GALTON WATSON FRACTAL SIGNALS

G. Decrouez, P-O. Amblard, J-M. Brossier

LIS/ENSIEG
Laboratoire des Images et des signaux
BP 46, 38402 Saint Martin d’Hères, France

O.D. Jones

The University of Melbourne
Department of Mathematics and Statistics
Victoria 3010, Australia

ABSTRACT
Iterated Function Systems (IFS) is a relevant model to produce fractal functions, whether deterministic (with strict self-similarity) or random (self-similar up to probability distribution). The basic idea of such a construction is to start with an initial function and then compress, dilate and translate it such that by doing so over and over again, we end up with a self-similar signal. This construction relies on a construction tree which has always been deterministic in the litterature for signals. Here we introduce new fractals, called Galton Watson fractals, as fixed points of IFS with a random underlying construction tree and random operators. We give a proof of the existence and uniqueness of a fixed point at the random and distribution level.

Index Terms— Fractals, Galton Watson Trees, Iterated Function Systems, Random fixed points, Self-Similarity.

1. INTRODUCTION
Since the discovery of the relevancy of fractal and $1/f$ processes to model natural phenomena, the fractal formalism has received increasing interests during the last 20 years, with applications to a wide variety of fields such as finance, turbulence, meteorology, image compression, network traffic [1, 2, 3]. There are many ways of producing fractal objects and one of the simplest ones has been studied by Barnsley [4], Hutchinson and Ruschendorff [5] and is called Iterated Function Systems (IFS). This formalism has been extended to random fractal sets with self similarity arising up to probability distribution. This has the merit to break the strict self-similarity emerging from deterministic fractals, giving more relevant fractal sets to model physical phenomena.

Roughly speaking, an IFS recursively apply a contractive operator $T$ (random or not) on an initial function $f_0$. The completeness of the metric space where the fractal lives assures the existence and uniqueness of a fixed point $f^*$, thanks to the well known Banach fixed point theorem. In other words, if we denote by $T^n$ the $n$-th iterate of $T$, one has:

$$T^n f_0 \to f^* \text{ as } n \to +\infty$$  \hspace{1cm} (1)

where $f^*$ is the only function which satisfies $f = Tf$. We are often concerned with IFS acting over the space of square integrable functions or finite energy signals in signal processing. This space is denoted $L_2(X)$ for deterministic signals and $L_2(X)$ for random signals. However, as it will be mentioned later, we will work with the more general spaces $L_p(X)$ and $L_p(X)$. $X$ is a compact subset of the real line and we will set $X = [0, 1]$ in the remainder without loss of generality. At each iteration, functions are stretched, compressed, translated by means of the contractive operator $T$. Generally, we decompose this operator into a set of $M$ operators $\phi_i : \mathbb{R} \times X \to \mathbb{R}$, $i \in \{1,...,M\}$. Each $\phi_i$ has its own way of deforming the signal, and the resulting signal will lie in a subinterval of $X$ (hence the compression). Mathematically, this can be written as

$$ (Tf)(x) = \sum_{i=1}^{M} \phi_i[f(g_i^{-1}(x)), g_i^{-1}(x))]1_{\phi_i(X)}(x) \hspace{1cm} (2)$$

for any $x \in X$. The $\phi_i$’s, $\phi_i : X \to X$, partition the interval $X$ into disjoint subintervals and $1_{\phi_i(X)}$ is the indicator function of the interval $\phi_i(X)$. In the random setting, equality is read as an equality in distribution and each $\phi_i$ acts on random functions $f^{(i)}$, where the $f^{(i)}$ are i.i.d. copies of the random function $f$. In figure (1), we depict the underlying construction tree of this deterministic operator.

Barnsley et. al. have recently generalised the IFS construction to $V$-variable superfractals [6]. In their approach, they work with $V$ different sets applying on them randomly selected IFSs. The $V$ fixed points obtained after convergence are called the $V$-variable fractals and the family of the $V$-variable fractals together with their probability distribution is called a superfractal. The number of branches in each tree of the forest is still constant in this construction.

In this study, we generalise the construction fractal functions to the case when the construction tree and the operators are random: at each node of the tree, the number of offspring is random. This construction tree is of Galton Watson type. Falconer [7], Mauldin and Williams [8] obtained results about the fractal dimension on random sets relying on a Galton Watson tree. Results valid for measures are also well known. Moreover, the generalized IFS construction is using different maps at each iterations and converges to an interpolating function, but operators considered are acting over the space of compact subsets of $\mathbb{R}^d$ [9]. Here we introduce a new approach valid for random functions and signals, where operators are directly acting over the space of functions. In the next section, we will briefly review the formalism of Galton Watson trees. Then we will describe a new class of fractal functions called Galton Watson functions and prove the convergence of the process at the random and distribution level.

2. GALTON WATSON TREES
A Galton Watson tree is a tree with a random number of branches at each node where the offspring distribution is independent and identically distributed at each node. A node can be identified by means of its label. If we denote by $\emptyset$ the root node, then the first generation of children will be denoted by $i$ where $1 \leq i \leq \nu_0$ if $\nu_0$ is the number of children at $\emptyset$. Then the second generation will be labelled $ij$, $1 \leq j \leq \nu_i$, and so on. More generally, a node is an element of
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3.1. Definition

We are concerned with the definition of a new fractal construction when the underlying tree structure is no more deterministic. Indeed, we cannot apply the same random operator at each node of the tree as the number of offspring is random. Instead, the operator applied depends on the position of the node on the tree and its number of maps match the number of offsprings of the node. The randomness in the construction comes therefore from the non-deterministic tree structure and from the random operators. Consider the space of $p$-integrable functions on a compact subset of the real line:

$$
L_p = \{ f | \int_X |f(t)|^p dt < +\infty \} \quad (6)
$$

Let $(\Sigma, \mathcal{F}, P)$ be any probability space and consider the more general space of $p$-integrable random functions:

$$
L_p = \{ f_\sigma(t), \sigma \in \Sigma, t \in X \mid \mathbb{E} \left[ \int_X |f_\sigma(t)|^p dt \right] < +\infty \} \quad (7)
$$

duomendowed with the metric

$$
d_p^* : \forall (f, g) \in L_p, d_p^*(f, g) = \mathbb{E} \left[ |f - g|^p \right] \quad (8)
$$

where $d_p(f, g) = \left( \int_X |f(t) - g(t)|^p dt \right)^{\frac{1}{p}}$ is the usual metric. Set $p = 2$ to work with finite energy signals. In the remainder we will be particularly interested in the probability space $\Sigma = \Omega$. We also need to extend the space of Galton Watson trees and associate an operator at each node of a given tree. Let $\kappa = (\omega; \{\phi_{u,j}, \rho_{u,j}\})_{u \in \Omega, j \in \mathbb{N}}$ be an extended tree: we associate to each branch $(u, u_j)$ of the tree an operator $\phi_{u,j}$ which will define the IFS transform of the signal on the subinterval $\rho_{u,j}(X)$. The operator $T$ is then defined by:

$$
(Tf)(x) = \sum_{j=1}^{\nu_u} \phi_{\rho_{u,j}} f_{\rho_{u,j}}^{-1}(\rho_{u,j}(x)) \quad (9)
$$

where $\rho_{u,j}$ partition $X$ into disjoint subintervals. We can consider for example uniform partitions: $\rho_{\kappa,j}(t) = t \frac{1}{\nu_{\kappa,j}} + j \frac{1}{\nu_{\kappa,j}}, t \in X, 1 \leq j \leq \nu_{\kappa,j}$. The contraction factor of $\phi_{\kappa,j}$ is denoted by $r_{\kappa,j}$. $\phi_{\kappa,j}$ are 2-variable maps Lipschitz in their first variable, with Lipschitz constant $K_{\kappa,j}$. Also note that $f_{\rho_{u,j}}$ are i.i.d. copies of $f_\sigma$. $\nu_u$ is a random variable with probability distribution $q$. Remark that in this notation we have used the subscript $\kappa$ to emphasize the dependence of the random function $f_\sigma$ to the probability space of extended Galton Watson trees. This dependence will be made precise in the next section. We arrive to the following definition:

\begin{figure}
\centering
\includegraphics[width=\textwidth]{galton_watson_tree.png}
\caption{Example of a Galton Watson tree $\omega$}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{galton_watson_diagram.png}
\caption{Underlying tree structure of fixed points of IFS}
\end{figure}

Proposition 1: For each probability $q = (q_j, j \in \mathbb{N})$ on $\mathbb{N}$, there exists a unique probability measure $P_q$ on $(\Omega, \mathcal{A})$ which gives to the random variable $\nu$ the law $q$ and for which, conditionally on the event $\nu = j$, the random variables $\nu_i, 1 \leq i \leq j$ are independent and identically distributed with distribution $P_q$.

$(\Omega, \mathcal{A}, P_q)$ is the space of Galton Watson trees.
DEFINITION 1 Let $u \in \omega$. Then $\{\phi_{u,j}, \theta_{u,j}\}; q)_{j=1…u}$ is a random function scaling system.

We say that a function is statistically self-similar for an IFS if it satisfies $f = Tf$ in distribution. The major result in the next part is to show the existence and uniqueness at random level and consideration of a random function $f^*$ which satisfies the random function scaling system.

3.2. Existence and Uniqueness of a fixed point

THEOREM 1 Let $(\Omega, A, P_0)$ be the space of Galton Watson trees and consider $\kappa$ an extended tree. Consider $\{\phi_{n,j}, \theta_{n,j}\}; q)_{j=1…n}$ a random function scaling system. Let $1 < p < +\infty$. If $\lambda = E \sum_{j=1}^{\kappa} r_{\theta,j} K_0^{\kappa} < 1$ and $E \sum_{j=1}^{\kappa} r_{\theta,j} \int |\phi_{\theta,j}(0, x)|^p dx < +\infty$ then for any $f_0 \in L_p(\mathbb{R})$, the exists a unique random function $f^*$ which satisfies $f^* = Tf^*$ and such that

$$d_p^p(T^n f_0, f^*) \leq \frac{\lambda^p}{1 - \lambda^p} d_p^p(f_0, Tf_0)$$

(10)

which tends to 0 as $n \to +\infty$.

This theorem states that the IFS converges to a random fixed point starting from any initial function $f_0$ under certain conditions. The fixed point exhibits self-similarity up to probability distribution. The proof is in two steps. First, we need to show that $\lambda < 1$ and consider the existence and uniqueness of a limit function at the random level.

We first make precise the notations $f^{(j)}$ and $g^{(j)}$. To prove theorem 1 we need to construct i.i.d. copies of the random function $f$. This can be achieved using the homogeneity property of Galton Watson trees: by $f_i^{(j)}$ we understand the function associated with the tree $T_i(\kappa)$. Since by proposition 1 the variables $T_i$ are independent and identically distributed with distribution $P_0$, the associated functions are also i.i.d.

Step 1: For $f \in L_p$, we show $E f_p [|(Tf)(x)|^p dx < +\infty$, that is $Tf \in L_p$. To do so, first notice that in the expression of $Tf$, the indicator function partitions $\mathbb{X}$ into disjoint subintervals, so that the absolute value of the sum equals the sum of absolute values. Furthermore, using $E(.) = E[E(\cdot |\nu_0)]$ (tower property of expectation) and contractive properties of $\phi_{\theta,j}$, it is straightforward to check that $E f_p [(Tf)(x)]^p dx$ is always smaller than

$$E \sum_{j=1}^{\kappa} r_{\theta,j} E[|\phi_{\theta,j} f^{(j)}(y),y|^p dy |\nu_0]$$

(11)

On the right hand side, we have set $y = \phi_{\theta,j}^{(j)}(x)$ and we have majorized the Jacobian of the transformation by $r_{\theta,j}$, the Lipschitz factor of $\phi_{\theta,j}$. Note that $E f_p [|\phi_{\theta,j} f^{(j)}(y),y|^p dy = d_p^p(\phi_{\theta,j}, f^{(j)} I(d), 0)$ where $I(d)$ stands for the identity function and 0 the zero function. Combining the triangle inequality of distance and the fact that for any positive $x$ and $y$: $(x + y)^p \leq 2^p (x^p + y^p)$, the left hand side is majorized by:

$$2^pE \sum_{j=1}^{\kappa} r_{\theta,j} d_p^p(\phi_{\theta,j}, I(d), 0, I(d), 0) +$$

$$2^pE \sum_{j=1}^{\kappa} r_{\theta,j} d_p^p(\phi_{\theta,j}, 0, I(d), 0)$$

(12)

The first part is bounded since $f \in L_p$. The second part can be written $E \sum_{j=1}^{\kappa} r_{\theta,j} \int |\phi_{\theta,j}(0, x)|^p dx$ and is bounded by assumption.

Step 2: Let $f$ and $g$ in $L_p$. Then,

$$d_p^p(Tf,Tg) = E d_p^p(Tf,Tg)$$

$$= E \int |(Tf)_n(x) - (Tg)_n(x)|^p dx$$

By replacing $(Tf)_n(x)$ and $(Tg)_n(x)$ by their own expression, the distance becomes:

$$= E \sum_{j=1}^{\kappa} r_{\theta,j} \int |\phi_{\theta,j} f^{(j)}(y),y - \phi_{\theta,j} g^{(j)}(y),y|^p dy$$

where we have made the same change of variable $y = \phi_{\theta,j}^{(j)}(x)$. Furthermore exploiting the Lipschitz property of $\phi_{\theta,j}$ and the i.i.d distribution of $f^{(j)}(y)$ and $g^{(j)}(y)$ $\forall j$, we obtain the inequality

$$d_p^p(Tf,Tg) \leq \lambda d_p^p(f,g)$$

(13)

where $\lambda = E \sum_{j=1}^{\kappa} r_{\theta,j} K_0^{\kappa}$. Since by assumption $\lambda$ is smaller than 1, the operator $T$ is contractive in the complete metric space $(L_p, d_p)$ and therefore admits a unique fixed point $f^*$ (at the random level) by the Banach fixed point theorem. Clearly:

$$d_p^p(T^n f_0, f^*) \leq \lambda d_p^p(T^{n-1} f_0, f^*)$$

(14)

which lead to

$$d_p^p(T^n f_0, f^*) \leq \lambda^p d_p^p(f_0, f^*)$$

(15)

Now using triangle inequality:

$$d_p^p(f_0, f^*) \leq d_p^p(f_0, T^n f_0) + \lambda^p d_p^p(f_0, f^*)$$

(16)

so that:

$$d_p^p(T^n f_0, f^*) \leq \frac{\lambda^p}{1 - \lambda^p} d_p^p(f_0, Tf_0)$$

(17)

which concludes the proof of the theorem.

Remark: As a consequence of the theorem, $T^n f \to f^*$ as $n \to +\infty$ almost surely. Suppose the contrary: let $n$ be the set of $\kappa$ such that $d_p(T^n f_0, f^*)$ do not tend to 0 as $n$ tends to $+\infty$. $\kappa$ is supposed not to be a neglected set. In such a case, $Ed_p(T^n f_0, f^*)$ does not tend to 0 as $n \to +\infty$, which is a contradiction with theorem 1. Hence the almost sure convergence. Note also that this convergence is exponentially fast.

Next, note that this equality at the random level is also true at the distribution level. However, equality in distribution does not implies equality at the random level. The following result can be proven in the same way as Hutchinson and Ruschendorf [5].
The idea is to define a new space of probability distribution of elements of $L_p$ and a new metric over this space which lead to a complete metric space. One can prove then that the operator $T$ seen at the distribution level is contractive in this space and therefore admits a unique fixed point.

To illustrate, we present in figure (3) a snapshot of the random fixed point and its mean estimated using 100 realisations of the fixed point. Here we only consider deterministic function scaling system; the same operators are applied once we have conditioned on the number of offsprings at a node. The function scaling system is in this context reduced to

$$\phi_{i,j}(u,v) = s_i u + \zeta_{i,j}(v)$$

with $s_i = 0.6$, $s_2 = 0.7$, $s_3 = 0.3$, $\zeta_{1,1}(t) = t(1-t)$, $\zeta_{2,1}(t) = t^2$, $\zeta_{2,2}(t) = 1-t^2$, $\zeta_{3,1}(t) = t$, $\zeta_{3,2}(t) = (t+1)(2-t)$ and $\zeta_{3,3}(t) = t(1-t)^3$.

The probability generating vector is $(0.2, 0.3, 0.5)$ in (a) and (b), $(0.2, 0.2, 0.6)$ in (c) and $(0.2, 0.1, 0.7)$ in (d).

**COROLLARY 1** $f^*$ is the unique fixed point of this IFS up to probability distribution.

The existence and uniqueness of a self-similar function when we allow a random construction tree and random operators. This construction does not force the number of offsprings to be bounded and the convergence results still hold for infinite number of operators. Moreover, the fixed points obtained all have a very erratic behaviour and we speculate that they might not have a density. An efficient tool to characterize such irregular objects is their multifractal spectrum introduced first by Frisch and Parisi [11] in the context of turbulence, and adapted to random processes and functions. The motivation to think of a multifractal spectrum for such fractal signals is due to its cascade construction. Cascade processes are indeed known to exhibit multifractal properties and results for random measures are known [12].

**REFERENCES**


