1. Complex Numbers.
   (a) Simplify \(\left|\frac{(\pi + i)^{100}}{(\pi - i)^{100}}\right|\).
   (b) If \(z_1 = -1 + i\) and \(z_2 = i\) find
   \[f(z_1, z_2) = \text{Arg } z_1 + \text{Arg } z_2 - \text{Arg}(z_1z_2)\]
   where \(\text{Arg}\) denotes the principal argument. What values are possible for \(f(z_1, z_2)\) with arbitrary \(z_1, z_2\)?
   (c) Find all solutions of \(e^z = -1\) with \(z \in \mathbb{C}\). How many solutions are there? Is the expected solution \(z = \log(-1)\) single-valued? How can we make the logarithm function single-valued?

2. Solving Equations.
   (a) By the fundamental theorem of algebra the complex polynomial equation
   \[z^3 - (2 + i)z^2 + 2(1 + i)z - 2i = 0\]
   has precisely three roots in \(\mathbb{C}\). Verify that \(z = i\) is one of the roots. Hence find all three roots expressing these in cartesian and exponential polar form.
   (b) Find all solutions for the complex variable \(z\) satisfying the equation
   \[z^2 = 1 + \text{Re } z\]
   How many solutions are there for \(z\)? Explain why the fundamental theorem of algebra does or does not apply.

3. Trig/Hyperbolic Identity. Prove the trigonometric/hyperbolic identity
   \[\sin(z + iw) = \sin z \cosh w + i \cos z \sinh w\]
   for arbitrary complex \(z, w \in \mathbb{C}\).

4. Planar Sets. Sketch the following loci or planar sets in the complex plane and decide if they are open, closed or neither:
   (a) \(|z + 2| = |z - 1|\)
   (b) \(|z - 1| \leq \text{Re } z + 1\)
   (c) \(0 < |z - 1| \leq 2\)
   (d) \(|z - 3| \geq 4\)
5. A Derivative.

(a) Find from first principles the derivative of \( f(z) = \sqrt{z} \). For what region in \( \mathbb{C} \) is this function analytic?

(b) Deduce for what region \( f(z) = \sqrt{z} \) is continuous.

(c) If \( z \) is on the unit circle, that is \( z = e^{i\theta} \) with \( -\pi < \theta \leq \pi \), what is the value of \( f(z) = \sqrt{z} \) when (i) \( \theta \to \pi^{-} \) and (ii) \( \theta \to -\pi^{+} \)? Is it really true that \( f(z) = \sqrt{z} \) is continuous at \( z = -1 \)?

6. Complex Limits. Evaluate the following limits if they exist

\[
\begin{align*}
(a) & \quad \lim_{z \to \infty} e^{-z} \\
(b) & \quad \lim_{z \to 0} \frac{\sinh 2z}{\tan 3z} \\
(c) & \quad \lim_{z \to i} \sqrt{\cosh \pi z} \\
(d) & \quad \lim_{z \to \infty} \frac{1}{z} \sinh 1/z
\end{align*}
\]

7. Cauchy-Riemann and Derivatives.

(a) If \( z = x + iy \), prove the following identities are valid for all \( x, y \in \mathbb{R} \)

\[
\begin{align*}
\sin z &= \sin x \cosh y + i \cos x \sinh y \\
\cos z &= \cos x \cosh y - i \sin x \sinh y
\end{align*}
\]

(b) Find the largest domain on which the real and imaginary parts of the complex functions \( \sin z \) and \( \cos z \) are continuous in \( \mathbb{C} \). Hence deduce the largest domain on which the complex functions \( \sin z \) and \( \cos z \) are continuous stating clearly any theorems you use.

(c) Also use the Cauchy-Riemann theorem (i) to find the largest domain on which \( \sin z \) and \( \cos z \) are analytic in \( \mathbb{C} \) and (ii) to find their derivatives.
8. **Harmonic Functions.** Show that the function of two real variables

\[ u(x, y) = x^2 - y^2 + x \]

is harmonic in \( \mathbb{R}^2 \). Suppose \( f(z) = u(x, y) + iv(x, y) \) is analytic in \( \mathbb{C} \). Use the Cauchy-Riemann equations and harmonic conjugates to find \( f(z) \) expressed only in terms of \( z \).

9. **Transcendental Powers.** Evaluate the following

(a) \( (1 - i)^{3/4} \)
(b) \( \lim_{z \to \pi i} \frac{\sinh z}{z^2 + \pi^2} \)
(c) \( \sinh(\log(i - 1)) \)
(d) \( \lim_{z \to -i} \log \left( \frac{i}{\cosh(\pi z/4)} \right) \)
(e) \( i^{1+i} \) (principal value)

10. **Arcsin and Arccos.**

(a) Find the domain and ranges of the functions \( \text{Arcsin} \ z \) and \( \text{Arccos} \ z \) and use the chain rule to find the derivatives.

(b) Hence show that

\[ \text{Arcsin} \ z + \text{Arccos} \ z = \frac{\pi}{2}, \quad z \in \mathbb{C}\setminus\{(-\infty, -1] \cup [1, \infty)\} \]

(c) Deduce the same result using the expressions

\[ \text{Arcsin} \ z = -i \log[iz + \sqrt{1 - z^2}], \quad \text{Arccos} \ z = -i \log[z + i\sqrt{1 - z^2}] \]

where the principal branch of the square root is taken.

11. **Laplace and Arctan.** Let \( z = x + iy \) and \( D \) be the first quadrant with \( x > 0 \) and \( y > 0 \).

(a) If \( v(x, y) \) is harmonic in \( D \), show that \( a \, v(x, y) + b \) (where \( a, b \) are constants) is also harmonic in \( D \).

(b) Show that \( \text{Arg} \ z = \text{Arctan}(y/x) \) in \( D \) and hence that \( v(x, y) = \text{Arg} \ z \) is harmonic in \( D \).

(c) The boundary of a very large metal sheet is kept at a constant temperature \( v = 100^\circ \) on the bottom edge and \( v = 50^\circ \) on the left edge. In equilibrium, the steady-state temperature distribution \( v(x, y) \) satisfies Laplace’s equation \( \nabla^2 v = 0 \). Find the solution \( v(x, y) \) of this boundary value problem.

(d) Use the fact that \( \log z \) is analytic in \( D \) to find the harmonic conjugate \( u(x, y) \) of \( v(x, y) = \text{Arctan}(y/x) \) in \( D \).
12. **Complex Series.**

(a) Find the largest region in which the following complex series converges and find its sum

$$\sum_{n=0}^{\infty} \left[ \left( \frac{2}{z} \right)^n + \left( \frac{z}{3} \right)^n \right]$$

(b) Find the radius of convergence for each of the complex power series

$$\sum_{n=0}^{\infty} \frac{n!}{(2n)!} z^{2n}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3^{n/2} + n}$$

13. **Term-by-Term Integration.**

(a) Show that the following series converges absolutely and uniformly in $|z| \leq 1$

$$\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}, \quad |z| \leq 1$$

(b) Find the sum of this series in $|z| \leq R < 1$ by integrating the geometric series twice. Explain carefully why this process is valid.

(c) Show that the result in (b) is not singular at $z = 0$ or $z = 1$. Hence show how the result in (b) can be extended to the domain $|z| \leq 1$?

14. **Differentiating Taylor Series.**

(a) Use the known Taylor series for $\log(1 + z)$ to obtain the Taylor series for the principal value of $\arctan z$

$$\arctan z := \frac{1}{2i} \log \left( \frac{1 + iz}{1 - iz} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n + 1}, \quad |z| < 1$$

(b) Use term-by-term differentiation to show that

$$\frac{d}{dz} \arctan z = \frac{1}{1 + z^2}, \quad |z| < 1$$

explaining why this process is valid.
15. **Cauchy Theorem.** Verify Cauchy’s theorem \( \oint_\Gamma f(z)dz = 0 \) for \( f(z) = z^2 - iz \) where

(a) \( \Gamma \) is the triangle with vertices at \( z = 0, 1, i \)

(b) \( \Gamma \) is the ellipse \( |z - i| + |z + i| = 6 \)

16. **Contour Integrals.**
   
   (a) Evaluate the contour integrals
   
   \[ \text{Re} \int_{-1}^{1} z \, dz, \quad \int_{-1}^{1} \text{Re} \, dz \]
   
   from \( z = -1 \) to \( z = 1 \) along the upper and lower semi-circles of the unit circle \( |z| = 1 \). Explain why the fundamental theorem of calculus can or cannot be applied in each case.

(b) Hence evaluate the closed contour integrals

   \[ \text{Re} \oint_{|z|=1} z \, dz, \quad \oint_{|z|=1} \text{Re} \, dz \]

   and explain why Cauchy’s theorem can or cannot be applied in each case.

17. **Fundamental Theorem of Calculus.**
   
   (a) Use the fundamental theorem of calculus to evaluate the closed contour integral
   
   \[ \oint_{|z|=1} \sqrt{z} \, dz \]
   
   where the principal value is used for the square root. Check your answer by parametrizing the contour to evaluate the integral.

(b) Explain why Cauchy’s theorem does or does not apply.
18. **Figure Eight.**

(a) Use Cauchy’s theorem to show that

\[ \oint_{\Gamma} \frac{dz}{z - a} = 2\pi i \]

for any simple curve \( \Gamma \) that encircles \( z = a \) once in an anti-clockwise direction.

(b) Let \( C \) be a figure of eight curve that encircles \( z = 1 \) once in an anti-clockwise direction and \( z = i \) once in a clockwise direction. Explain why \( C \) is or is not a simple curve. Use the result of part (a), Cauchy’s theorem and partial fractions to evaluate the closed integral

\[ \oint_{C} \frac{z \, dz}{(z - i)(z - 1)} \]

19. **Cauchy Integrals.** Clearly state Cauchy’s theorem and the general Cauchy integral formula. Hence evaluate the following closed contour integrals

(a) \[ \oint_{|z| = 1} \frac{z \, dz}{z - 2} \]

(b) \[ \oint_{|z| = 2} \frac{z \, dz}{z - 1} \]

(c) \[ \oint_{|z| = 2} \frac{z \, dz}{(z - 1)^3} \]

(d) \[ \oint_{|z| = 2} \frac{z \, dz}{(z - 1)(z - 3)} \]

20. **Trigonometric Integral.** Use the generalized Cauchy integral formula to evaluate the trigonometric integral

\[ I = \int_{0}^{2\pi} \frac{dt}{(2 + \cos t)^2} \]
21. **Laurent Series.** Find the Laurent series of the function

\[ f(z) = \frac{1}{z(2z + 1)(z - 1)} \]

in each of the regions

(a) \(0 < |z| < \frac{1}{2}\), \quad (b) \(\frac{1}{2} < |z| < 1\), \quad (c) \(|z| > 1\)

22. **Singularities.**

(a) Identify all singularities of the following functions \(f(z)\) and evaluate the residues at any poles:

(i) \(\frac{(z - 1)^2}{(z + 1)(z + 3)}\), \quad (ii) \(\frac{\text{Arctan } z}{z}\), \quad (iii) \(\frac{\text{Log}(1 - z)}{z}\), \quad (iv) \(\frac{z}{(1 - z)^3}\)

(b) A function \(f(z)\) is said to have a singularity of a given type at \(z = \infty\) if \(f\left(\frac{1}{z}\right)\) has a singularity of that type at \(z = 0\). Identify any singularities in the functions (i)–(iv) at \(z = \infty\).

23. **Residues.**

(a) Evaluate the residues at the poles in the upper-half complex plane of the meromorphic function

\[ f(z) = \frac{1}{(z^2 + 1)(z^2 + 4)} \]

(b) Find the residues at all poles inside the unit circle of the function

\[ f(z) = \frac{z^4 + 1}{z^2(z^2 + 4z + 1)} \]
24. **Trigonometric Integral.**

   (a) Use the generalized Cauchy integral formula to evaluate the trigonometric integral

   \[
   I = \int_0^{2\pi} \frac{1 + \sin t}{(2 + \cos t)^2} \, dt
   \]

   (b) Use the residue theorem to evaluate the same trigonometric integral \( I \).

25. **Improper Integral.**

   (a) Evaluate the residues at the poles in the upper-half complex plane of the meromorphic function

   \[
   f(z) = \frac{1}{z^4 + z^2 + 1}
   \]

   (b) Hence use residue calculus, with a contour closed by a semi-circle in the upper-half plane, to calculate the real improper integral

   \[
   I = \int_0^\infty \frac{dx}{x^4 + x^2 + 1}
   \]
26. Integrals by Residues. Use residue calculus to evaluate the following real integrals

(a) \[ \int_{0}^{\infty} \frac{x^2 \, dx}{(x^2 + 1)(x^2 + 4)} \]  
(b) \[ \int_{0}^{\infty} \frac{x^6 \, dx}{(x^4 + 1)^2} \]

27. Limiting Contours. Use residue calculus and limiting contour theorems to evaluate the improper real integral

\[ I = \int_{0}^{\infty} \frac{\sin x \, dx}{x(1 + x^2)} \]