MAST 90056  Problem set 1  Due Wednesday, the 9th of September

1. Prove: Given a non-constant holomorphic function \( f \) in the region \( \Omega \) and a point \( z_0 \in \Omega \) with \( w_0 = f(z_0) \), there exists an open neighbourhood \( V \) of \( z_0 \) in \( \Omega \), a real number \( r > 0 \) and a positive integer \( m \) such that \( f : V - \{ z_0 \} \to D(w_0, r) - \{ w_0 \} \) is an \( m \)-fold covering map.

2. A germ of a holomorphic function at 0 is a pair \((f, D)\) where \( D \) is an open disk centered at zero and \( f \in \mathcal{H}(D) \), up to the equivalence relation generated by the relation that if \( D' \subset D \) then the germ \((f, D)\) is equal to the germ \((f, D')\). Let \( \mathcal{O} \) denote the set of all germs of holomorphic functions. Show that \( \mathcal{O} \) is a local commutative \( \mathbb{C} \)-algebra isomorphic to a subalgebra of the ring of formal power series \( \mathbb{C}[z] \). Finally prove that \( \mathcal{O} \) is a principal ideal domain.

3. The Riemann sphere \( \mathbb{CP}^1 \) is the union of the complex plane and one point \( \infty \) ”at infinity”, made a topological space by declaring the collection of sets \( D(\infty, r) = \{ z \in \mathbb{C} | |z| > r \} \cup \{ \infty \} \) for \( r > 0 \) a neighbourhood basis of infinity. Show that the Riemann sphere carries a unique structure as complex manifold (and hence, Riemann surface). Prove that every holomorphic function on the Riemann sphere is constant, and that every meromorphic function is rational and in particular has only finitely many poles. (Do not use general results about abstract Riemann surfaces to do this.)

4. EDIT: This problem will be due with the final assignment on October 23. Let \( S \) be a compact Riemann surface and \( \mathcal{O}_S \) the sheaf of holomorphic functions on \( S \). Let \( \mathcal{F} \) be a coherent sheaf of \( \mathcal{O}_S \)-modules, that is, a sheaf of modules such that every point \( s \in S \) admits an open neighbourhood \( V \subset S \) and a finite presentation

\[
\mathcal{O}_V^n \to \mathcal{O}_V^m \to \mathcal{F} \to 0
\]

of the restriction of \( \mathcal{F} \) to \( V \). Show that the vector space of global sections \( \Gamma(S, \mathcal{F}) \) is finite-dimensional. (This will require some reading. Hint: if \( B \) is a Banach or Frechet space whose unit ball is compact, then \( B \) is finite-dimensional; use the Azela - Ascoli theorem.)

5. Let \( \Lambda \subset \mathbb{C} \) be a complete lattice, that is the abelian subgroup of \( \mathbb{C} \) generated by some real basis \( \{ \omega_1, \omega_2 \} \). Show that the factor group \( \mathbb{E}_\Lambda = \mathbb{C}/\Lambda \) endowed with the quotient topology has a unique structure as Riemann surface such that the canonical projection \( \mathbb{C} \to \mathbb{E}_\Lambda \) is holomorphic. Moreover, demonstrate that \( \mathbb{E}_\Lambda \) has an abelian group structure such that the addition and inverse maps are holomorphic. (That is, it is a commutative complex Lie group.)

6. Show that every \( \mathbb{E}_\Lambda \) as in problem 5. is isomorphic as complex Lie group to one where \( \Lambda \) is generated by a basis \( \{ 1, \tau \} \) with \( \tau \) a complex number with positive imaginary part; call this surface \( \mathbb{E}_\tau \). Under which circumstances will \( \mathbb{E}_\tau \) be isomorphic to \( \mathbb{E}_{\tau'} \)?