Topology: solutions to practice problem sets #1

1. We need to show that homotopy is reflexive, symmetric, and transitive.
   - For reflexivity, a homotopy H from f to f is $H_t(X) = f(x), \forall t$.
   - For symmetry, if J is a homotopy from f to g, then $J_0 = f$ and $J_1 = g$. Define a homotopy $K_t := J_{1-t}$; K is a homotopy from g to f.
   - For transitivity, if H is a homotopy from f to g, and J is a homotopy from g to h, then define a homotopy K from f to h by

     $$K_t(x) := \begin{cases} H(2t), & 0 \leq t \leq 1/2 \\ J(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

2. Same plan as for #1.
   - The identity $\mathrm{id} : X \to X$ is a homotopy equivalence.
   - If $f : X \to Y$ is a homotopy equivalence, then by definition, there is a map $f' : Y \to X$ so that $f \circ f'$ and $f' \circ f$ are homotopic to the appropriate identities. Thus f' is a homotopy equivalence.
   - Let $f : X \to Y$ and $g : Y \to Z$ be homotopy equivalences. Then, by definition, there are maps $f' : Y \to X$ and $g' : Z \to Y$ which are inverse homotopy equivalences. We claim that $g \circ f : X \to Z$ is a homotopy equivalence (with homotopy inverse $f' \circ g'$), because

     $$(g \circ f) \circ (f' \circ g') = g \circ (f \circ f') \circ g' \simeq g \circ \mathrm{id} \circ g' = g \circ g' \simeq \mathrm{id}$$

     (and similarly for the other direction of composition).

3. The inclusion $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ given by $i(z) = z$ has a homotopy inverse $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ given by $r(z) = z/|z|$. In fact, r is a deformation retraction: $r \circ i = \mathrm{id}_{S^{n-1}}$. On the other hand, $i \circ r(z) = z/|z|$. A homotopy from $i \circ r$ to $\mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ is given by $H_t(z) = z/(t + (1-t)|z|)$.

4. (a) First notice that ± 1 are *central*: they commute with all elements of the group. So when a or b are ± 1 all commutators $[a, b] = 1$. Secondly, notice that $[a, a] = 1, \forall a$.

   Third, the inverses of i, j, and k are their negatives (e.g., $i \cdot (-i) = -i \cdot i = -(-1) = 1$). Then compute:

   $$[i, j] = iji^{-1}j^{-1} = ij(-i)(-j) = k^2 = -1$$

   Similarly, the brackets of all other elements involving only $\pm i, \pm j$, and $\pm k$ are ± 1. This proves that the commutator subgroup $[Q_8, Q_8]$ is generated by $\{\pm 1\}$. Since that is already a normal subgroup, we have $[Q_8, Q_8] = \{\pm 1\}$.

   (b) Q_8 has 8 elements, and $[Q_8, Q_8]$ has 2. Therefore $Q_8/[Q_8, Q_8]$ has four elements and is, by definition, abelian. There are exactly two abelian groups of order 4, $\mathbb{Z}/4$ and $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. All classes in $Q_8/[Q_8, Q_8]$ have order either 1 or 2; e.g.:

   $$i^2 = -1 = 1 \text{ in } Q_8/[Q_8, Q_8]$$

   Therefore $Q_8/[Q_8, Q_8] = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, since $\mathbb{Z}/4$ has an element of order 4.

5. If X is contractible, it is homotopy equivalent to a point $*$. Therefore $\pi_1(X, x_0) \cong \pi_1(*, *) = 0$.