1. Let $\Lambda \subset \mathbb{C}$ be a complete lattice, write $\Lambda' = \Lambda - \{0\}$ and define a function in the complex plane by

$$p(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda'} \left( \frac{1}{(z + \lambda)^2} - \frac{1}{\lambda^2} \right).$$

Show that this function is meromorphic, $\Lambda$-periodic, and holomorphic outside $\Lambda$.

2. Using the notation and terminology of problem 5 of the second assignment, show that the $\mathbb{C}$-vector space $L(X)$ of global sections is finite-dimensional. (This problem replaces the original problem 4 of the first assignment, but is actually much easier. This can be done without hard analysis.)

3. Let $X$ be a topological space and $\mathcal{F}$ a presheaf on $X$. Let $p : |\mathcal{F}| \to X$ be the associated space and local homeomorphism (as defined in 10.7 in the lecture notes). Consider the presheaf $a\mathcal{F}$ defined as follows: an element of $a\mathcal{F}(U)$ is a continuous map $s : U \to |\mathcal{F}|$ such that $p \circ s = id_U$ (that is, a section to $p$ over $U$), and restriction of $s$ to $V \subseteq U$ is simply restriction of the map $s$. Show:

(a) $a\mathcal{F}$ is a sheaf.

(b) Assigning to $f \in \mathcal{F}(U)$ the function $s : U \to |\mathcal{F}|$ such that $s(u) = f_u$ determines a map of presheaves $\iota : \mathcal{F} \to a\mathcal{F}$, which is an isomorphism if and only if $\mathcal{F}$ is a sheaf.

(c) If $\mathcal{G}$ is a sheaf on $X$ and $\phi : \mathcal{F} \to \mathcal{G}$ is a map of presheaves then there is a unique map of sheaves $\psi : a\mathcal{F} \to \mathcal{G}$ such that $\psi \circ \iota = \phi$.

4. Continuing from problem 1, suppose $f$ is a $\Lambda$-periodic meromorphic function in the complex plane that has poles only at lattice points and such that these poles are of order 2. Show that there are complex numbers $c \neq 0$ and $d$ such that $f = cp + d$.

5. Let $X$ be a compact Riemann surface and assume that $U \subset X$ is an open subset with finite complement biholomorphically equivalent to $\mathbb{C}$. Show that $X \cong \mathbb{CP}^1$.

6. Let $X$ and $Y$ be two compact Riemann surfaces, each admitting a non-constant meromorphic function (they all do, but that is not so easy to prove). Assume that $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are isomorphic as $\mathbb{C}$-algebras. Show that $X$ and $Y$ are biholomorphically equivalent. (Hint: prove that $X$ and $Y$ can both be realized as Riemann surfaces of an algebraic function satisfying the same algebraic equation.)