q_j Y_1 x^{n_0} r_i x^{n_1} r_i^{-1} x^{n_2} \\
&\equiv x^{m_2} r_i x^{m_1} r_i^{-1} x^{m_0} u(s_b) F_i^{\#} q_{i_1} G_i v(s_b) x^{n_0} r_i x^{n_1} r_i^{-1} x^{n_2} \\
&= x^{m_2} r_i x^{m_1} r_i^{-1} u(s_b x) F_i^{\#} q_{i_1} G_i v(s_b x) r_i x^{n_1} r_i^{-1} x^{n_2} \\
&= x^{m_2} r_i x^{m_1} u(s_b x^{-1}) H_i^{\#} q_{i_2} K_i v(s_b x^{-1}) x^{n_1} r_i^{-1} x^{n_2} \\
&= x^{m_2} r_i u(s_b x^{-1}) H_i^{\#} q_{i_2} K_i v(s_b x^{-1}) r_i^{-1} x^{n_2} \\
&= x^{m_2} u(s_b x) F_i^{\#} q_{i_1} G_i v(s_b x) x^{n_2} \\
&= X_1 q_j Y_1 \equiv \Sigma_2
\end{aligned}$$

first by r_1 then by r_1^{-1}. Here $X_1^{\#} \equiv u(s_b) F_i^{\#}$ and $Y_1 \equiv G_i v(s_b)$, where $u(s_b)$ and $v(s_b)$ are s-words. The word $u(s_b x)$ is the word obtained from u by replacing each s_b by $s_b x$, and similarly for v.

Also all equations in this calculation except for the pinches are either identities or are consequences of the relations in $\mathcal{B}_1 * Q$. In particular, in $\mathcal{B}_1 * Q$

$$x^{m_1} u(s_b x^{-1}) H_i^{\#} q_{i_2} K_i v(s_b x^{-1}) x^{n_1} = u(s_b x^{-1}) H_i^{\#} q_{i_2} K_i v(s_b x^{-1})$$

and so $x^{m_1} u(s_b x^{-1}) H_i^{\#} = u(s_b x^{-1}) H_i^{\#}$ and $K_i v(s_b x^{-1}) x^{n_1} = K_i v(s_b x^{-1})$ in $\mathcal{B}_1$. Thus $x^{m_1} = x^{n_1} = 1$, and so $m_1 = n_1 = 0$ and L_1 and L_2 are not freely reduced. This is a contradiction and the proof of the lemma is complete. □

Combining this lemma with Lemma 2.5 we obtain the following result which reduces the Main Lemma to a fact about proofs in $\Gamma(T)$:

Corollary 3.2. *Assume there is some non-empty freely reduced word W in the generators $\{q^{-1}tq, x, r_i, i \in I\}$ such that $W = 1$ in $\mathcal{B}_3$. Then there is a non-trivial, reversal-free proof in $\Gamma(T)$*

$$q \to w_2 \to \dots w_{n-1} \to q.$$

□

In the next section we will show that, because T is deterministic, there is no such non-trivial, reversal-free proof.

4. Reversals and the determinism of T

We need to look at the construction due to Post [4] of $\Gamma(T)$ from the Turing machine T in some detail. Suppose the Turing machine T has alphabet $s_0, s_1, \dots, s_M$ and internal states $q_0, q_1, \dots, q_N$ with q_1 as the start state and q_0 as the unique halting state.

Let $\gamma(T)$ be the semigroup with presentation

$$\gamma(T) = \langle h, s_0, s_1, \dots, s_M, q, q_0, q_1, \dots, q_N \mid R(T)\rangle$$

where the relations $R(T)$ are

$$q_i s_j = q_l s_k \text{ if } q_i s_j s_k q_l \in T.$$

and for all $b = 0, 1, \dots, M$:

$$\begin{array}{rcll} q_i s_j s_b & = & s_j q_l s_b & \text{if } q_i s_j R q_l \in T, \\ q_i s_j h & = & s_j q_l s_0 h & \text{if } q_i s_j R q_l \in T, \\ s_b q_i s_j & = & q_l s_b s_j & \text{if } q_i s_j L q_l \in T, \\ h q_i s_j & = & h q_l s_0 s_j & \text{if } q_i s_j L q_l \in T, \end{array}$$

$$\begin{array}{rcl} q_0 s_b & = & q_0 \\ s_b q_0 h & = & q_0 h \\ h q_0 h & = & q \end{array}$$

We recall that intuitively the symbol h marks the ends of the tape the Turing machine T is reading.

The semigroup $\Gamma(T)$ used in the previous sections is obtained from $\gamma(T)$ by regarding the h as the last s-letter and reindexing these s-letters so that $h = s_M$. (So $\Gamma(T)$ is just $\gamma(T)$ in a slightly revised notation.)

A word of $\gamma(T)$is *h-special* if it has the form huq_jvh where u and v are positive s-words and $q_j \in \{q, q_0, q_1, \dots, q_N\}$. Since only the last relation $hq_0h = q$ creates or destroys h, we observe the following:

Lemma 4.1. *If w_1 and w_2 are words of $\gamma(T)$ and neither if them is the word q, and if $w_1 \to w_2$ is an application of a relation, then w_1 is h-special if and only if w_2 is h-special.* □

Lemma 4.2. *Let $w \equiv huq_jvh$ be an h-special word of $\gamma(T)$. Then at most one of the relations of $\gamma(T)$ has a forward application to w.*

Proof. Clearly we may assume $q_j \not\equiv q$. If $q_j \not\equiv q_0$ the conclusion is immediate from the deterministic nature of the Turing machine T. So suppose $q_j \equiv q_0$. If u and v are empty then there is a forward application of $hq_0h = q$, but clearly this is the only possibility. If u is non-empty and v is empty, then just one of the relations $s_bq_0h = q_0h$ has a forward application. Finally if v is non-empty, just one of the relations $q_0s_b = q_0$ has a forward application. This completes the proof. □

Lemma 4.3. *Let $w \equiv huq_jvh$ be an h-special word of $\gamma(T)$. If*

$$w \equiv w_1 \to w_2 \to \cdots \to w_{n-1} \to q$$

is a proof in $\gamma(T)$, then either this proof contains a reversal or all the applications of relations are forward.

Proof. Clearly the application $w_{n-1} \to q$ is forward and $w_{n-1} \equiv hqh$. Assume that not all applications of relations in the given proof are forward. Suppose that $w_{j-1} \to w_j$ is the last backward application of a relation in the proof. Then $w_j \to w_{j+1}$ and all subsequent applications are forward. But $w_{j-1} \to w_j$ backward implies that $w_j \to w_{j-1}$ is a forward application of the same relation.

Now w_{n-1} is h-special and none of $w_j, \ldots, w_{n-1}$ could be q since only a backward application applies to q. So each of $w_j, \ldots, w_{n-1}$ must be h-special. Hence at most one relation has a forward application to w_j by the previous result. Thus $w_{j+1} \equiv w_{j-1}$ and hence $w_{j-1} \to w_j \to w_{j+1}$ is a reversal. This proves the lemma. □

Corollary 4.4. *Every non-trivial proof in* $\Gamma(T)$ *of the form*

$$q \to w_2 \to \cdots w_{n-1} \to q$$

contains a reversal.

Proof. Clearly $w_2 \equiv hq_0h$. So $w_2 \to \cdots w_{n-1} \to q$ either contains a reversal (as claimed), or consists of all forward applications. Assume all the applications $w_2 \to \cdots w_{n-1} \to q$ are forward. But the only forward application of a relation to $w_2 \equiv hq_0h$ is $hq_0h \to q$ so the given proof begins with a reversal. This proves the corollary. □

Corollaries 3.2 and 4.4 together immediately imply the Main Lemma 2.4, and so the proof of our theorem is complete. □

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