THE WORD PROBLEM IN QUOTIENTS OF A GROUP

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Every finitely presented group G has a quotient group with solvable word problem – namely the trivial group. We will construct a finitely presented group G such that every non-trivial quotient of G has unsolvable word problem (of degree $\mathbf{0}'$ in case the quotient is recursively presented) and such that every countable group is embedded in some quotient of G.

A presentation of a group is an ordered pair $\langle S; D\rangle$ where S is a set and D a collection of words on the elements of S and their inverses. The group G presented by $\langle S; D\rangle$ is the quotient of the free group on S by the normal closure of the words in D, written $G = \langle S; D\rangle$. We usually will not distinguish between a group (as an abstract algebraic object) and its presentation.

$G = \langle S; D\rangle$ is said to be *finitely generated* if S is finite and *finitely presented* when S and D are both finite. In case S is finite and D is a recursively enumerable (r.e.) set of words, G is said to be *recursively presented*. (See Rogers [5] for all logical terminology.) In case G is finitely generated, the *word problem* for G is the algorithmic problem of deciding for arbitrary words w of G whether or not $w = 1$ in G.

Suppose that $G = \langle S; D\rangle$ is finitely generated and let F denote the free group on S, K the normal closure of the words in D so that $G = F/K$. Let M be a normal subgroup of G (written $M \triangleleft G$), N_M the preimage of M in F. Then $K \subset N_M \triangleleft F$ and $G/M \cong F/N_M$. The normal subgroup M of G is said to be r.e. if and only if N_M is an r.e. set of words in F. Note that N_M is an r.e. set of words if and only if N_M is the normal closure in F or an r.e. set of words. Since G is finitely generated, M is r.e. if and only if G/M can be recursively presented. Finally, we remark that the degree of the word problem for $G = \langle S; D\rangle$ is just the degree of K as a subset of the words of F. For G recursively presented, the degree of the word problem is an r.e. degree. Also observe that G has solvable word problem if and only if K and F/K are both r.e. sets of words.

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Lemma 1. *There exists a group A given by two generators and an r.e. set of defining relations and a fixed word $1 \neq w \in A$ such that*

(*) *if $N \lhd A$, N r.e. and $w \notin N$,*
then A/N has word problem of degree $\mathbf{0}'$.

Proof. Let X, Y be two disjoint r.e. sets of natural numbers such that $0 \in X$, $1 \in Y$ and X and Y are effectively inseparable. Then (i) X and Y are creative and (ii) if S, T are two disjoint r.e. sets such that $S \supset X$, $T \supset Y$ then S and T are creative and effectively inseparable. (See Smullyan [7, pp. 95, 98, 109, 110].)

Let F be the free group generated by two elements c, d; denote by E the subgroup of F generated by the elements

$$e_{i-1} = c^{-i}dcd^{-1}c^{i}d^{2}c^{-i}d^{-1}c^{-1}dc^{i}d^{-2}$$

for $i = 1, 2, \ldots$. Then $e_0, e_1, \ldots$ freely generate E and are a recursive subset of the words of F.

Let R denote the normal closure in E of the following r.e. collection of words:

$$\text{(1)} \qquad \begin{cases} e_0 e_j^{-1} & \text{for all } j \in X \\ e_1 e_k^{-1} & \text{for all } k \in Y \end{cases}$$

Finally let R^F denote the normal closure of R in F and define $A = F/R^F$. Observe that R^F is also the normal closure in F of the r.e. set of words given in (1). Thus A can be given by two generators and an r.e. set of defining relations.

By a theorem of Higman, Neumann and Neumann (see [2] or [3, pp. 540-541]), $R^F \cap E = R$ and so E/R is naturally embedded in $A = F/R^F$ by the map $e_i \mapsto e_i$.

But E/R is obtained from E by identifying e_0 and e_j when $j \in X$ and identifying e_1 and e_k when $k \in Y$ (so E/R is again a free group. Hence

$$\text{(2)} \qquad \begin{cases} e_0 = e_j \text{ in } A & \Leftrightarrow j \in X \\ e_1 = e_k \text{ in } A & \Leftrightarrow k \in Y \end{cases}$$

Now, since $0 \in X$, $1 \in Y$ and X and Y are disjoint, $e_0 \neq e_1$ in A. Define $w = e_0 e_1^{-1}$ so $w \neq 1$ in A.

Let $N \lhd A$, N r.e. and $w \notin N$. Define $S = \{j \mid e_0 e_j^{-1} \in N\}$ and $T = \{k \mid e_1 e_k^{-1} \in N\}$. Since N is r.e. and A is recursively presented, both S and T are r.e. sets of natural numbers. From (2) it follows that $S \supset X$ and $T \supset Y$. Moreover, S and T are disjoint. For suppose $\ell \in S \cap T$. Then $e_0 e_\ell^{-1} \in N$, $e_1 e_\ell^{-1} \in N$ and so $w = e_0 e_1^{-1} \in N$ contrary to the hypothesis on N. Therefore, by property (ii) of X and Y, S and T are creative and effectively inseparable.

Since $e_0 = e_j$ in A/N if and only if $j \in S$ and since $e_0, e_1, e_2, \ldots$ are a recursive set of words, it follows that S is (Turing) reducible to the word problem for A/N. But A/N is finitely generated, recursively presented and $S \in \mathbf{0}'$, so the word problem in A/N has degree $\mathbf{0}'$. This completes the proof. □

Theorem 2. *There exists a finitely presented group G with word problem of degree $\mathbf{0}'$ and a fixed word $1 \neq w \in G$ such that:*

1. *If $\phi : G \to H$ is any epimorphism from G onto H then*
   (a) *$\phi(w) = 1$ in H implies that H is the trivial group.*
   (b) *$\phi(w) \neq 1$ in H and H can be recursively presented imply that H has unsolvable word problem of degree $\mathbf{0}'$.*
   (c) *$\phi(w) \neq 1$ in H implies that H has unsolvable word problem (not necessarily of r.e. degree).*
2. *If $\overline{G}$ is any quotient of G which is non-trivial, then $\overline{G}$ and $\overline{w}$ (the image of w) have the properties listed in (1) for G and w.*
3. *G is SQ–universal, i.e. given a countable group C, there is a normal subgroup N_C of G such that G/N_C contains subgroup isomorphic to C.*

Proof. Let A and w be the group and word of the previous lemma. By a theorem of Higman [1], A can be embedded in a finitely presented group L. Put $M = L \times F_2$ (direct product) where F_2 is the free group on two generators. Now the image of A in L and hence M is determined by the images of the two generators of A, so we may write A and w again for their images in M.

Let G be the finitely presented group constructed by Rabin [4, Theorem 1.2] from the pair (M, w). Since $w \neq 1$ in M, M is embedded in G. By inspecting Rabin's construction it is easy to see that G also has the following properties: (i) if $N \lhd G$ and $w \in N$, then $N = G$; (ii) if $B \lhd F_2$ (so that $B \lhd M$ and $w \notin B$) then $w \notin B^G$ and $L \times (F_2/B)$ is naturally embedded in G/B^G. Indeed G/B^G is isomorphic to the group obtained by applying Rabin's construction to the pair $(L \times (F_2/B), w)$. We continue to write A for its image in G.

Now we must verify that G has properties (1)-(3) of the theorem. By a well known theorem of Higman, Neumann and Neumann (see [2] or [3, p. 540]), F_2 is SQ–universal. Hence by property (ii) of G above, G is SQ–universal and (3) is verified.

Let $\phi : G \to H$ be any epimorphism from G onto H. If $\phi(w) = 1$, then $w \in \ker \phi$ and so, by property (i) above, $\ker \phi = G$ and H is the trivial group. Thus, (1a) is verified.

Next suppose that $\phi(w) \neq 1$ and that H can be recursively presented. Then $\ker \phi$ is an r.e. normal subgroup of G since G is recursively presented. Put $N = A \cap \ker \phi$. since the image of A is determined by two words, A is an r.e. subset of G. Thus N is an r.e. normal subgroup of A. But $w \notin N$ since $\phi(w) \neq 1$ and $w \in A$, so by the previous lemma, A/N has word problem of degree $\mathbf{0}'$. Now the word problem for A/N is reducible to the word problem for H since A/N is a 2-generator subgroup of H. Thus (1b) is verified.

Now suppose that $\phi(w) \neq 1$ and that H can not be recursively presented. Then (trivially) H has unsolvable word problem (not necessarily of r.e. degree). Thus (1c) and hence (1) is verified.

Property (2) is immediate because $\overline{G}$ non-trivial implies $\overline{w} \neq 1$ in $\overline{G}$ and every quotient of $\overline{G}$ is also a quotient of G. This completes the proof. □

Let G and w be as in the above theorem. Now G has a finite presentation, say

$$G = \langle x_1, \ldots, x_n; R_1, \ldots, R_m \rangle$$

(generators x_i, relations R_j) and assume w is written as a word on the x_i. Let ETG denote the elementary theory of groups, as described for instance in Shoenfield [6, pp. 22, 78]. Consider the following formula Q of ETG:

$$R_1 = 1 \wedge R_2 = 1 \wedge \cdots \wedge R_m = 1 \wedge w \neq 1.$$

A model of ETG is just a group. Let M be a model of ETG in which Q is satisfiable. From the above theorem, it follows that M contains a finitely generated subgroup H which is a non-trivial quotient of G - namely the subgroup of M generated by the elements which, when substituted for the x_i in Q, satisfy Q. Hence M has unsolvable word problem and, if M is recursively presented, the word problem for M has degree $\mathbf{0}'$. Denote by $\exists Q$ the existential closure of Q, that is $(\exists x_1) \ldots (\exists x_n) Q$. The following is now immediate:

Corollary 3. *Let EGT* be obtained from ETG by adding the single axiom $\exists Q$. Then any model of ETG* has unsolvable word problem. Any recursively presented model of ETG* has word problem of degree $\mathbf{0}'$.* □

Remark: ETG and ETG* are finitely axiomatizable theories.

The proofs of the above theorem and lemma could be simplified if there were a recursively presented simple group with unsolvable word problem. However, no such group can exist as was pointed out to the author by R. Thompson (for finitely presented groups):

Proposition 4. *Let F be a free group on some (possibly infinite) recursive set of symbols $x_1, x_2, \ldots$. Let N be an r.e. normal subgroup of F such that F/N is a simple group. Then N is a recursive set of words, i.e. F/N has a solvable word problem.*

Proof. We need only show that $\{w \in F \mid w \notin N\}$ is r.e. If $F = N$ this is trivial, so we may suppose $F \neq N$. Let x_ℓ be any fixed generator of F such that $x_\ell \notin N$. Let w be arbitrary and denote by R_w the normal closure in F of $N \cup \{w\}$. Since N is r.e., R_w is r.e. uniformly in w. Now $x_\ell \in R_w$ if and only if $w \notin N$ since $x_\ell \notin N$ and F/N is simple. But $\{w \mid x_\ell \in R_w\}$ is r.e. since the R_w are r.e. uniformly in w. Hence $\{w \mid w \notin N\}$ is r.e. □

POSTSCRIPT 5 DECEMBER, 1980: This paper was written in February 1970, and was circulated as a preprint but never submitted for publication. Fortunately A. Macintyre was able to make use of these results in his work on algebraically closed groups. It follows from the Corollary above that any (non-trivial) algebraically closed group has a finitely generated subgroup with unsolvable word problem. Consequently if A is a (non-trivial) algebraically closed group, then A cannot be recursively presented (see [8]) nor can A be embedded in a finitely presented group (see [10]).

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