The $L(h,1,1)$-labelling problem for trees

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Abstract

Let $h \geq 1$ be an integer. An $L(h,1,1)$-labelling of a (finite or infinite) graph is an assignment of nonnegative integers (labels) to its vertices such that adjacent vertices receive labels with difference at least $h$, and vertices distance two or three apart receive distinct labels. The span of such a labelling is the difference between the maximum and minimum labels used, and the minimum span over all $L(h,1,1)$-labellings is called the $\lambda_{h,1,1}$-number of the graph. We prove that, for any integer $h \geq 1$ and any finite tree $T$ of diameter at least three or infinite tree $T$ of finite maximum degree, $\Delta_2(T) - 1 \leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1$, and both lower and upper bounds are attainable, where $\Delta_2(T)$ is the maximum total degree of two adjacent vertices. Moreover, if $h$ is less than or equal to the minimum degree of a non-pendant vertex of $T$, then $\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2$. In particular, $\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T)$ and the chromatic number of the third power of $T$ is equal to $\Delta_2(T)$. Furthermore, if $T$ is a caterpillar and $h \geq 2$, then $\Delta_2(T) - 1 \leq \lambda_{h,1,1} \leq \Delta_2(T) + h - 2$ with both lower and upper bounds achievable.

Key words: channel assignment, frequency assignment, $L(h,1,1)$-labelling, $L(h,k)$-labelling, $\lambda_{h,1,1}$-number, tree, graph, power graphs of a tree, chromatic number

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1 Introduction

Motivated by the problem [13] of assigning frequencies to transmitters in a radio communication network, various channel assignment problems have received extensive attention in recent years. Usually, such problems can be formulated as graph labelling problems, and a major concern is to minimize the span of a channel assignment subject to a set of constraints involving pairs of vertices within a given distance. Among others the following model has been studied widely, especially in the case when $d = 2$. Given a finite or infinite graph $G = (V(G), E(G))$ and a
sequence $h_1, h_2, \ldots, h_d$ of nonnegative integers, an $L(h_1, h_2, \ldots, h_d)$-labelling of $G$ is a mapping $\phi : V(G) \to \{0, 1, 2, \ldots\}$ such that, for $t = 1, 2, \ldots, d$ and any $u, v \in V(G)$ with $d(u, v) = t$, 

$$|\phi(u) - \phi(v)| \geq h_t$$

where $d(u, v)$ is the distance in $G$ between $u$ and $v$. (In this paper an infinite graph means a graph with countably infinitely many vertices.) In practical terms, the label of $u$ under $\phi$, $\phi(u)$, is the channel assigned to the transmitter corresponding to $u$. Without loss of generality we will always assume $\min_{v \in V(G)} \phi(v) = 0$. Under this assumption the span of $\phi$ is defined as $\max_{v \in V(G)} \phi(v)$. Define [12, 13]

$$\lambda_{h_1, h_2, \ldots, h_d}(G) := \min_{\phi} \max_{v \in V(G)} \phi(v)$$

to be the $\lambda_{h_1, h_2, \ldots, h_d}$-number of $G$, where the minimum is taken over all $L(h_1, h_2, \ldots, h_d)$-labellings of $G$. In practice [13] this parameter corresponds to the minimum bandwidth required by the radio communication network under the constraints above.

The $L(h_1, h_2, \ldots, h_d)$-labelling problem above is interesting in both theory and practical applications. For instance, when $d = 1$, it becomes the ordinary vertex-colouring problem since $\lambda_h(G) = h(\chi(G) - 1)$, where $\chi(G)$ is the chromatic number of $G$. In the case when $d = 2$, many interesting results (see e.g. [4, 6, 7, 10, 12, 16, 17]) have been obtained for various families of finite graphs, especially when $(h_1, h_2) = (2, 1)$. The reader is referred to [1] for an extensive bibliography on the $L(h_1, h_2)$-labelling problem and [19] for a short survey on Hamming graphs and hypercubes. In the following we just mention a few results for finite trees since they are more relevant to this article. In [12] it was proved that, for any finite tree $T$, $\lambda_{2,1}(T)$ is either $\Delta(T) + 1$ or $\Delta(T) + 2$, where $\Delta(T)$ is the maximum degree of $T$. A polynomial time algorithm for determining $\lambda_{2,1}(T)$ was given in [4], which was adapted [8] later to give the fixed-parameter complexity of the $L(2,1)$-labelling problem for a larger family of graphs. In [5] it was proved that $\Delta(T) + h - 1 \leq \lambda_{h,1}(T) \leq \min\{\Delta(T) + 2h - 2, 2\Delta(T) + h - 2\}$ with both lower and upper bounds attainable. In the case when $h_1 \geq h_2 > 1$, the complexity of determining $\lambda_{h_1, h_2}$ for finite trees remains open, and it is conjectured [9] that the problem is NP-hard. In [11] the $\lambda_{h_1, h_2}$-number was derived for infinite regular trees when $h_1 \geq h_2$, and for $h_1 < h_2$ the authors of [3] studied the smallest integer $\lambda$ such that every tree of maximum degree $\Delta \geq 2$ admits an $L(h_1, h_2)$-labelling of span at most $\lambda$.

More recently, researchers began to investigate the $L(h_1, h_2, h_3)$-labelling problem. For example, in [18] the second-named author studied the problem for hypercubes $Q_n$ by using a group-theoretic approach, leading to upper bounds on $\lambda_{h_1, h_2, h_3}(Q_n)$ which are tight in certain cases. In [2] the $L(h, 1, 1)$-labelling problem (where $h \geq 1$) for outerplanar graphs was investigated. Nevertheless, in contrast to $L(h_1, h_2)$, we know only very little about $L(h_1, h_2, h_3)$-labellings even for some basic graphs such as trees.

In this paper we study the $L(h, 1, 1)$-labelling problem for finite and infinite trees, where $h \geq 1$. Define

$$\Delta_2(G) := \max_{u \neq v \in E(G)} (d(u) + d(v))$$

for any graph $G$, where $d(u)$ is the degree of $u$ in $G$. Note that, if $G$ is infinite, then $\Delta_2(G) = \infty$ if and only if there exists no positive integer $N$ such that $d(u) \leq N$ for all $u \in V(G)$, and in this
case we have $\lambda_{h,1,1}(G) = \infty$. Thus, we consider only finite trees and infinite trees with finite maximum degree. We obtain the following bounds on $\lambda_{h,1,1}(T)$ in terms of $\Delta_2(T)$, which will be proved in Section 2 along with an algorithm for finding an $L(h,1,1)$-labelling of $T$ with span $\Delta_2(T) + h - 1$. When $T$ is finite, the running time of this algorithm is $O(|V(T)|^2)$.

**Theorem 1** Let $h \geq 1$ be an integer. Let $T$ be a finite tree with diameter at least three or an infinite tree with finite maximum degree. Then

$$\Delta_2(T) - 1 \leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1. \quad (1)$$

Moreover, the lower bound is attainable for any $h \geq 1$ and the upper bound is attainable for any $h \geq 3$.

The lower bound above is achieved by any tree $T$ with diameter three and any $h$ with $1 \leq h \leq \min\{d(u), d(v)\}$, where $u, v$ are the two vertices of $T$ with degree greater than one. In fact, if we assign 0 to $u$, $\Delta_2(T) - 1$ to $v$, $d(v), d(v) + 1, \ldots, \Delta_2(T) - 2$ to the neighbors of $u$ other than $v$, and $1, 2, \ldots, d(v) - 1$ to the neighbors of $v$ other than $u$, then we get an $L(h,1,1)$-labelling of $T$ with span $\Delta_2(T) - 1$, and hence $\lambda_{h,1,1}(T) = \Delta_2(T) - 1$. Let $T'$ be the infinite tree obtained from $T$ by attaching an infinite path (with one closed end) to a neighbor of $u$. It is easy to check that $\lambda_{h,1,1}(T') = \Delta_2(T') - 1$.

In the next section we will give for any $h \geq 3$ a family of trees which achieve the upper bound in (1). Our next result, to be proved in Section 3, implies that this upper bound can be improved when $h = 1, 2$. Define

$$\delta^*(T) := \min_{u \in V(T), d(u) \geq 2} d(u).$$

**Theorem 2** Let $T$ be a finite tree with diameter at least three or an infinite tree with finite maximum degree. Then for any positive integer $h \leq \delta^*(T)$ we have

$$\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2. \quad (2)$$

A tree is called a **caterpillar** if the removal of all degree-one vertices results in a path, called the spine. Thus the spine of an infinite caterpillar is an infinite path with at least one open end. The next result, to be proved in Section 3, shows that for caterpillars the upper bound in (1) can also be reduced by one for any $h \geq 2$.

**Theorem 3** Let $h \geq 2$ be an integer. Let $T$ be a finite caterpillar of diameter at least three or an infinite caterpillar of finite maximum degree. Then

$$\Delta_2(T) - 1 \leq \lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2$$

and both lower and upper bounds are achievable. Moreover, if there exists no vertex on the spine with degree $\Delta_2(T) - 2$, then $\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 3$; if there exist consecutive vertices $u, v, w$ on the spine such that $d(u) = d(w) = \Delta_2 - 2$ and $d(v) = 2$, then $\lambda_{h,1,1}(T) = \Delta_2(T) + h - 2$.

Note that $\delta^*(T) \geq 2$ for any tree $T$ with diameter at least three. Thus, in the case when $h = 2$, Theorems 1 and 2 give the following corollary, which can be viewed as the counterpart of the result $\Delta(T) + 1 \leq \lambda_{2,1}(T) \leq \Delta(T) + 2$ mentioned above.
Corollary 4 Let $T$ be a finite tree with diameter at least three or an infinite tree with finite maximum degree. Then

$$\Delta_2(T) - 1 \leq \lambda_{2,1,1}(T) \leq \Delta_2(T).$$

The third power of a graph $G$, $G^3$, is the graph with the same vertices as $G$ such that two vertices are adjacent if and only if the distance in $G$ between them is at most three. From the definition of $\lambda_{1,1,1}$ it is evident that $\lambda_{1,1,1}(G) = \chi(G^3) - 1$. Thus when $h = 1$, Theorems 1 and 2 together give the following corollary, which is of interest for its own sake. (See, for example, [14, 15, 17] for related results on the chromatic number of power graphs.)

Corollary 5 Let $T$ be a finite tree with diameter at least three or an infinite tree with finite maximum degree. Then

$$\lambda_1(T^3) = \lambda_{1,1,1}(G) + 1 = \Delta_2(T).$$

The $\lambda_{h_1,h_2}$-number of a graph is often bounded by its maximum degree $\Delta$. For example, motivated by the conjecture [12] that $\lambda_{2,1}(G) \leq \Delta(G)^2$ for any graph $G$ with $\Delta(G) \geq 2$, a number of results in the literature relate $\lambda_{h_1,h_2}(G)$ to $\Delta(G)$ (see the survey paper [1]). Our results above suggest that distance-three labelling problems (not necessarily for trees) are more relevant to $\Delta_2$.

We will use the following notations: for a vertex $v$ of a tree $T$,

$$N(v) := \{u \in V(T) : uv \in E(T)\}$$
$$\overline{N}(v) := N(v) \cup \{v\}$$
$$N_3(v) := \{u \in V(T) : 1 \leq d(u, v) \leq 3\}.$$ 

An edge $uv$ of $T$ is called heavy if it achieves $\Delta_2(T)$, that is, $d(u) + d(v) = \Delta_2(T)$.

2 Proof of Theorem 1

In this section we always assume that $T$ is a finite tree with diameter at least three or an infinite tree with finite maximum degree. The following lemma gives the lower bound in Theorem 1.

Lemma 6 Let $h \geq 1$ be an integer. Then

$$\lambda_{h,1,1}(T) \geq \Delta_2(T) - 1.$$ 

Proof Let $uv \in E(T)$ be a heavy edge. Then $N(u) \cup N(v)$ contains $\Delta_2(T)$ vertices with mutual distance at most three. Since these $\Delta_2(T)$ vertices require $\Delta_2(T)$ distinct labels in any $L(h,1,1)$-labelling, we have $\lambda_{h,1,1}(T) \geq \Delta_2(T) - 1$ immediately. \hfill $\square$

In the following we abbreviate $\Delta_2(T)$ to $\Delta_2$ and fix a heavy edge $uv$ of $T$. Let $T - uv$ be the graph obtained from $T$ by deleting the edge $uv$. Denote by $T_u, T_v$ the connected components of $T - uv$ containing $u, v$ respectively. Let

$$l_u := \max_{w \in V(T_u)} d(u, w), \quad l_v := \max_{w \in V(T_v)} d(v, w).$$
Note that, if $T$ is infinite, then at least one of $T_u, T_v$ must be infinite. Moreover, if $T_u$ ($T_v$, respectively) is infinite, then we define $l_u = \infty$ ($l_v = \infty$, respectively). Define

$$L_i(u) := \{w \in V(T_u) : d(u, w) = i\}, \quad i = 0, 1, \ldots, l_u$$

$$L_i(v) := \{w \in V(T_v) : d(v, w) = i\}, \quad i = 0, 1, \ldots, l_v.$$ 

In particular, $L_0(u) = \{u\}$ and $L_0(v) = \{v\}$. To facilitate our labelling we index the vertices of $L_i(u)$ by sequences of positive integers of length $i$ in the following way. First, we index the vertices in $L_1(u) (= N(u) \setminus \{v\})$ by $1, 2, \ldots, d(u) - 1$ (sequences of length 1) respectively in a fixed but arbitrary order. Then for each vertex $a_1 \in \{1, 2, \ldots, d(u) - 1\}$ we index its neighbors other than $u$ by $a_1 a_2$ in an arbitrary order, where $a_2 = 1, 2, \ldots, d(a_1) - 1$. Inductively, for a vertex $a_1 a_2 \cdots a_{i-1} a_i$ in level $L_i(u)$, if it is not a degree-one vertex, then we index its neighbors other than $a_1 a_2 \cdots a_{i-1}$ by $a_1 a_2 \cdots a_{i-1} a_i a_{i+1}$ in an arbitrary order, where $a_{i+1} = 1, 2, \ldots, d(a_1 a_2 \cdots a_{i-1} a_i) - 1$. In this way each vertex of $T_u$ other than $u$ is indexed by a unique sequence whose length is the distance between the vertex and $u$. Moreover, the unique path between $u$ and a vertex $a_1 a_2 \cdots a_{i-1} a_i \in L_i(u)$ is

$$u, a_1, a_1 a_2, a_1 a_2 a_3, \ldots, a_1 a_2 \cdots a_{i-1} a_i.$$ 

In the same fashion, we index the vertices of $T_v$ other than $v$ by sequences, and we use $b_1 b_2 \cdots b_{i-1} b_i$ to denote a typical vertex in level $L_i(v)$ in order to avoid confusion with vertices of $T_u$. In the following, if $i = 1$ then $a_1 \cdots a_{i-1}, b_1 \cdots b_{i-1}$ are interpreted as $u, v$ respectively, and $a_1 \cdots a_{i-2}, b_1 \cdots b_{i-2}$ are interpreted as $v, u$ respectively. The following observations will be used without further explanation in the proof of Lemma 8.

**Lemma 7**

(a) The following equalities (3)-(6) hold for $i = 1, 2, \ldots, l_u - 1$ and (5)-(6) for $i = 1, 2, \ldots, l_v - 1$:

$$L_{i+1}(u) = \bigcup_{a_1 \cdots a_i \in L_i(u)} (N(a_1 \cdots a_i) \setminus \{a_1 \cdots a_{i-1}\}) \quad (3)$$

$$N_3(a_1 \cdots a_i a_{i+1}) \cap \bigcup_{j=0}^i L_j(u) = N(a_1 \cdots a_{i-1}), \quad \forall a_1 \cdots a_i a_{i+1} \in L_{i+1}(u) \quad (4)$$

$$L_{i+1}(v) = \bigcup_{b_1 \cdots b_i \in L_i(v)} (N(b_1 \cdots b_i) \setminus \{b_1 \cdots b_{i-1}\}) \quad (5)$$

$$N_3(b_1 \cdots b_i b_{i+1}) \cap \bigcup_{j=0}^i L_j(v) = N(b_1 \cdots b_{i-1}), \quad \forall b_1 \cdots b_i b_{i+1} \in L_{i+1}(v). \quad (6)$$

(b) Any two vertices of $T$ which are in the same level $L_i(u)$ ($L_i(v)$, respectively) but not adjacent to the same vertex in level $L_{i-1}(u)$ ($L_{i-1}(v)$, respectively) are distance 4 apart.

The next lemma gives the upper bound in (1). For integers $x < y$, let

$$[x, y] := \{x, x+1, \ldots, y-1, y\}.$$ 

For a labelling $\phi$ of $T$ and a subset $U$ of $V(T)$, denote

$$\phi(U) := \{\phi(u) : u \in U\}.$$ 

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Lemma 8 Let $h \geq 1$ be an integer. Then
\[
\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 1.
\]

Proof We construct recursively an $L(h, 1, 1)$-labelling $\phi$ of $T$ with span $\Delta_2(T) + h - 1$. Recall that $uv$ is a fixed heavy edge of $T$. If $T$ is finite, then both $l_u$ and $l_v$ are finite; otherwise either $l_u$ or $l_v$ is $\infty$.

Part 1 (Initialization): Define
\[
\phi(u) = 0, \quad \phi(v) = \Delta_2 + h - 1; \quad (7)
\]
\[
\phi(a_1) = \Delta_2 + h - 1 - a_1, \quad a_1 = 1, 2, \ldots, d(u) - 1; \quad (8)
\]
\[
\phi(b_1) = b_1, \quad b_1 = 1, 2, \ldots, d(v) - 1. \quad (9)
\]
Since $\Delta_2 + h - 1 - (d(u) - 1) = d(v) + h$, we have
\[
\phi(N(u) \setminus \{v\}) = [d(v) + h, \Delta_2 + h - 2]
\]
\[
\phi(N(v) \setminus \{u\}) = [1, d(v) - 1].
\]
Since $d(u) \geq 2$ and $d(v) \geq 2$, the labelling above satisfies the $L(h,1,1)$-conditions among the vertices of $N(u) \cup \{N(v)\}$.

Part 2 (Labelling $T_u$): If $l_u = 1$, then $T_u$ has been labelled fully. Otherwise, for each $a_1 \in L_1(u)$ we label (in an arbitrary manner) the vertices of $N(a_1) \setminus \{u\} \subseteq L_2(u)$ by $1, 2, \ldots, d(a_1) - 1$ respectively, so that $\phi(N(a_1) \setminus \{u\}) = [1, d(a_1) - 1]$. We do this for all vertices $a_1 \in L_1(u)$ independently, and in this way all vertices in $L_2(u)$ are labelled. Since $d(a_1) \leq \Delta_2 - d(u) = d(v)$ by the definition of $\Delta_2$, in view of Lemma 7(b) the labelling so far satisfies the $L(h,1,1)$-conditions up to level $L_2(u)$.

If $l_u = 2$ then $T_u$ has been labelled; otherwise we label $L_3(u)$ as follows: If $d(a_{12}) \geq a_1 + 2$ (that is, $\phi(a_1) > \Delta_2 - d(a_{12}) + 1$), then we label (in an arbitrary manner) the vertices of $N(a_{12}) \setminus \{a_1\}$ by $\Delta_2 + h - d(a_{12}), \ldots, \Delta_2 + h - 2 - a_1, \Delta_2 + h - a_1, \ldots, \Delta_2 + h - 1$ respectively, so that $\phi(N(a_{12}) \setminus \{a_1\}) = [\Delta_2 + h - d(a_{12}), \Delta_2 + h - 1] \setminus \{\Delta_2 + h - 1 - a_1\}$. If $d(a_{12}) \leq a_1 + 1$, then we label arbitrarily the vertices of $N(a_{12}) \setminus \{a_1\}$ by $\Delta_2 + h - 1 - d(a_{12}), \ldots, \Delta_2 + h - 1$ respectively, so that $\phi(N(a_{12}) \setminus \{a_1\}) = [\Delta_2 + h - 1 - d(a_{12}), \Delta_2 + h - 1]$. Since $d(a_1) + d(a_{12}) \leq \Delta_2$ by the definition of $\Delta_2$ and $\phi(a_{12}) \in [1, d(a_1) - 1]$ by the labelling above, in both cases these new labels satisfy the $L(h,1,1)$-conditions with existing labels up to level $L_2(u)$. Moreover, in view of Lemma 7(b), we can label $N(a_{12}) \setminus \{a_1\}$ for all vertices $a_{12} \in L_2(u)$ independently, and thus label all vertices in $L_3(u)$ without violation of the $L(h,1,1)$-conditions.

If $l_u = 3$ then $T_u$ has been labelled; otherwise we label $L_4(u)$ as follows. Note that $\phi(a_1) = \Delta_2 + h - 1 - a_1, \phi(a_{12}) \in [1, d(a_1) - 1]$ and $\phi(a_{12}, a_{23}) \in [\Delta_2 + h - d(a_{12}), \Delta_2 + h - 1] \setminus \{\Delta_2 + h - 1 - a_1\}$ or $[\Delta_2 + h - 1 - d(a_{12}), \Delta_2 + h - 1]$ by the labelling above. We distinguish the following two cases for level $L_4(u)$.

We first consider the case where $d(a_{12}, a_{23}) \geq \phi(a_{12}) + 1$. In this case, if $d(a_{12}, a_{23}) \leq \Delta_2 + h - 1 - a_1$, then we label arbitrarily the vertices of $N(a_{12}, a_{23}) \setminus \{a_1, a_{12}\}$ by $0, 1, \ldots, \phi(a_{12}) - 1, \phi(a_{12}) + 1, \ldots, d(a_{12}, a_{23}) - 1$ (that is, $\phi(N(a_{12}, a_{23}) \setminus \{a_1, a_{12}\}) = [0, d(a_{12}, a_{23}) - 1]\{\phi(a_{12})\}$); if $d(a_{12}, a_{23}) \geq \Delta_2 + h - a_1$, then we label these vertices arbitrarily by $0, 1, \ldots, \phi(a_{12}) - 1, \phi(a_{12}) + 1, \ldots, d(a_{12}, a_{23}) - 1$ (that is, $\phi(N(a_{12}, a_{23}) \setminus \{a_1, a_{12}\}) = [0, d(a_{12}, a_{23}) - 1]\{\phi(a_{12})\}$).
we use the following simplified notations: 

\[ \phi(a_{1}a_{2}a_{3}) \] (that is, \( \phi(N(a_{1}a_{2}a_{3}) \cup \{a_{1}a_{2}\}) = [0, d(a_{1}a_{2}a_{3}) \cup \{a_{1}a_{2}\}, L_{2} + h - 1 - a_{1}] \)). Since \( d(a_{1}a_{2}) + d(a_{1}a_{2}a_{3}) \leq \Delta_{2} \), in each possibility these new labels satisfy the \( L(h, 1, 1) \)-conditions with existing labels up to level \( L_{3}(u) \).

Next we assume \( d(a_{1}a_{2}a_{3}) \leq \phi(a_{1}a_{2}) \). In this case, we have \( d(a_{1}a_{2}a_{3}) \leq \phi(a_{1}a_{2}) \leq d(a_{1}) - 1 \leq d(v) - 1 < \phi(a_{1}) + 1 = \Delta_{2} + h - a_{1} \). Thus, we label the vertices of \( N(a_{1}a_{2}a_{3}) \cup \{a_{1}a_{2}\} \) by \( 0, 1, \ldots, d(a_{1}a_{2}a_{3}) - 2 \) (that is, \( \phi(N(a_{1}a_{2}a_{3}) \cup \{a_{1}a_{2}\}) = [0, d(a_{1}a_{2}a_{3}) - 2] \)). Again, since \( d(a_{1}a_{2}) + d(a_{1}a_{2}a_{3}) \leq \Delta_{2} \), these new labels satisfy the \( L(h, 1, 1) \)-conditions with the vertices up to level \( L_{3}(u) \).

By Lemma 7(b) we can label \( N(a_{1}a_{2}a_{3}) \cup \{a_{1}a_{2}\} \) for all \( a_{1}a_{2}a_{3} \in L_{3}(u) \) independently in the above way, and thus label \( L_{4}(u) \), without violating the \( L(h, 1, 1) \)-conditions.

In general, we prove by induction that the following hold for \( i = 1, \ldots, l_{u} \) when \( T \) is finite and for all integers \( i \geq 1 \) when \( T \) is infinite:

(a) if \( i \) is odd, then for all \( a_{1} \cdots a_{i-1} \in L_{i-1}(u) \) we can label independently the vertices of \( N(a_{1} \cdots a_{i-1}) \cup \{a_{1} \cdots a_{i-2}\} \) by the \( d(a_{1} \cdots a_{i-1}) - 1 \) largest available integers in \([\Delta_{2} + h - 1 - d(a_{1} \cdots a_{i-1}), \Delta_{2} + h - 1]\) such that the \( L(h, 1, 1) \)-conditions are satisfied among vertices of \( T_{u} \) up to level \( L_{i}(u) \);

(b) if \( i \) is even, then for all \( a_{1} \cdots a_{i-1} \in L_{i-1}(u) \) we can label independently the vertices of \( N(a_{1} \cdots a_{i-1}) \cup \{a_{1} \cdots a_{i-2}\} \) by the \( d(a_{1} \cdots a_{i-1}) - 1 \) smallest available integers in \([0, d(a_{1} \cdots a_{i-1})]\) such that the \( L(h, 1, 1) \)-conditions are satisfied among vertices of \( T_{u} \) up to level \( L_{i}(u) \).

The discussion above established these statements for \( i = 1, 2, 3, 4 \). Suppose that (a) and (b) are true for all levels up to \( i = l_{u} - 1 \), implying that we have labelled all vertices of \( T_{u} \) up to level \( L_{i}(u) \) without violating the \( L(h, 1, 1) \)-conditions. In the following we prove that they are true for level \( i + 1 \) as well. We will repeatedly use the property that \( d(a_{1} \cdots a_{i-1}) + d(a_{1} \cdots a_{i}) \leq \Delta_{2} \) (by the definition of \( \Delta_{2} \)) without mentioning it explicitly. Since there is no danger of confusion, we use the following simplified notations:

\[ A_{i} := N(a_{1} \cdots a_{i-1}) \cup \{a_{1} \cdots a_{i-2}\}, \ x_{i} := \phi(a_{1} \cdots a_{i}), \ i - 3 \leq t \leq i. \]

Case 1. \( i \) is even.

Since \( i \) is even, we have \( \phi(A_{i}) \subset [0, d(a_{1} \cdots a_{i-1})] \) by the induction hypothesis. Thus, \( x_{i} \leq d(a_{1} \cdots a_{i-1}) \leq \Delta_{2} - d(a_{1} \cdots a_{i-2}) \leq x_{i-1} \), where the second inequality is from the definition of \( \Delta_{2} \) and the last one is from (a) applied to \( i - 1 \). Similarly, \( x_{i-1} \geq \Delta_{2} - d(a_{1} \cdots a_{i-2}) \geq d(a_{1} \cdots a_{i-3}) \geq x_{i-2} \) and \( x_{i-3} \geq \Delta_{2} - d(a_{1} \cdots a_{i-4}) \geq d(a_{1} \cdots a_{i-3}) \geq x_{i-2} \). Thus, by the \( L(h, 1, 1) \)-conditions we have

\[ x_{i-3} \geq x_{i-2} + h, \ x_{i-1} \geq x_{i-2} + h, \ x_{i-1} \geq x_{i} + h \] (10)

and \( x_{i-3}, x_{i-2}, x_{i-1}, x_{i} \) are pairwise distinct.

In the case where \( x_{i-1}, x_{i-2} \leq \Delta_{2} + h - d(a_{1} \cdots a_{i}) \), we can label the vertices of \( A_{i+1} \) by the integers in \([\Delta_{2} + h + 1 - d(a_{1} \cdots a_{i}), \Delta_{2} + h - 1]\) without violating the \( L(h, 1, 1) \)-conditions. Henceforth we assume that at least one of \( x_{i-1} \) and \( x_{i-2} \) is at least \( \Delta_{2} + h + 1 - d(a_{1} \cdots a_{i}) \). Since \( x_{i-1} > x_{i-2} \) by (10), this implies that \( x_{i-1} \geq \Delta_{2} + h + 1 - d(a_{1} \cdots a_{i}) \). If \( x_{i-2} \geq \Delta_{2} + h - 1 -
\(d(a_1 \cdots a_i)\), then \(x_{i-3} \geq \Delta_2 + 2h - 1 - d(a_1 \cdots a_i)\) by (10) and hence \(\phi(A_i) = [0, d(a_1 \cdots a_{i-1}) - 2]\) by the induction hypothesis. In this case we can label the vertices of \(A_{i+1}\) by the integers in \([\Delta_2 + h - 1 - d(a_1 \cdots a_i), \Delta_2 + h - 1] \setminus \{x_{i-1}, x_{i-2}\}\) without violation of the \(L(h, 1, 1)\)-conditions. If \(x_{i-2} \leq \Delta_2 + h - 2 - d(a_1 \cdots a_i)\), then since \(\phi(A_i) \subseteq [0, d(a_1 \cdots a_{i-1})]\) we can label the vertices of \(A_{i+1}\) by the integers in \([\Delta_2 + h - d(a_1 \cdots a_i), \Delta_2 + h - 1] \setminus \{x_{i-1}\}\) without violation of the \(L(h, 1, 1)\)-conditions.

Case 2. \(i\) is odd.

Since \(i\) is odd, by the induction hypothesis we have \(\phi(A_i) \subseteq [\Delta_2 + h - 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]\). Applying the induction hypothesis to \(L_{i-3}(u)\) and \(L_{i-1}(u)\), we get \(x_{i-3} \leq d(a_1 \cdots a_{i-4}) \leq \Delta_2 - d(a_1 \cdots a_{i-3}) \leq x_{i-2}, x_{i-1} \leq d(a_1 \cdots a_{i-2}) \leq \Delta_2 - d(a_1 \cdots a_{i-3}) \leq x_{i-2}\) and \(x_{i-1} \leq d(a_1 \cdots a_{i-2}) \leq \Delta_2 - d(a_1 \cdots a_{i-1}) \leq x_i\). Hence

\[
x_{i-2} \geq x_{i-3} + h, \ x_{i-2} \geq x_{i-1} + h, \ x_i \geq x_{i-1} + h
\]

and \(x_{i-3}, x_{i-2}, x_{i-1}, x_i\) are pairwise distinct.

In the case where \(x_{i-1}, x_{i-2} \geq d(a_1 \cdots a_{i-2}) - 1\), we label the vertices of \(A_{i+1}\) by the integers in \([0, d(a_1 \cdots a_{i}) - 1]\). So we may assume that at least one of \(x_{i-1}\) and \(x_{i-2}\) is smaller than \(d(a_1 \cdots a_{i}) - 1\), which implies \(x_{i-1} \leq d(a_1 \cdots a_i) - 2\) in view of (11). If \(x_{i-2} \leq d(a_1 \cdots a_i)\), then \(x_{i-3} \leq d(a_1 \cdots a_i) - h\) by (11), and hence \(\phi(A_i) = [\Delta_2 + h + 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]\) by the induction hypothesis. In this case we label \(A_{i+1}\) by \([0, d(a_1 \cdots a_i)] \setminus \{x_{i-1}, x_{i-2}\}\). If \(x_{i-2} \geq d(a_1 \cdots a_i) + 1\), then since \(\phi(A_i) \subseteq [\Delta_2 + h - 1 - d(a_1 \cdots a_{i-1}), \Delta_2 + h - 1]\) we can label \(A_{i+1}\) by \([0, d(a_1 \cdots a_i) - 1]\) \setminus \{x_{i-1}\}\. In each possibility the \(L(h, 1, 1)\)-conditions are satisfied by the labels for \(A_{i+1}\).

Up to now we have proved (a) and (b) by induction and thus finished labelling \(T_u\).

Part 3 (Labeling \(T_v\)): We label \(T_v\) by using similar techniques to those above. Note first that the vertices in \(L_1(v)\) were labelled in the initialization. If \(l_v \geq 2\), then for each \(b_1 \in L_1(v)\), we label the vertices of \(N(b_1) \setminus \{v\} \subseteq L_2(v)\) by \(\Delta_2 + h - d(b_1), \ldots, \Delta_2 + h - 2\) respectively, so that \(\phi(N(b_1) \setminus \{v\}) = [\Delta_2 + h - d(b_1), \Delta_2 + h - 2]\). We do this for all \(b_1 \in L_1(v)\) independently, and in this way all vertices in \(L_2(v)\) are labelled. Since \(\Delta_2 - d(b_1) \geq d(v)\) and \(\phi(N(v) \setminus \{u\}) = [1, d(v) - 1]\), by Lemma 7 this labelling satisfies the \(L(h, 1, 1)\)-conditions with vertices in \(\{u, v\} \cup L_1(v)\).

Note that for each vertex \(w \in \bigcup_{l \geq 3} L_i(v)\) we have \(N_3(w) \subseteq V(T_v)\) and hence we can label \(w\) without considering the labels used by \(T_u\). Similarly to (a) and (b), by induction we can prove the following for \(i = 1, 2, \ldots, l_v\) if \(T\) is finite and for all integers \(i \geq 1\) if \(T\) is infinite:

(c) if \(i\) is odd, then for all \(b_1 \cdots b_{i-1} \in L_{i-1}(v)\) we can label independently the vertices of \(N(b_1 \cdots b_{i-1}) \setminus \{b_1 \cdots b_{i-2}\}\) by the \(d(b_1 \cdots b_{i-1}) - 1\) smallest available integers in \([0, d(a_1 \cdots a_{i-1})]\) such that the \(L(h, 1, 1)\)-conditions are satisfied among vertices of \(T_v\) up to level \(L_i(v)\);

(d) if \(i\) is even, then for all \(b_1 \cdots b_{i-1} \in L_{i-1}(v)\) we can label independently the vertices of \(N(b_1 \cdots b_{i-1}) \setminus \{b_1 \cdots b_{i-2}\}\) by the \(d(b_1 \cdots b_{i-1}) - 1\) largest available integers in \([\Delta_2 + h - 1 - d(b_1 \cdots b_{i-1}), \Delta_2 + h - 1]\) such that the \(L(h, 1, 1)\)-conditions are satisfied among vertices of \(T_v\) up to level \(L_i(v)\).
The proof of these statements is similar to that of (a) and (b) and hence is omitted.

In summary, we have proved that $T$ admits an $L(h, 1, 1)$-labelling with span $\Delta_2 + h - 1$. Therefore, $\lambda_{h, 1, 1}(T) \leq \Delta_2 + h - 1$. \hfill $\square$

The proof of Lemma 8 is valid for both finite and infinite cases. Clearly, it gives an algorithm for constructing an $L(h, 1, 1)$-labelling of $T$ with span $\Delta_2 + h - 1$. In the case when $T$ is finite, this algorithm takes $O(n)$ time (where $n = |V(T)|$) for vertex-indexing and initialization, and $O(n)$ time for each of the $O(n)$ rounds of labelling as described in (a)-(d). Therefore, the algorithm runs in $O(n^2)$ time for finite trees.

The truth of (1) follows from Lemmas 6 and 8 immediately. In the introduction we have shown that the lower bound in (1) is achievable. To complete the proof of Theorem 1 we now prove that the upper bound in (1) is attainable as well when $h \geq 3$. Let $T^*$ be the tree defined by

$$V(T^*) = \{w\} \cup \{w_i, v_i : 1 \leq i \leq h + 2\} \cup \{v_{i,j}, z_{i,j} : 1 \leq i \leq h + 2, 1 \leq j \leq h + 1\}$$

$$E(T^*) = \{ww_i, w_iu_i, u_iv_i : 1 \leq i \leq h + 2\} \cup \{v_{i,j}, v_{i,j}z_{i,j} : 1 \leq i \leq h + 2, 1 \leq j \leq h + 1\}.$$

Lemma 9 Let $h \geq 3$ be an integer. Let $T$ be a finite tree or an infinite tree of finite maximum degree such that $\Delta(T) = \Delta(T^*) (= h + 2)$, $\Delta_2(T) = \Delta_2(T^*) (= h + 4)$ and $T$ contains $T^*$ as a subtree. Then

$$\lambda_{h, 1, 1}(T) = \Delta_2(T) + h - 1 = 2h + 3.$$

Proof Since $\lambda_{h, 1, 1}(T^*) \leq \lambda_{h, 1, 1}(T) \leq \Delta_2(T) + h - 1$ by Lemma 8, it suffices to prove $\lambda_{h, 1, 1}(T^*) \geq 2h + 3$.

Suppose to the contrary that $\lambda_{h, 1, 1}(T^*) \leq 2h + 2$ and let $\phi$ be an $L(h, 1, 1)$-labelling of $T^*$ with span $2h + 2$. We first prove:

Claim. If $v$ is a maximum degree vertex of $T^*$, then $\phi(v) \in \{0, 1, 2h + 1, 2h + 2\}$.

Suppose otherwise (that is, $\phi(v) \in [2, 2h]$) and let $A$ be the set of available labels for the neighbours of $v$. If $2 \leq \phi(v) < h$, then $A \subseteq \{\phi(v)+h, \ldots, 2h+2\}$ and so $|A| \leq h + 3 - \phi(v) \leq h + 1$. If $h \leq \phi(v) \leq h + 2$, then $A \subseteq \{0, \ldots, \phi(v) - h\} \cup \{\phi(v) + h, \ldots, 2h + 2\}$ and hence $|A| \leq 4 \leq h + 1$. If $h + 2 < \phi(v) \leq 2h$, then $A \subseteq \{0, \ldots, \phi(v) - h\}$ and hence $|A| \leq \phi(v) - h + 1 \leq h + 1$. Since $|N(v)| = h + 2$, in each of these cases there is not enough labels for the vertices in $N(v)$. This contradiction establishes the claim.

Since $w$ is a maximum degree vertex, by using the dual labelling $\lambda_{h, 1, 1}(T^*) - \phi(z)$ instead of $\phi(z)$ ($z \in V(T^*)$) when necessary, by the claim above we may assume without loss of generality that $\phi(w) \in \{0, 1\}$. Assume first that $\phi(w) = 1$. Then $\phi(\{w_1, \ldots, w_{h+2}\}) = \{h + 1, \ldots, 2h + 2\}$. Suppose without loss of generality that $\phi(w_1) = h + 1$. Then the only label available for $u_1$ is 0, that is, $\phi(u_1) = 0$. Now $v_1$ is a maximum degree vertex so that $\phi(v_1) \in \{2h + 1, 2h + 2\}$ by the claim above. If $\phi(v_1) = 2h + 1$, then $\phi(\{v_{1,1}, \ldots, v_{1,h+1}\}) \subseteq \{0, \ldots, h + 1\}\{0, h + 1\} = \{1, \ldots, h\}$. This is a contradiction because we need at least $h + 1$ labels for $v_{1,1}, \ldots, v_{1,h+1}$. Therefore, $\phi(v_1) = 2h + 2$ and so we must have $\phi(\{v_{1,1}, \ldots, v_{1,h+1}\}) = \{1, \ldots, h, h + 2\}$. Assume without loss of generality that $\phi(v_{1,1}) = h + 2$. Then $\phi(z_{1,1}) \in \{0, 1, 2, 2h + 2\}$. But this is a contradiction since $0, 2h + 2, 1, 2$ have been used by $u_1, v_1$ and two vertices in $\{v_{1,1}, \ldots, v_{1,h+1}\}$ respectively.
Assume next that \( \phi(w) = 0 \). Then \( \phi(\{w_1, \ldots, w_{h+2}\}) \subseteq \{h, \ldots, 2h+2\} \) and exactly one label in this set is not used by these vertices. Suppose that \( h+1 \) is not used and assume without loss of generality that \( \phi(w_1) = h \). Then there is no label available for \( u_1 \), a contradiction. Therefore, \( h+1 \) is used and without loss of generality we may assume \( \phi(w_1) = h+1 \). The labels available for \( u_1 \) are \( 1, 2h+1 \) and \( 2h+2 \), except possibly at most one of these labels. We consider the case \( \phi(u_1) = 1 \) only since the other two cases are similar. Since \( v_1 \) is a maximum degree vertex, by our claim it must be labelled \( 2h+1 \) or \( 2h+2 \). If \( \phi(v_1) = 2h+1 \), then the available label set for \( v_1, \ldots, v_{h+1} \) is \( \{0, \ldots, h+1\} \setminus \{1, h+1\} \), which contains less than \( h+1 \) labels, a contradiction. If \( \phi(v_1) = 2h+2 \), then the available label set for \( v_1, \ldots, v_{h+1} \) is \( \{0, \ldots, h+2\} \setminus \{1, h+1\} \), which has cardinality \( h+1 \). So we may assume without loss of generality that \( \phi(v_{1,1}) = h+2 \). However, there is no label available for \( z_{1,1} \), again a contradiction.

So far we have completed the proof of Theorem 1.

### 3 Proof of Theorems 2 and 3

As before we abbreviate \( \Delta(T), \Delta_2(T) \) to \( \Delta, \Delta_2 \) respectively. For a set \( X \) of integers, denote by \( \max X \) (\( \min X \)) the maximum (minimum) integer in \( X \).

**Proof of Theorem 2**

Let \( T \) be a finite tree with diameter at least three or an infinite tree of finite maximum degree. Let \( h \leq \delta^*(T) \). Choose a maximum degree vertex \( w \) as the root of \( T \) and set

\[
L_i := \{v \in V(T) : d(w, v) = i\}, \quad i = 0, 1, \ldots
\]

For any \( v \in V(T) \) we use \( p(v) \) to denote the parent of \( v \), and \( c_1(v), \ldots, c_{d(v)-1}(v) \) the children of \( v \).

**Claim.** There exists an \( L(h, 1, 1) \)-labelling \( \phi \) of \( T \) such that, for any \( k \geq 2 \) and \( v \in L_k \),

\[
\phi(\{p(p(v)), c_1(p(v)), \ldots, c_{d(p(v))-1}(p(v))\}) = \{a \mod (\Delta_2 + h - 1), \ldots, (a + d(p(v)) - 1) \mod (\Delta_2 + h - 1)\}
\]

for some \( a \in [0, \Delta_2 + h - 2] \).

We prove this claim by constructing \( \phi \) inductively. To begin with we define

\[
\phi(w) = 0
\]

\[
\phi(c_i(w)) = \Delta_2 + h - i - 1, \quad i = 1, \ldots, \Delta.
\]

For each \( i \) such that \( c_i(w) \) has at least one child, define

\[
\phi(c_j(c_i(w))) = j, \quad j = 1, \ldots, d(c_i(w)) - 1.
\]

Clearly, (12) holds for \( k = 2 \) and \( v \in L_2 \) (with \( a = 0 \)). Observe that the smallest label used by a child of \( w \) is \( \Delta_2 + h - \Delta - 1 \leq \Delta_2 + 2h - \Delta - 1 = h + 1 \). Note also that \( \phi(c_i(w)) - \phi(c_j(c_i(w))) = (\Delta_2 + h - i - 1) - j \geq (\Delta_2 + h - \Delta - 1) - (d(c_i(w)) - 1) \geq (\Delta_2 + h - \Delta - 1) - (\Delta_2 - \Delta - 1) = h \).

Thus \( \phi \) satisfies the \( L(h, 1, 1) \)-conditions among vertices in \( L_0 \cup L_1 \cup L_2 \).
Assume that we have labelled the vertices of $T$ up to some level $k \geq 2$ such that (12) holds for $k$ and $v \in L_k$ and the $L(h, 1, 1)$-conditions are satisfied among vertices up to $L_k$. We extend $\phi$ to level $L_{k+1}$ in the following way.

For any $u \in L_k$, let $C := \phi(\{p(p(u)), c_1(p(u)), \ldots, c_j(p(u))\})$, where $j = d(p(u)) - 1$. Then, by our induction hypothesis, $C = \{a_1 \mod (\Delta_2 + h - 1), \ldots, (a_1 + j) \mod (\Delta_2 + h - 1)\}$ for some $a_1 \in [0, \Delta_2 + h - 2]$. Let $A := [0, \Delta_2 + h - 2] \setminus C$. Then $A = \{a_2 \mod (\Delta_2 + h - 1), \ldots, (a_2 + b_1) \mod (\Delta_2 + h - 1)\}$, where $a_2 = (a_1 + j + 1) \mod (\Delta_2 + h - 1)$ and $b_1 = \Delta_2 + h - j - 3$. All labels in $A \setminus \{\phi(p(u))\}$ are available for the children of $u$, except the ones in $B := [\phi(u) - h + 1, \phi(u) - 1] \cup [\phi(u) + 1, \phi(u) + h - 1]$. Since $\phi(u) \in C$ and $h \leq \delta^*(T) \leq d(p(u)) = j + 1 = |C|$, there are at least

$$((\phi(u) - \min C) + \min\{h - 1, \min C + j - \phi(u)\}) \geq h - 1$$

labels in $B$ which are either in $C$ or not in $[0, \Delta_2 + h - 2]$ at all. So $|B \cap A| \leq h - 1$ as $|B| = 2(h - 1)$. Therefore, the label set available for the children of $u$ is $(A \setminus \{\phi(p(u))\}) \setminus B$ which has cardinality at least $(\Delta_2 + h - j - 3) - (h - 1) = \Delta_2 - j - 2$. Since $d(u) + d(p(u)) \leq \Delta_2$, $u$ has at most $\Delta_2 - j - 2$ children and so there are enough labels in $(A \setminus \{\phi(p(u))\}) \setminus B$ to label them without violating the $L(h, 1, 1)$-conditions. Note that $A \setminus B$ has the form $\{a_3 \mod (\Delta_2 + h - 1), \ldots, (a_3 + b_2) \mod (\Delta_2 + h - 1)\}$ for some $a_3 \in [0, \Delta_2 + h - 2]$ and $b_2 \geq \Delta_2 - j - 2$. Because of this we may select legal labels in $\{a_3 \mod (\Delta_2 + h - 1), \ldots, (a_3 + b_2) \mod (\Delta_2 + h - 1)\}$ around $\phi(p(u))$ to label the children of $u$ such that (12) holds for each child $v$ of $u$. (Note that $p(p(v)) = p(u)$.) In this way, we have extended $\phi$ to level $L_{k+1}$ and hence completed the proof of the claim. (If $T$ is finite then we stop in a finite number of inductive steps. If $T$ is infinite then we continue the labelling process indefinitely.)

Since $\phi$ has span $\Delta_2 + h - 2$, we have $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 2$. \hfill $\Box$

**Proof of Theorem 3** Because of Lemma 6 it suffices to prove the upper bounds.

We first consider the case where $T$ is a finite caterpillar. Let $v_1, v_2, \ldots, v_n$ be the spine of $T$, where $n \geq 2$. Then $\Delta_2 \geq 4$ and $2 \leq d(v_i) \leq \Delta_2 - 2$ for $i = 1, 2, \ldots, n$. Let $v_0$ be a fixed neighbor of $v_1$ other than $v_2$, and $v_{n+1}$ a fixed neighbor of $v_n$ other than $v_{n-1}$. Let

$$V_i := N(v_i) \setminus \{v_{i-1}, v_{i+1}\}, \ i = 1, 2, \ldots, n.$$  

In the case when $n = 2$, assigning 0 to $v_1$, $\Delta_2 + h - 3$ to $v_2$, $d(v_2) + h - 2, d(v_2) + h - 1, \ldots, \Delta_2 + h - 4$ to the neighbors of $v_1$ other than $v_2$, and $1, 2, \ldots, d(v_2) - 1$ to the neighbors of $v_2$ other than $v_1$, we get an $L(h, 1, 1)$-labelling of $T$ with span $\Delta_2 + h - 3$. Hence $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 3$. Thus we assume $n \geq 3$ in the following.

**Case 1. There exists no $v_i$ on the spine such that $d(v_i) = \Delta_2 - 2$.**

In this case $\Delta_2 - d(v_i) \geq 3$ for $i = 1, 2, \ldots, n$ and so $\Delta_2 \geq 5$. If $\Delta_2 = 5$, then there exists a heavy edge on the spine whose end-vertices have degrees 2 and 3 ($= \Delta_2 - 2$) respectively, a
contradiction. Hence $\Delta_2 \geq 6$. We first define, for $i = 0, 1, \ldots, n + 1$,
\[
\phi(v_i) = \begin{cases} 
0, & i \equiv 0 \pmod{4} \\
\Delta_2 + h - 3, & i \equiv 1 \pmod{4} \\
1, & i \equiv 2 \pmod{4} \\
\Delta_2 + h - 4, & i \equiv 3 \pmod{4}.
\end{cases} \tag{13}
\]
Then for each $i = 1, \ldots, n$ with $d(v_i) > 2$ we assign $|V_i|$ distinct labels to the vertices in $V_i$, one label for each vertex but in an arbitrary manner, such that
\[
\phi(v_i) = \begin{cases} 
[\Delta_2 + h - 2 - d(v_i), \Delta_2 + h - 5], & i \equiv 0 \pmod{2} \\
[2, d(v_i) - 1], & i \equiv 1 \pmod{2}.
\end{cases} \tag{14}
\]
Since $\Delta_2 \geq 6$, it is clear that the vertices on the spine satisfy the $L(h, 1, 1)$-conditions. For $u_i \in V_i, u_j \in V_j, d(u_i, u_j)$ is 2 if $i = j$, 3 if $|i - j| = 1$, and at least 4 if $|i - j| \geq 2$. From the definition of $\phi$ it follows that $|\phi(u_i) - \phi(u_{i+1})| \geq \Delta_2 + h - 1 - (d(v_i) + d(v_{i+1})) \geq h - 1 \geq 1$ for $i = 1, 2, \ldots, n - 1$. For $i \equiv 1 \pmod{4}, \phi(v_i) - \max \phi(V_i) = (\Delta_2 + h - 3) - (d(v_i) - 1) \geq h + 1$
since $\Delta_2 - d(v_i) \geq 3$. For $i \equiv 2 \pmod{4}, \min \phi(V_i) - \phi(v_i) = \Delta_2 + h - 3 - d(v_i) \geq h$. Similarly,
for $i \equiv 3 \pmod{4}, \phi(v_i) - \max \phi(V_i) = \Delta_2 + h - 3 - d(v_i) \geq h, and for i \equiv 0 \pmod{4}, \min \phi(V_i) - \phi(v_i) = \Delta_2 + h - 2 - d(v_i) \geq h + 1$. Since $h \geq 2$, by the definition of $\phi$ for any $i$ between 0 and $n + 1$ and any vertex $u$ not on the spine such that $d(u, v_i) = 2$ or 3, we have $\phi(u) \neq \phi(v_i)$. Therefore, $\phi$ is an $L(h, 1, 1)$-labelling of $T$ with span $\Delta_2 + h - 3$, and hence $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 3$.

Case 2. There exists $v_{i^*}$ on the spine such that $d(v_{i^*}) = \Delta_2 - 2$, where $1 \leq i^* \leq n$.

In this case we have $d(v_{i^*-1}) = 2$ (if $i^* > 1$) and $d(v_{i^*+1}) = 2$ (if $i^* < n$) by the definition of $\Delta_2$. Define, for $i = 0, 1, \ldots, n + 1$,
\[
\phi(v_i) = \begin{cases} 
0, & i - i^* \equiv 0 \pmod{4} \\
\Delta_2 + h - 2, & i - i^* \equiv 1 \pmod{4} \\
1, & i - i^* \equiv 2 \pmod{4} \\
\Delta_2 + h - 3, & i - i^* \equiv 3 \pmod{4}.
\end{cases} \tag{15}
\]
Then for each $i = 1, \ldots, n$ with $d(v_i) > 2$ assign $|V_i|$ distinct labels to the vertices in $V_i$, one label per vertex, such that
\[
\phi(V_i) = \begin{cases} 
[\Delta_2 + h - 1 - d(v_i), \Delta_2 + h - 4], & i - i^* \equiv 0 \pmod{2} \\
[2, d(v_i) - 1], & i - i^* \equiv 1 \pmod{2}.
\end{cases} \tag{16}
\]
Similar to Case 1, one can verify that $\phi$ is an $L(h, 1, 1)$-labelling of $T$ with span $\Delta_2 + h - 2$. Hence $\lambda_{h,1,1}(T) \leq \Delta_2 + h - 2$.

Under any $L(h, 1, 1)$-labelling $\phi$ of $T$, the vertices in $V_{i^*} \cup \{v_{i^*-1}, v_{i^*}, v_{i^*+1}\}$ receive distinct labels, and moreover the label of $v_{i^*}$ must differ by at least $h$ from the labels of the other $\Delta_2 - 2$ vertices in this set. This is possible only when the span is at least $\Delta_2 + h - 3$. Moreover, if the span of $\phi$ is $\Delta_2 + h - 3$, then we must have $\phi(v_{i^*}) = 0$ or $\Delta_2 + h - 3$, and both $\phi(v_{i^*-2})$ (if
\(i^* > 1\) and \(\phi(v_{i^*+2})\) (if \(i^* < n\)) are at least 1 or at most \(\Delta_2 + h - 4\), respectively. Thus, if the span of \(\phi\) is \(\Delta_2 + h - 3\), we must have \(d(v_{i^*-2}) < \Delta_2 - 2\) and \(d(v_{i^*+2}) < \Delta_2 - 2\), for otherwise \(\phi(v_{i^*-2})\) and \(\phi(v_{i^*+2})\) are 0 or \(\Delta_2 + h - 3\) by the previous sentence, a contradiction. In other words, if there exist consecutive vertices \(u,v,w\) on the spine such that \(d(u) = d(w) = \Delta_2 - 2\) and \(d(v) = 2\), then \(\lambda_{h,1,1}(T) \geq \Delta_2 + h - 2\) and hence \(\lambda_{h,1,1}(T) = \Delta_2 + h - 2\) by the upper bound in the previous paragraph.

Now we assume that \(T\) is an infinite caterpillar with finite maximum degree. Then either (i) the spine of \(T\) has one open end, or (ii) it has two open ends. In the former case let the spine be \(v_1, v_2, \ldots\) and let \(v_0\) be a neighbor of \(v_1\) other than \(v_2\), and in the latter case let the spine be \(\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\). In both cases we extend the rules (13) and (15) to all vertices \(v_i\), where \(i \geq 0\) in case (i) and \(i = \ldots, -2, -1, 0, 1, 2, \ldots\) in case (ii). Then we apply (14) and (16) to all \(V_i\), where \(i \geq 1\) in (i) and \(i = \ldots, -2, -1, 0, 1, 2, \ldots\) in (ii). The results follow from the same argument as in the finite case.

\[\square\]

## 4 Remarks and questions

If all vertices on the spine of a finite caterpillar \(T\) have maximum degree \(\Delta\), then \(\lambda_{h,1,1}(T) = \max\{h, \Delta - 1\} + \Delta = \max\{h + \Delta_2(T)/2, \Delta_2(T) - 1\}\) as shown by Jinjiang Yuan. (We are grateful to Jinjiang for informing us of this result.) This indicates that the upper bound in Theorem 3 is far away from the actual value of \(\lambda_{h,1,1}\) in certain cases, although it is attainable in some other cases.

The condition \(h \leq \delta^*(T)\) is sufficient but not necessary to guarantee (2). In fact, if a finite tree \(T\) of diameter at least three has only one heavy edge, then we can prove \(\lambda_{h,1,1}(T) \leq \Delta_2(T) + h - 2\) by modifying the proof of Lemma 8. To achieve this we simply decrease the labels of the vertices in \(L_i(u)\) (\(i \geq 1\) is odd) and \(L_i(v)\) (\(i \geq 0\) is even) by one. Since \(T\) has only one heavy edge, we have \(d(a_1 \cdots a_i) + d(a_1 \cdots a_{i-1}) < \Delta_2\) and \(d(b_1 \cdots b_i) + d(b_1 \cdots b_{i-1}) < \Delta_2\) for \(i \geq 1\), and these inequalities ensure the validity of modified statements (a)-(d).

In view of Theorem 1 and Corollaries 4 and 5, we may ask the following questions naturally.

**Question 10** (a) Given \(h \geq 3\), characterise those finite trees \(T\) with diameter at least three such that \(\lambda_{h,1,1}(T) = \Delta_2(T) + h - 1\).

(b) Characterise finite trees \(T\) with diameter at least three such that \(\lambda_{2,1,1}(T) = \Delta_2(T)\).

Similar questions may be asked for infinite trees with finite maximum degree.

We speculate that ‘most’ finite trees of diameter at least three would have \(\lambda_{2,1,1}\)-number \(\Delta_2 - 1\). To make this precise let \(N(n)\) be the number of pairwise non-isomorphic trees with \(n\) vertices and diameter at least three, and let \(N_1(n)\) be the number of such trees with \(\lambda_{2,1,1} = \Delta_2 - 1\).

**Conjecture 11** \(\lim_{n \to \infty} \frac{N_1(n)}{N(n)} = 1\).

We finish this article by the following question.

**Question 12** Given \(h \geq 1\), is the problem of determining \(\lambda_{h,1,1}\) for finite trees solvable in polynomial time?
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References


