1. Let $D \subseteq \mathbb{C}$ be the open unit disk; $D = \{z \mid |z|^2 < 1\}$. Define a metric on $D$ by the first fundamental form

$$4 \frac{1}{(1 - |z|^2)^2} |dz|^2.$$ 

Here, if $z = x + iy$, then $|dz|^2 = dx^2 + dy^2$.

(a) Show that any diffeomorphism $f : D \to D$ is an isometry iff, when we write $w = f(z)$, then

$$4 \frac{1}{(1 - |w|^2)^2} |dw|^2 = 4 \frac{1}{(1 - |z|^2)^2} |dz|^2.$$ 

(b) Show that the Möbius transformation

$$f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z}$$

is an isometry of $D$, for any $a \in D$ and $\theta \in \mathbb{R}$.

(c) I claim that the line given by the $y$-axis is a geodesic in $D$. Specifically, if you define $\gamma : \mathbb{R} \to D$ by

$$\gamma(t) = \left(0, \tanh \left(\frac{t}{2}\right)\right),$$

then $\gamma$ is parameterized according to arclength, and satisfies the geodesic equations. Show that this is in fact true.

(d) Show that any other line through the origin is the image of a geodesic.

(e) **Harder:** Show that any semicircle meeting the unit circle $\partial D$ at right angles is a geodesic.

(f) Compute the distance from 0 to $z \in D$. **Warning:** we haven’t yet proved that a geodesic minimizes distance. You’ll have to do this by hand.

2. Let $X$ be a surface with constant Gaussian curvature $K(p) = -1$ for every $p \in X$. If the genus of $X$ is $g$, what is its area?

3. Show that if $X$ is an oriented surface with Gaussian curvature $K(p) > 0$ for every $p \in X$, then $X$ is homeomorphic to a sphere. Conversely, show that if $K(p) < 0$ for every $p \in X$, then the genus of $X$ is at least 2.

4. Let $S_1$ and $S_2$ be the (geodesic) semicircles in the hyperbolic plane $\mathbb{H}$ which are centered on the origin and have radius 1 and 2:

$$S_1 = \{(x, y) \mid x^2 + y^2 = 1, \ y > 0\}, \quad \text{and} \quad S_2 = \{(x, y) \mid x^2 + y^2 = 4, \ y > 0\}.$$ 

Show that there is a unique shortest path connecting $S_1$ and $S_2$. **Hint:** Two things to consider when trying to prove this:

- What sort of angle does such a path make with $S_1$ and $S_2$?
- You may find the lemma about triangles that we used to prove Gauss-Bonnet useful.