Finitely presented residually free groups

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Outline

Recall that a group G is said to be *residually free* if for every element $1 \neq g \in G$ there is a homomorphism $\phi : G \to F$ from G onto a free group F such that $\phi(g) \neq 1$.

A group is residually free if and only if it is isomorphic to a subgroup of an unrestricted direct product of free groups.

Examples of residually free groups include free abelian groups $\mathbb{Z}^n$ and direct products of free groups of finite rank and surface groups. Since being free is a hereditary property, the subgroups of these are residually free as well.

Unfortunately residually free groups can be badly behaved.

Theorem

Let $D = F_1 \times F_2$ of where F_i is free of rank greater than 1. Then

- *(Mihailova) D has finitely generated subgroups with unsolvable membership problem.*
- *(CFM) D has finitely generated subgroups with unsolvable conjugacy problem.*
- *(CFM) Deciding whether a finitely generated subgroup is all of D is unsolvable.*
- *(CFM) The isomorphism problem for finitely generated subgroups of D is unsolvable.*
- *(Baumslag-Roseblade) D has subgroups with continuously many non-isomorphic $H_1(-, \mathbb{Z})$.*
- *(Grunewald, Baumslag-Roseblade) If $S \leq D$ intersects each factor and projects onto each factor, then either S has finite index or S is not finitely presented.*

For a direct product with more factors, there are the following examples:

Theorem (Stallings, Bieri)

Let $D_n = F_1 \times \cdots \times F_n$ where the F_n non-abelian free groups of rank 2. Let S be the kernel of the map $D_n \to \mathbb{Z}$ sending each of the given basis elements to 1. Then S is finitely presented and of type FP_{n-1}, but $H_n(S, \mathbb{Z})$ is not finitely generated (so S is not of type FP_n).

It turns out to be useful to generalise this situation to subgroups of a direct product $\Gamma_1 \times \cdots \times \Gamma_m$ where the Γ_i are *finitely generated, fully residually free groups* (= *limit groups*).

A group G is a *fully residually free* if for every finite subset $X \subset G$ there is a homomorphism $\phi : G \to F$ from G to a free group F such that $\phi|_X : X \to F$ is injective, that is, the images of the elements of X are all different. Clearly fully residually free groups are residually free. A finitely generated, fully residually free group is called a *limit group*.

Limit groups play a crucial role in the work of Kharlampovich and Miasnikov and the work of Sela on the Tarski problem.

Examples of limit groups are free abelian groups $\mathbb{Z}^n$, free groups of finite rank and surface groups. If G is a limit group and C is a maximal abelian subgroup of G then $G *_C (C \times \mathbb{Z}^n)$ is a limit group.

Proposition (Some properties of limit groups)

Let Γ *be a limit group. Then*

- Γ *is torsion free, finitely presented, CAT(0) and has a finite* $K(\Gamma, 1)$.
- *Finitely generated subgroups of* Γ *are limit groups.*
- *For* $a, b, c \in \Gamma \setminus \{1\}$ *commutativity is transitive, that is,* $[a, b] = [b, c] = 1$ *implies* $[a, c] = 1$.
- Γ *has the same universal theory as a free group.*
- *If* S *is a subgroup of* Γ *with* $H_1(S, \mathbb{Q})$ *finite dimensional, then* S *is finitely generated (and hence is a limit group).*
- *Limit groups have good algorithms.*

A key fact for us is a result from algebraic geometry over groups:

Theorem (Baumslag-Miasnikov-Remeslennikov)

A finitely generated residually free group can be embedded into a direct product of finitely many limit groups.

For finitely presented residually free groups, Kharlampovich and Miasnikov give an algorithm for finding such an embedding. We will later describe a new algorithm for finding such an embedding.

A subgroup of a direct product is called a *subdirect product* if its projection to each factor is surjective. If we are interested in a subgroup of a direct product of $\mathcal{P}$-groups where $\mathcal{P}$ is a hereditary property, we can often reduce to the case of a subdirect product. This is the case for finitely generated subgroups of a direct products of limit groups.

We usually also assume a subgroup for a direct product intersects each factor non-trivially since otherwise it embeds in a product of fewer factors. A *full subdirect product* is a subdirect product which intersects each factor non-trivially.

Our first result is an extension of the theorems of Baumslag-Roseblade and Stallings-Bieri to products of several limit groups.

Theorem (BHMS)

Let $\Gamma_1, \ldots, \Gamma_n$ be non–abelian limit groups and let $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ be a finitely generated subdirect product which intersects each factor non-trivially. Then either :

1. *S is of finite index;* or
2. *S is of infinite index and has a finite-index subgroup $S_0 < S$ such that $H_j(S_0; \mathbb{Q})$ has infinite $\mathbb{Q}$-dimension for some $j \leq n$.*

Here is one consequence of the above result.

Theorem (BHMS)

If $\Gamma_1, \ldots, \Gamma_n$ are limit groups and $S \subset \Gamma_1 \times \cdots \times \Gamma_n$ is a subgroup of type $FP_n(\mathbb{Q})$, then S is virtually a direct product of n or fewer limit groups.

Our next result provides the advertised algorithm for embedding a finitely presented residually free group as a subdirect product of limit groups. The embedding constructed is quite canonical.

Theorem (BHMS)

There is an algorithm that, given a finite presentation of a residually free group S, will construct an embedding $\iota : S \hookrightarrow \exists\mathrm{Env}(S)$, so that

1. $\exists\mathrm{Env}(S) = \Gamma_{\mathrm{ab}} \times \exists\mathrm{Env}_0(S)$ *where* $\Gamma_{\mathrm{ab}} = H_1(S, \mathbb{Z})/(\mathrm{torsion})$ *and* $\exists\mathrm{Env}_0(S) = \Gamma_1 \times \cdots \times \Gamma_n$ *is a direct product of non-abelian limit groups* Γ_i*. The intersection of* S *with the kernel of the projection* $\rho : \exists\mathrm{Env}(S) \to \exists\mathrm{Env}_0(S)$ *is the centre* $Z(S)$ *of* S*, and* $\rho(S)$ *is a full subdirect product.*
2. *Each* $L_i := \Gamma_i \cap S$ *contains a term of the lower central series of a subgroup of finite index in* Γ_i *and so* $\exists\mathrm{Env}(S)/(L_1 \times \cdots \times L_n)$ *is virtually nilpotent.*
3. [Universal Property] *For every map* $\phi : S \to D = \Lambda_1 \times \cdots \times \Lambda_m$*, with* $\phi(S)$ *subdirect and* Λ_i *non-abelian limit groups, there exists a unique homomorphism* $\hat{\phi} : \exists\mathrm{Env}_0(S) \to D$ *with* $\hat{\phi} \circ \rho|_S = \phi$*;*
4. [Uniqueness] *If* $\phi : S \hookrightarrow D$ *embeds* S *as a full subdirect product, then* $\hat{\phi} : \exists\mathrm{Env}_0(S) \to D$ *is an isomorphism respecting direct sums.*

The group $\exists\mathrm{Env}(S)$ in the previous theorem is called the *existential envelope* of S and the associated factor $\exists\mathrm{Env}_0(S)$ is the *reduced existential envelope*. The projection ρ embeds $S/Z(S)$ in $\exists\mathrm{Env}_0(S)$, and $\rho(S) \subset \exists\mathrm{Env}_0(S)$ is always a full subdirect product. The subgroup $S \subset \exists\mathrm{Env}(S)$ is always a subdirect product but it is full if and only if S has a non-trivial centre.

We next turn to the question of determining which subgroups of a direct product of limit groups are finitely presented.

In order to state our next theorem concisely we introduce the following temporary definition: an embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ of a residually free group S as a full subdirect product of limit groups is said to be *neat* if Γ_0 is abelian (possibly trivial), $S \cap \Gamma_0$ is of finite index in Γ_0, and Γ_i is non-abelian for $i = 1, \ldots, n$.

Theorem (BHMS)

Let S be a finitely generated residually free group. Then the following conditions are equivalent:

1. *S is finitely presentable;*
2. *S is of type $\mathrm{FP}_2(\mathbb{Q})$;*
3. *$\dim H_2(S_0;\mathbb{Q}) < \infty$ for all subgroups $S_0 \subset S$ of finite index;*
4. *there exists a neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$ such that the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $1 \leq i < j \leq n$;*
5. *for every neat embedding $S \hookrightarrow \Gamma_0 \times \cdots \times \Gamma_n$, the image of S under the projection to $\Gamma_i \times \Gamma_j$ has finite index for $1 \leq i < j \leq n$.*

Corollary

For all $n \in \mathbb{N}$, a residually free group S is of type $\mathrm{F_n}$ if and only if it is of type $\mathrm{FP_n}(\mathbb{Q})$.

The property of projecting virtually onto pairs has some interesting consequences in more general contexts.

Theorem (BHMS)

Let $S \subset G_1 \times \cdots \times G_n$ be a subgroup of a direct product of finitely presented groups. If the projection $p_{ij}(S) \subset G_i \times G_j$ is of finite index for all $i, j \in \{1, \ldots, n\}$, then S is finitely presented.

The proof of this result involves establishing an Asymetric 1-2-3-Theorem which says that certain fibre products are finitely presented. We also give an effective version of this result which enables one to compute the presentation.

Proposition (BHMS)

Let $G_1, \ldots, G_n$ be finitely generated groups and let $S \subset G_1 \times \cdots \times G_n$ be a subgroup. If $p_{ij}(S) \subset G_i \times G_j$ is of finite index for all $i, j \in \{1, \ldots, n\}$, then there exist finite-index subgroups $G_i^0 \subset G_i$ such that $\gamma_{n-1}(G_i^0) \subset S$.

In case the projections of S are surjective onto pairs one can see this as follows. Consider an $(n-1)$-fold commutator in G_1, say $([x_1, \ldots, x_{n-1}], 1 \ldots, 1)$. By surjective on pairs, there are elements $(x_1, 1, *, \ldots, *)$, $(x_2, *, 1, *, \ldots, *)$, and so on in S where the $*$ entries are unnamed elements. Then their $(n-1)$-fold commutator is $([x_1, \ldots, x_{n-1}], 1 \ldots, 1)$ which therefore lies in S.

We now turn to some novel examples of finitely presented residually free groups. Let Φ_i be the free group with basis $\{a_i, b_i\}$. Then for any finite subset $E \subset \mathbb{Z}$ and $c > 1$ we define certain finitely generated subgroups $S(E, c)$ of the direct product of $|E|$ copies of Φ_i.

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As a concrete example we have $S = S(\{1,2,3,4\},3)$ is the subgroup of $\Phi_1 \times \Phi_2 \times \Phi_3 \times \Phi_4$ generated by the following 12 elements: the four images of the generators of Γ

$$(a_1, a_2, a_3, a_4)\ ,\ (b_1, b_2, b_3, b_4)$$

$$(a_1, a_2^2, a_3^3, a_4^4)\ ,\ (b_1, b_2^2, b_3^3, b_4^4)$$

together with the eight elements

$$([[a_1, b_1], a_1], 1, 1, 1)\ ,\ ([[a_1, b_1], b_1], 1, 1, 1)\ ,\ (1, [[a_2, b_2], a_2], 1, 1)\ ,\ \ldots$$

$$\ldots\ ,\ (1, 1, 1, [[a_4, b_4], a_4])\ ,\ (1, 1, 1, [[a_4, b_4], b_4])$$

which are normal generators for the subgroups $\gamma_3(\Phi_i)$ for $1 \leq i \leq 4$.

Observe that $(1, a_2, a_3^2, a_4^3)$ and $(a_1^3, a_2^2, a_3, 1)$ are in S.

Theorem (BHMS)

For any positive integer c, and any finite subset $E \subset \mathbb{Z}$ of cardinality at least $c+1$, the group $S(E, c)$ is a finitely presentable subdirect product of the non-abelian free groups Φ_n $(n \in E)$ and $S(E, c) \cap \Phi_n = \gamma_c(\Phi_n)$ for each $n \in E$.

Motivated by these examples, we construct a collection of examples of finitely presentable residually free groups which is complete up to commensurability.

Definition

Let $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_n\}$ be a finite collection of 2 or more limit groups, let $c \geq 2$ be an integer, and let $\underline{g} = \{(g_{k,1}, \ldots, g_{k,n}),\ 1 \leq k \leq m\}$ be a finite subset of $\Gamma := \Gamma_1 \times \cdots \times \Gamma_n$. Define $T = T(\mathcal{G}, \underline{g}, c)$ to be the subgroup of Γ generated by $\underline{g}$ together with the c-th term $\gamma_c(\Gamma)$ of the lower central series of Γ.

Theorem

Let $T(\mathcal{G}, \underline{g}, c)$ be defined as above.

1. *If, for all $1 \leq i < j \leq n$, the images in $H_1\Gamma_i \times H_1\Gamma_j$ of the ordered pairs $(g_{k,i}, g_{k,j})$ generate a subgroup of finite index, then the residually free group $T(\mathcal{G}, \underline{g}, c)$ is finitely presentable.*
2. *Every finitely presentable residually free group is either a limit group or else is commensurable with one of the groups $T(\mathcal{G}, \underline{g}, c)$.*

We now turn briefly to algorithmic properties.

Theorem (BHMS)

Every finitely presented residually free group has a solvable conjugacy problem.

Theorem (BHMS)

If G is a finitely presented residually free group and $H \subset G$ is a finitely presentable subgroup, then there is an algorithm that, given a word in the generators of G, will determine whether or not the element of G it defines belongs to H.

Theorem (BHMS)

The class of finitely presented residually free groups is recursively enumerable.

The isomorphism problem for finitely presented residually free groups remains open.

Our canonical embedding algorithm may be helpful in this regard. In the special case of those finitely presented groups whose envelop has at most two non-abelian factors, there is such an algorithm.