

Loops, matchings and alternating-sign matrices ^{*}

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Abstract

The appearance of numbers enumerating alternating sign matrices in stationary states of certain stochastic processes on matchings is reviewed. New conjectures concerning nest distribution functions are presented as well as a bijection between certain classes of alternating sign matrices and lozenge tilings of hexagons with cut off corners.

1 Introduction

Following an observation of Razumov and Stroganov [36] for the XXZ spin chain (most terms will be defined below), it was observed by Batchelor et al. [2] that the groundstates of loop Hamiltonians with different boundary conditions are related to symmetry classes of the alternating sign matrices introduced by Mills et al. [29, 30, 41]. The following is an account of this surprising new relation between physics and combinatorics.

The connection can be described using the action of the Temperley-Lieb algebra on matchings of $\{1, \dots, n\}$. This action, which will be described in detail in Section 2, will be used to define the dense $O(1)$ loop model. In Section 3 it is shown that matchings define an equivalence relation on alternating sign matrices. These two ingredients have led several authors [31, 34, 37, 38] to formulate conjectures relating symmetry classes of alternating sign matrices to boundary conditions in the dense $O(1)$ loop model. The exact correspondence is stated at the end of Section 3.

Many of the alternating sign matrix numbers factorise into small primes. The appearance of such numbers in stationary states has a nice application. As shown in Section 4, exact closed form formulae of certain expectation values and correlation functions can be guessed from exact calculations on a few examples. This is extremely useful since one obtains exact conjectural results for physical quantities. In fact, in some cases one may argue the validity of the conjectured

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formulae, or at least their asymptotics, because they make sense physically [14]. Identification of numbers can also be useful mathematically, since relations between different combinatorial objects may be discovered. For example, a direct bijection between a class of alternating sign matrices and hexagons with cut off corners was discovered this way. This bijection is described in Section 5.

2 A stochastic process on matchings

In the first two subsections we define a stochastic process on non-crossing perfect matchings [34]. A simple example of such a process is given at the end of subsection 2.2. In the remaining subsections similar stochastic processes on more general matchings will be defined.

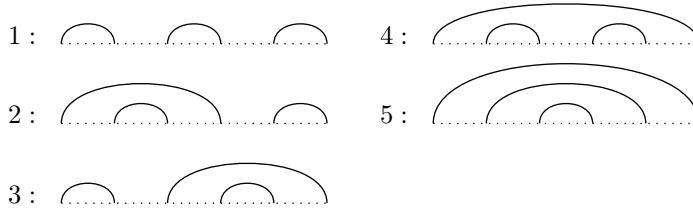
2.1 Matchings and matchmakers

A p -**matching**, or simply matching, of the **vertex set** $[n] = \{1, \dots, n\}$ is an unordered collection of p pairs of vertices, or **edges**, and $n - 2p$ single vertices. A matching F is called **crossing** if it contains an edge $\{i, j\}$ and a vertex k such that $i < k < j$ or if it contains edges $\{i, j\}$ and $\{k, l\}$ such that $i < k < j < l$. A matching is **perfect** if $n = 2p$ and **near-perfect** if $n = 2p + 1$. Let \mathcal{F}_{2n} denote the set of all non-crossing perfect matchings of $[2n]$, and \mathcal{F}_{2n+1} the set of all non-crossing near-perfect matchings of $[2n + 1]$.

Example 1 For $n = 6$ there are five non-crossing perfect matchings.

$$\mathcal{F}_6 = \{\{1, 2\}\{3, 4\}\{5, 6\}, \{1, 4\}\{2, 3\}\{5, 6\}, \{1, 2\}\{3, 6\}\{4, 5\}, \\ \{1, 6\}\{2, 3\}\{4, 5\}, \{1, 6\}\{2, 5\}\{3, 4\}\}.$$

The edges of each matching are written here in a particular order but it is to be understood that no particular order is preferred. A notation for the matchings that will be useful later is to depict them as loop segments connecting vertices. The five matchings comprising \mathcal{F}_6 will thus be denoted by the following pictures,

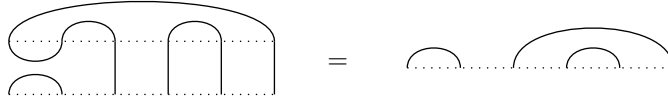


Instead of this graphical notation, a sometimes more convenient typographical notation for matchings of $[n]$ is obtained by using parentheses for paired vertices [31],

$$\mathcal{F}_6 = \{(())(), (())(), ()(), (()), (((())))\}.$$

We will use the graphical and parenthesis notation interchangeably.

The action of e_1 on $((\))$ is given by



2.2 Hamiltonian and stationary state

The loops in the graphical representation of the Temperley-Lieb algebra have the physical interpretation of boundaries of percolation clusters. A much studied object in physics is the loop energy operator, or loop **Hamiltonian**, which is defined as

$$H_{2n}^C = \sum_{j=1}^{2n-1} (1 - e_j) \quad (3)$$

The superscript C stands for **closed boundary conditions**, which will be explained in section 2.3. In the representation $H_{2n}^C : \text{Span}(\mathcal{F}_{2n}) \rightarrow \text{Span}(\mathcal{F}_{2n})$, H_{2n}^C is called the Hamiltonian of the **Temperley-Lieb loop model**, or **dense O(1) loop model** with closed boundary conditions. We will refer to this representation as the **loop representation** of H_{2n}^C .

In the loop representation H_{2n}^C is a matrix whose off-diagonal entries are all non-positive and whose columns add up to zero [34]. Such a matrix is called an **intensity matrix** and defines a **stochastic process** in continuous time given by the master equation,

$$\frac{d}{dt} P_{2n}(t) = -H_{2n}^C P_{2n}(t), \quad P_{2n}(t) = \sum_{F \in \mathcal{F}_{2n}} a_F(t) F,$$

where $a_F(t)$ is the unnormalized probability to find the system in the state F at time t . Since H_{2n}^C is an intensity matrix it has at least one zero eigenvalue. Its corresponding left eigenvector P_{2n}^L is trivial and its right eigenvector P_{2n} is called the **stationary state**,

$$\begin{aligned} P_{2n}^L H_{2n}^C &= 0, & P_{2n}^L &= (1, 1, \dots, 1), \\ H_{2n}^C P_{2n} &= 0, & P_{2n} &= \lim_{t \rightarrow \infty} P_{2n}(t). \end{aligned}$$

Properties of the stationary state from a physical perspective are described in [13].

Example 3 For $n = 3$ the action of H_6^C on $\text{Span}(\mathcal{F}_6)$ can be calculated by its action on the five basis states $()()()$, $((\))()$, $()(())$, $((\))()$, $((\))()$ (see also Example 1). We find for example,

$$H_6^C ()()() = 2()()() - ((\))() - ()(()).$$

Similarly calculating the action of H_6^C on the other basis states yields

$$H_6^C = - \begin{pmatrix} -2 & 2 & 2 & 0 & 2 \\ 1 & -3 & 0 & 1 & 0 \\ 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}.$$

The stationary state P_6 of H_6^C is given by

$$P_6^T = (11, 5, 5, 4, 1),$$

where T denotes transposition.

The stationary state turns out to have a surprising combinatorial interpretation. Let us look at a few more explicit solutions of $H_{2n}^C P_{2n} = 0$,

$2n$	P_{2n}^T
2	(1)
4	(2, 1)
6	(11, 5, 5, 4, 1)
8	(170, 75, 75, 71, 56, 56, 50, 30, 14, 14, 14, 14, 6, 1)

We can now make the surprising observation [2] that the largest components of P_{2n} , i.e. $\{1, 2, 11, 170, \dots\}$, enumerate cyclically symmetric transpose complement plane partitions, and that $P_{2n}^L P_{2n}$ which is the sum of elements of P_{2n} , i.e. $\{1, 3, 26, 646, \dots\}$, enumerate vertically symmetric alternating sign matrices.

Alternating sign matrices (ASMs) were introduced by Mills et al. [29, 30] and are matrices with entries in $\{-1, 0, 1\}$ such that the entries in each column and each row add up to 1 and the non-zero entries alternate in sign. Mills et al. conjectured that the number of ASMs is given by

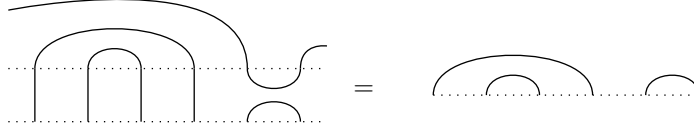
$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, \dots, \quad (4)$$

which was proved more than a decade later by Zeilberger [49] and Kuperberg [24]. Conjectured enumerations of symmetry classes were given by Robbins [41], many of which were subsequently proved by Kuperberg [25]. Proofs of some remaining conjectures were announced recently by Okada [32]. The properties and history of ASMs are reviewed in a book by Bressoud [6], as well as by Robbins [40] and Propp [35].

In the following we will see that not only the largest components and the sum of components of the stationary state have a meaning, but that the other integers also have a combinatorial interpretation [37]. It turns out that other symmetry classes of ASMs appear when we use different boundary conditions for the loop Hamiltonian. To show that, we first need to extend some definitions and introduce a few more concepts.

Furthermore, the definition of f_1 and f_n is such that we can erase those parts of composite pictures that connect the left and right external vertices but are otherwise disconnected.

Example 6 $e_5((()))(=(())() \in \mathcal{F}_6^e$ can be displayed graphically as



Instead of the Hamiltonian (3) which does not contain boundary operators one may also consider the Hamiltonians $H_n^M : \text{Span}(\mathcal{F}_n^{\text{re}}) \rightarrow \text{Span}(\mathcal{F}_n^{\text{re}})$ and $H_n^O : \text{Span}(\mathcal{F}_n^e) \rightarrow \text{Span}(\mathcal{F}_n^e)$

$$H_n^M = (1 - f_n) + \sum_{j=1}^{n-1} (1 - e_j)$$

$$H_n^O = (1 - f_1) + (1 - f_n) + \sum_{j=1}^{n-1} (1 - e_j)$$

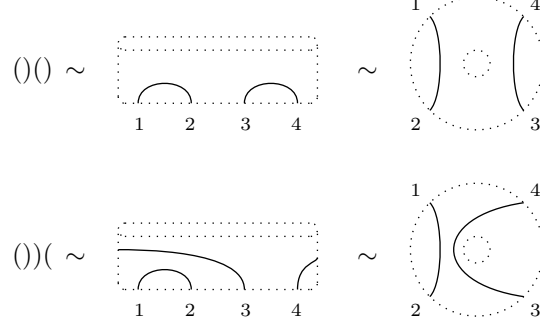
where M denotes **mixed boundary conditions** [31] and O denotes **open boundary conditions**. In the case of open boundaries loop segments can “penetrate the boundary” and end on an external point, as in Example 5. For closed boundaries loops are not allowed to end on either external point, while for mixed boundaries, the loops can only penetrate the right boundary but not the left.

2.3.2 Periodic boundary conditions

The action of the generators e_j can be extended to directed matchings using the graphical representation and keeping track of the orderings of i, j, k in (1). Let us denote all non-crossing directed (near-)perfect matchings of $[n]$ by \mathcal{F}_n^* . For directed perfect matchings we can still use the parentheses notation, e.g. $() \sim (1, 2)$ and $) \sim (2, 1)$, and graphically they can be conveniently depicted on a cylinder. A matching (i, j) for $i < j$ is represented by a loop segment over the front of the cylinder and for $i > j$ by a loop segment over the back. There is an equivalently graphical representation on an annulus, the annulus being the top-view of the cylinder. The matching (i, j) is represented by a loop segment keeping the center of the annulus to its left.

Example 7 The directed matchings $(1,2)(3,4)=()()$ and $(1,2)(4,3)=()()$ are

graphically represented on the cylinder and annulus by



The definitions (1) for the generators e_j extended to directed matchings are,

$$e_j : \begin{cases} (j, j+1), (j+1, j) & \mapsto (j, j+1) \\ (i, j)(j+1, k) & \mapsto (i, j)(j, j+1) \\ (j, i)(j+1, k), (i, j)(k, j+1) & \mapsto (k, i)(j, j+1) \end{cases}, \quad (5)$$

for $i < j < k$.

If n is odd, there will be one unpaired vertex in \mathcal{F}_n^* . In the parentheses notation we denote this by a vertical line, e.g. $(1, 2)\{3\} \sim ()|$, while graphically we think of it as a defect line running from the top to the bottom of the cylinder. The action of e_j extended to near-perfect directed matchings is

$$e_j : (i, j)\{j+1\}, \{j\}(j+1, i) \mapsto \{i\}(j, j+1). \quad (6)$$

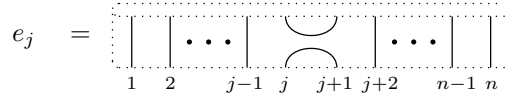
Graphically it is natural to introduce an additional generator e_n for directed matchings, see Levy [26] and Martin [27]. Its non-trivial action on elements $F \in \mathcal{F}_n^*$ containing 1 or n is similar to (5) and is described by,

$$e_n : \begin{cases} (n, 1), (1, n) & \mapsto (n, 1) \\ (1, i)(j, n) & \mapsto (j, i)(n, 1) \\ (1, i)(n, j), (i, 1)(j, n) & \mapsto (i, j)(n, 1) \\ \{1\}(i, n), (1, i)\{n\} & \mapsto \{i\}(n, 1) \end{cases} \quad (7)$$

Lemma 3 *The generator e_n satisfies the relations*

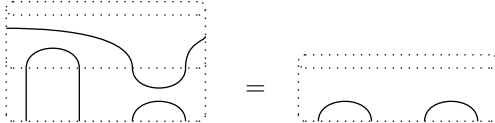
$$\begin{aligned} e_n^2 &= e_n, & e_n e_1 e_n &= 1, & e_1 e_n e_1 &= e_1 \\ e_n e_j &= e_j e_n & j &\notin \{1, n-1\}, \end{aligned}$$

and the generators e_j , $j \in \{1, \dots, n\}$ can be represented graphically on a cylinder as

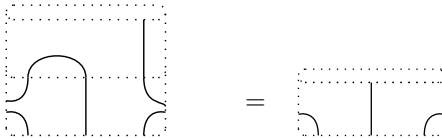


The definitions (5) and (7) imply that for perfect matchings non-contractible loops running around the cylinder can be erased. For near-perfect matchings they allow to neglect the winding of the defect line. A thorough discussion how to capture this in algebraical rules is given in [28], see also [34].

Example 8 $e_3()() = ()() \in \mathcal{F}_4^*$ is displayed graphically as



and $e_3()| =)|(\in \mathcal{F}_3^*$ corresponds to



We define a new Hamiltonian $H_n^{\text{P}*} : \text{Span}(\mathcal{F}_n^*) \rightarrow \text{Span}(\mathcal{F}_n^*)$ by

$$H_n^{\text{P}*} = \sum_{j=1}^n (1 - e_j), \quad (8)$$

where P stands for **periodic boundary conditions**. For even n , the Hamiltonian (8) also has an action on non-directed non-crossing perfect matchings. Graphically this corresponds to closing the top of the cylinder, or removing the inner disk of the annulus. The two distinct matchings in Example 7 then become equal. This case we denote by $H_{2n}^{\text{P}} : \text{Span}(\mathcal{F}_{2n}) \rightarrow \text{Span}(\mathcal{F}_{2n})$

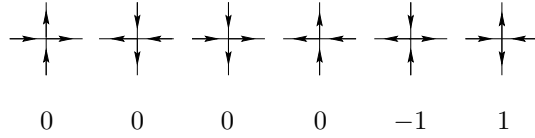
$$H_{2n}^{\text{P}} = \sum_{j=1}^{2n} (1 - e_j).$$

3 Fully packed loop diagrams

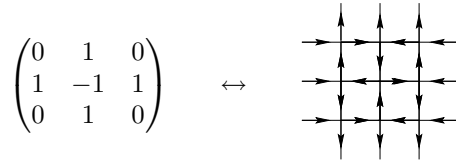
The **fully packed loop model** (FPL) [1, 46] is a model of polygons on a lattice such that each vertex is visited exactly once by a polygon. The model is an alternative representation of the **six-vertex model** on the square lattice, as will be described below. Here we consider the six-vertex model with **domain wall** boundary conditions which were introduced by Korepin [21], as well as related boundary conditions.

It is well known that ASMs are in bijection with configurations of the six-vertex model with domain wall boundary conditions, see for example Elkies et al. [8]. The correspondence between entries in an ASM and the six vertex

configurations is given by

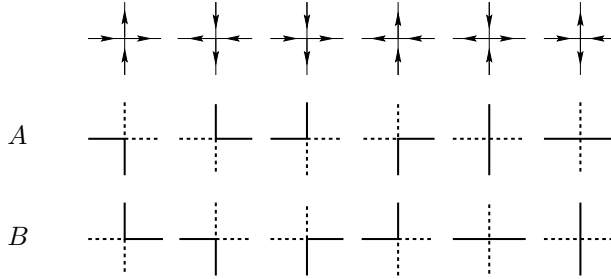


Example 9 There are seven 3×3 ASMs. Six of these are the 3×3 permutation matrices and only one of them contains a -1 . Its corresponding six-vertex configuration is given by



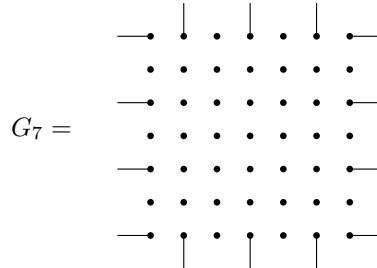
The six-vertex model was used extensively by Kuperberg [24, 25] to prove many enumerations of ASMs and their symmetry classes using and generalizing the Izergin and Tsuchiya determinants [16, 17, 22, 45]. It was also used by Zeilberger [50] in his proof of the refined alternating-sign matrix conjecture.

A fully packed loop configuration is obtained from a six-vertex configuration by dividing the square lattice into its even and odd sublattice denoted by A and B respectively. Instead of arrows, only those edges are drawn that on sublattice A point inward and on sublattice B point outward.



We take the vertex in the upper left corner to belong to sublattice A . The domain wall boundary condition translates into a boundary condition for the loops. Loops either form closed circuits, or begin and end on boundary sites which are prescribed by the boundary in- and out-arrows on sublattice A and B respectively. Such sites are called **designated boundary sites**. From the above it follows that ASMs are in bijection with fully packed loop diagrams on a square grid. The **grid** for $n \times n$ ASMs is denoted by G_n .

Example 10 The 7×7 ASMs are in bijection with FPL diagrams on G_7



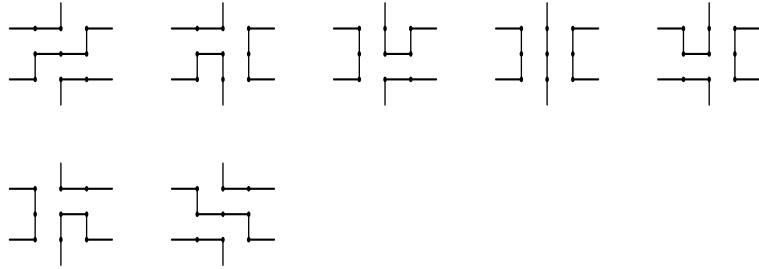
3.1 Matchings of FPL diagrams

In each FPL diagram every designated boundary site is connected to another such site. Number the designated boundary sites, starting with 1 at the left boundary in the upper left corner and ending with $2n$ at the top boundary in the same corner. We then have

Lemma 4 *Each FPL diagram on a grid G_n defines a perfect matching of $[2n]$. A matching therefore defines an equivalence relation on FPL diagrams.*

The (G, F) -cardinality $M_F(G)$ is the number of FPL diagrams with matching F on grid G .

Example 11 There are 7 FPL diagrams on G_3 ,



The 2nd and 5th diagram have the same matching and likewise with the 3rd and 6th diagram. The others have distinct matchings. The set of matchings is therefore $\{()()(), ()()(), ()()(), ()()(), ()()(), ()()()\}$ and the corresponding set of cardinalities is $\{1, 2, 2, 1, 1\}$.

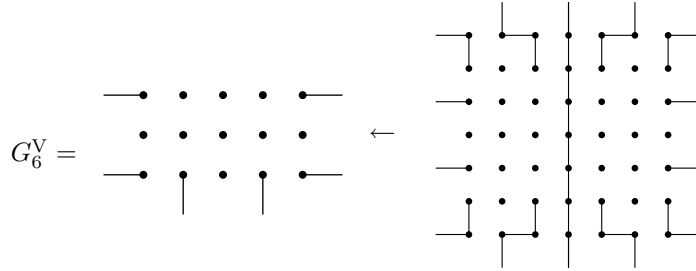
3.2 Symmetry classes

Requiring ASMs to be symmetric with respect to one of the symmetries of the square puts constraints on the loop configurations at certain edges. For $(2n + 1) \times (2n + 1)$ **vertically symmetric** ASMs there must be a loop running

from top to bottom along the symmetry axis, and it is also not hard to see that the edges in the top and bottom row are fixed. Requiring vertical symmetry reduces the configuration space of FPL diagrams on G_{2n+1} to that of FPL diagrams on a rectangular grid of size $n \times (2n - 1)$. Let G_{2n}^V denote this grid.

Lemma 5 *Each FPL diagram on a grid G_{2n}^V defines a perfect matching on $[2n]$.*

Example 12 The 26 vertically symmetric ASMs of size 7×7 are in bijection with FPL diagrams on G_6^V ,

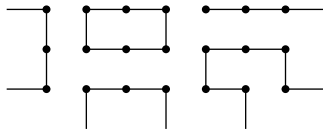


A class of FPL diagrams closely related to those on G_{2n}^V are diagrams with an odd number of designated boundary sites. This class is defined by FPL diagrams on an $n \times (2n + 1)$ grid, denoted by G_{2n+1}^V , such that the unpaired loop line may end anywhere on the top boundary.

Lemma 6 *Each such FPL diagram on a grid G_{2n+1}^V defines a near-perfect matching on $[2n + 1]$, or equivalently, a perfect right extended $(n, 1)$ -matching on $[2n + 1]$.*

Remark 5 The upper boundary of G_{2n+1}^V plays a role analogous to the vertex $\{n + 1\}$.

Example 13 An FPL diagram on G_7^V with matching $()()()()$ is

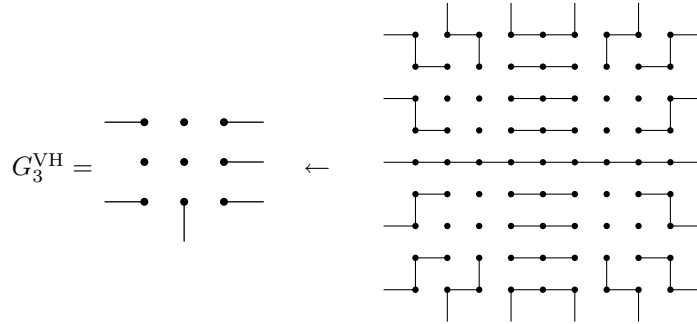


For $(2n + 1) \times (2n + 1)$ **vertically and horizontally symmetric** ASMs the horizontal symmetry axis contains loop segments as well as all horizontal edges crossing the vertical symmetry axis. Furthermore, the complete boundary layer is fixed. Requiring horizontal and vertical symmetry reduces the configuration space of FPL diagrams on G_{2n+3} to that of FPL diagrams on a square grid of size $n \times n$, which is denoted by G_n^{VH} .

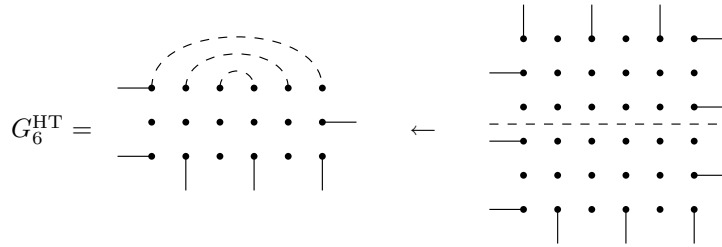
Lemma 7 *Each FPL diagram on a grid G_n^{VH} defines a perfect right extended matching on $[n]$.*

Remark 6 The boundary sites on G_n^{VH} coming from the horizontal edges on the vertical symmetry axis of G_{2n+3} play the role of the extra site at $\{n+1\}$.

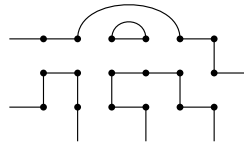
Example 14 The 6 vertically and horizontally symmetric ASMs of size 9×9 are in bijection with FPL diagrams on G_3^{VH}



Half-turn symmetric ASMs, or HTASMs, are symmetric under rotation by 180 degrees. For HTASMs of size $2n$ one has only to consider FPL diagrams on the lower half. In contrast to the case of vertically symmetric ASMs, the sites on the top boundary of this rectangular grid can now also be connected via one of n arcs, or HT boundaries [25]. Loops crossing the horizontal symmetry axis of the square, map to loops on such arcs. Requiring half turn symmetry reduces the configuration space of FPL diagrams on G_{2n} to that of FPL diagrams on a rectangular grid of size $2n^2$ with HT boundaries, which is denoted by G_{2n}^{HT} ,



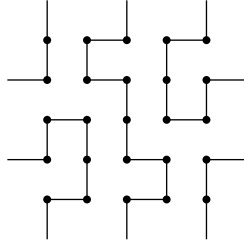
Example 15 An FPL diagram on G_6^{HT} with matching $)()()$ is



For odd-sized HTASMs the situation is a little more complicated. In this case there is one loop segment running across the FPL diagram dividing it in two

identical parts. However, the shape of these parts depend on the way the loop segment runs across the cylinder and there is no unique reduced graph to accommodate for all possible link patterns. In the is case we define G_{2n+1}^{HT} as G_{2n+1} , the graph for odd unrestricted FPL diagrams, with the proviso that it has to be half-turn symmetric.

Example 16 An FPL diagram on G_5^{HT} with matching $)()|($ is



3.3 Boundary conditions and symmetry classes

Finally we can state a conjecture that relates the stationary state for different boundary conditions in the loop model to symmetry classes of FPL diagrams. This conjecture is a collection of results obtained for special cases by several authors [2, 31, 34, 36, 37, 38].

Conjecture 1 Let $\{H, \mathcal{F}, G\}$ be any of the following four triples,

H	\mathcal{F}	G
H_n^{C}	\mathcal{F}_n	G_n^{V}
H_n^{M}	$\mathcal{F}_n^{\text{re}}$	G_n^{VH}
H_{2n}^{P}	\mathcal{F}_{2n}	G_n
$H_n^{\text{P}*}$	\mathcal{F}_n^*	G_n^{HT}

and let $P \in \text{Span}(\mathcal{F})$ be a solution of

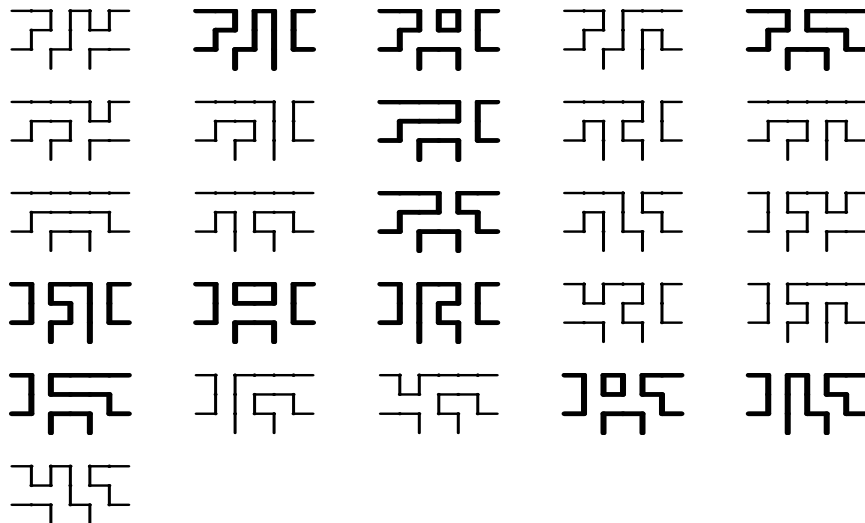
$$HP = 0, \quad P = \sum_{F \in \mathcal{F}} a_F F,$$

then a_F is equal to the (F, G) -cardinality $M_F(G)$, i.e., the number of fully packed loop diagrams on G with matching $F \in \mathcal{F}$.

There are also some conjectures for open boundary conditions [31], but it is not known to which grid that case corresponds.

Example 17 The stationary state for closed boundaries and $n = 6$ is given by $P_6^T = (11, 5, 5, 4, 1)$ on the basis $\mathcal{F}_6 = \{()()(), (())(), ()(())(), ((())), (((()))\}$, see

Example 3. Indeed one finds among the 26 FPL diagrams on G_6^V precisely 11 with matching $()()()$, 5 with matching $(())()$, 5 with matching $()(())$, 4 with matching $(())()$ and 1 with matching $((()))$. The 11 diagrams with matching $()()()$ are printed bold,



4 Correlation functions and expectation values

In physics one would rather like to know expectation values and correlation functions than the stationary state itself. An expectation value is nothing else but the ratio of a refined to a full enumeration. If the refinement is with respect to two or more constraints one usually speaks of a correlation function rather than expectation value. A well known example of an expectation value is given by the refined ASM conjecture, which was formulated by Mills et al. [29, 30] and proved by Zeilberger [50]. Recent progress has been made by Stroganov for double refined ASM correlations [43].

For other correlation functions or expectation values closed formulae have been conjectured on the basis of explicit calculations for small system sizes. The formula for the emptiness formation probability (see e.g. [22]) in the XXZ spin chain representation for periodic and twisted boundary conditions was conjectured by Razumov and Stroganov in [36] and [39]. The asymptotic behaviour of that conjecture was proved by Kitanine et al. in [20] using the integrability of the XXZ spin chain. In the loop representation several other conjectures for expectation values are stated in [31, 51].

Below we will give examples of explicit formulas for nest distribution functions in the case of closed and periodic boundary conditions, or vertically and half-turn symmetric FPL diagrams respectively.

4.1 Nests

The **nests** of a non-crossing perfect matching F are the sequences of consecutive parentheses of F that close among themselves and are not enclosed by other parentheses. The number of nests of F is denoted by c_F . In terms of Dyck paths (see Remark 1), the number of nests is one less the number of times the path touches the real axis.

Example 18 The number of nests for the non-crossing perfect matchings $F = ()()(), (())(), ()(), ((())), (((()))$ is $c_F = 3, 2, 2, 1, 1$.

The nests of an FPL diagram are the nests defined by its matching.

Let \mathcal{F}_G denote the set of non-crossing perfect matchings defined by the FPL diagrams on the grid G . The **nest distribution function** $P_G(k)$ is defined as the number of FPL diagrams on G with k nests,

$$P_G(k) = \sum_{F \in \mathcal{F}_G} M_F(G) \delta_{k, c_F}, \quad Z_G = \sum_k P_G(k),$$

and the **average number of nests** is defined as

$$\langle k \rangle_G = \frac{1}{Z_G} \sum_k k P_G(k). \quad (9)$$

Below we show that in some cases an exact formula for these quantities can be guessed from explicit data for small sizes. As a rule, a number can only be guessed easily if it factorises into small primes. It should be mentioned that in such cases there are several helpful software utilities that could be used, see e.g. Appendix A in [23].

Table 1 *The average number of nests on $G = G_{2n}^{\text{HT}}$ for $n \in \{2, \dots, 7\}$ and the prime factorisations of its denominator and its numerator.*

$2n$	4	6	8	10	12	14
$\langle k \rangle_G$	$\frac{8}{5}$	$\frac{21}{10}$	$\frac{28}{11}$	$\frac{65}{22}$	$\frac{624}{187}$	$\frac{3485}{935}$
den	5	$2 \cdot 5$	11	$2 \cdot 11$	$11 \cdot 17$	$5 \cdot 11 \cdot 17$
num	2^3	$3 \cdot 7$	$2^2 \cdot 7$	$5 \cdot 13$	$2^4 \cdot 3 \cdot 13$	$2 \cdot 7 \cdot 13 \cdot 19$

It is not difficult to derive a formula for $\langle k \rangle_{G_{2n}^{\text{HT}}}$ that exactly reproduces the numbers in Table 1. We have

Conjecture 2 *The average number of nests on $G = G_{2n}^{\text{HT}}$ is*

$$\langle k \rangle_G = n \prod_{j=1}^{n-1} \frac{3j+1}{3j+2} \sim \frac{\Gamma(5/6)}{\sqrt{\pi}} (2n)^{2/3} \quad (n \rightarrow \infty).$$

This guessing does not always work as we can see from the next table where the same quantities are given for G_{2n}^V ,

Table 2 *The average number of nests on $G = G_{2n}^V$ for $n \in \{2, \dots, 7\}$ and the prime factorisations of its denominator and its numerator.*

$2n$	4	6	8	10	12	14
$\langle k \rangle_G$	$\frac{5}{3}$	$\frac{29}{13}$	$\frac{52}{19}$	$\frac{913}{285}$	$\frac{1693}{465}$	$\frac{69769}{17205}$
den	3	13	19	$3 \cdot 5 \cdot 19$	$3 \cdot 5 \cdot 31$	$3 \cdot 5 \cdot 31 \cdot 37$
num	5	29	$2^2 \cdot 13$	$11 \cdot 83$	1693	$7 \cdot 9967$

The appearance of large prime factors in the numerator makes it hard to conjecture a formula. However, for the case of nests we are very fortunate because while we may not be able to guess the average (9) for $G = G_{2n}^V$, we can find the complete distribution function.

Conjecture 3 *Let $(a)_k = \Gamma(a+k)/\Gamma(a)$ and $G = G_{2n}^V$. The nest distribution function $P_G(k)$ is given by*

$$P_G(k) = k \frac{4^{n+k}}{27^n} \frac{(1/2)_{n+k}}{(1/3)_{2n}} \frac{(3n+1)!(2n-k-1)!}{n!(n-k)!(2n+k+1)!} A_{2n+1}^V,$$

where

$$A_{2n+1}^V = \prod_{j=0}^{n-1} (3j+2) \frac{(6j+3)!(2j+1)!}{(4j+2)!(4j+3)!} = 1, 3, 26, 646, \dots$$

is the number of $(2n+1) \times (2n+1)$ vertically symmetric ASMs.

This conjecture has been checked for n up to 8. Assuming the conjecture the average number of nests can be calculated. It turns out that (9) with Conjecture 3 is summable (see below) and we find,

Corollary 1 *The average number of nests on $G = G_{2n}^V$ is*

$$\begin{aligned} \langle k \rangle_G &= \frac{1}{A_{2n+1}^V} \sum_{k=1}^n k P_G(k) = \frac{1}{3} \left(\prod_{j=0}^{n-1} \frac{(2j+1)(3j+4)}{(j+1)(6j+1)} - 1 \right) \\ &\sim \frac{\Gamma(1/3)\sqrt{3}}{2\pi} (2n)^{2/3} \quad (n \rightarrow \infty). \end{aligned}$$

Corollary 1 fits the data in Table 2. While we obtain a nice formula for $\langle k \rangle_{G_{2n}^V}$, the simple subtraction of $1/3$ in Conjecture 1) gives rise to large prime factors in Table 2 and makes it virtually impossible to guess the result directly from there. A result similar to Conjecture 3 is obtained for nests on G_{2n}^{HT} .

Conjecture 4 Let $G = G_{2n}^{\text{HT}}$. The nest distribution function $P_G(k)$ is given by

$$P_G(k) = 3nk \frac{(2k)!(n+k-1)!(2n-k-1)!}{k!2(n-k)!(2n+k)!} A_{2n}^{\text{HT}}. \quad (10)$$

where

$$A_{2n}^{\text{HT}} = A_n^2 \prod_{j=0}^{n-1} \frac{3j+2}{3j+1} = 2, 10, 140, \dots$$

is the number of $2n \times 2n$ half-turn symmetric ASMs. A_n is the number of $n \times n$ ASMs defined in (4).

This conjecture has again been checked for n up to 8 and is consistent with Conjecture 2.

4.2 Strange evaluations

Corollary 1 has been obtained from Conjecture 3 using the Mathematica implementation of the Gosper-Zeilberger algorithm [15, 47, 48] by Paule and Schorn [33]. Using a similar procedure, the consistency of Conjectures 2 and 4 was checked. Both results can also be derived using the following hypergeometric summation formula,¹

$${}_5F_4 \left[\begin{matrix} a, 1+2a/3, 1-2d, 1/2+a+m, -m \\ 2a/3, 1/2+a+d, -2m, 1+2a+2m \end{matrix} ; 4 \right] = \frac{(1-d)_m (a+1)_m}{(1/2)_m (1/2+a+d)_m}, \quad (11)$$

which can be derived from one of the strange evaluations of Gessel and Stanton [11]. A detailed derivation is beyond the scope of this paper, but one may for example derive Conjecture 2 by writing (9) in hypergeometric notation using Conjecture 4, and taking $a = 3/2$, $m = n - 1$ and $d = -1/3$ in (11). The derivation of Corollary 1 is similar but a bit more complicated since one has to use an additional contiguous relation.

5 Hexagons with cut off corners

The following conjecture stated by Mitra et al. [31] claims that certain elements of the stationary state of H_{2n}^C are given by enumerations of hexagons with cut off corners calculated by Ciucu and Krattenthaler [7]. Using the notation of [31], we denote p repeated opening parentheses by $(^p$ and p repeated opening parentheses followed by p closing parentheses as $(\dots)_p = (^p \dots)^p$. Furthermore, we use the special notation $[s, t, p]$ for matchings of the form $((\)_s (\)_t)_p$.

¹We are grateful to Christian Krattenthaler for pointing this out.

Conjecture 5 *The coefficient a_F of matching $F = [s, t, p]$ in the stationary state of $H_{2p+2s+2t}^C$ is given by*

$$\begin{aligned} a_{[s,t,p]} &= \det_{1 \leq i, j \leq s} \left(\binom{2(s+t+p) + j - 2i}{s+t-j} - \binom{2(s+t+p) + j - 2i}{s+t-j-2i+1} \right) \\ &= \prod_{j=1}^s \frac{(j-1)!(2t+2p+2j-1)!(2p+2j)_j(3t+2p+3j)_{s-j}}{(t+2p+s+2j-1)!(t+s-j)!} \end{aligned}$$

The evaluation of the determinant in this conjecture is due to Ciucu and Krattenthaler [7].

The numbers appearing in Conjecture 5 enumerate lozenge tilings, or dimer configurations, on hexagons $H_a(s, 2p+s+t-1, t)$ with maximal staircases removed from adjacent vertices (see [7] for definitions). With Conjecture 1 in mind, Conjecture 5 implies that FPL diagrams with matching $[s, t, p]$ are equinumerous with lozenge tilings on $H_a(s, 2p+s+t-1, t)$. Here we give the explicit bijection.

Theorem 1 *Let $n = p + s + t$. Then*

$$M_{[s,t,p]}(G_{2n}^V) = \prod_{j=1}^s \frac{(j-1)!(2t+2p+2j-1)!(2p+2j)_j(3t+2p+3j)_{s-j}}{(t+2p+s+2j-1)!(t+s-j)!}$$

In the rest of this section we give a sketch of the proof of Theorem 1. We first note that every matching fixes the loops of the corresponding FPL diagrams to pass through a particular set of edges. Let us call these edges **fixed edges**. The way these edges are fixed is prescribed in

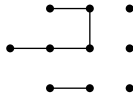
Lemma 8 *The implication*



holds if the top and bottom loop segments do not belong to the same loop and if either of the following holds,

- i) *The middle loop segment belongs to a third loop.*
- ii) *If the middle loop segment belongs to the same loop as the top or bottom segment, it is connected to it via one of the leftmost edges.*

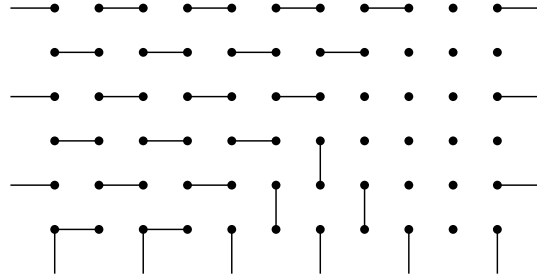
In case i) the third loop has to pass between the top and bottom loop segments while the condition in case ii) excludes possibilities like



Remark 7 The fixed edges can be adjacent and therefore do not necessarily form a perfect matching. Furthermore, for a matching $[s, t, p]$ every site of G_{2n}^V is part of a fixed edge.

Remark 8 None of the FPL diagrams with matching $[s, t, p]$ contains an internal closed loop.

Example 19 For the matching $(2, 1, 3) = (((((())))))))$ the fixed edges resulting from the first 5 parentheses, and the parentheses 6, 7 and 8 are

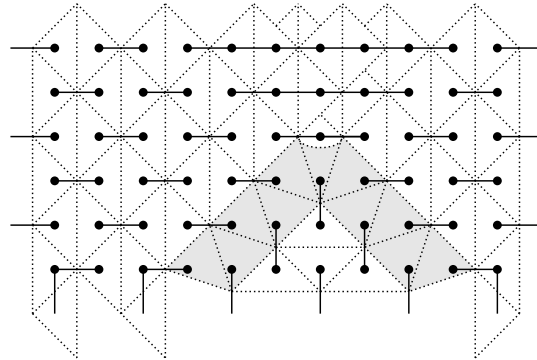


For clarity of the picture, the fixed edges resulting from the last 4 parentheses are not drawn. That region of fixed edges partly overlaps the region of fixed edges corresponding to the first 5 parentheses.

Remark 9 More edges may in fact be fixed due to the specific geometry of the grid G_{2n}^V .

Since for matchings $[s, t, p]$ all sites correspond to a fixed edge, the problem of counting all FPL diagrams with this matching reduces to a dimer problem which is equivalent to a lozenge tiling problem on a graph defined by the fixed edges. This graph is called the **fixed graph**. It forms a folded piece of triangular lattice and is obtained by joining the midpoints of all the fixed edges.

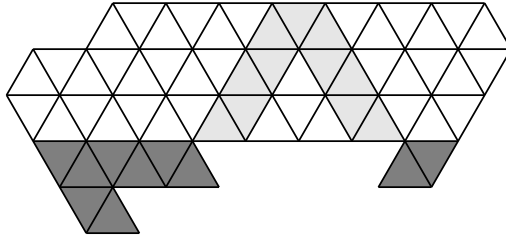
Example 20 For the matching $(2, 1, 3)$, the set of fixed edges and its corresponding fixed graph is



The graph consists of three homogeneous regions arising from the fixed edges of the first 5 parentheses, parentheses 6,7 and 8 and the last 4 parentheses respectively. The first and last region partially overlap in the top middle of the picture, where the fixed edges are adjacent. To guide the eye, the region between the three homogeneous regions is shaded.

As already hinted at in Remark 9, the particular geometry of the fixed graph completely determines dimer or tiling configurations on certain regions. These regions are the overlapping parts of the fixed graph and patches in the lower left and right corners. They do therefore not contribute to the enumeration and may as well be removed.

Example 21 By removing the overlapping part of the graph in Example 20 and deforming it so that it fits on the triangular lattice, it follows that the fixed graph for the matching $(2, 1, 3)$ is



The lightly shaded area corresponds to the shaded area of Example 20, and is a guide to the eye only. The regions on which dimer configurations are enforced by the geometry are shaded darker.

When the regions with fixed dimer configurations are removed, the resulting graph is precisely that of a hexagon with maximal staircases removed from adjacent corners [7] and whose sides are given by s , $2p + s + t - 1$ and t . The number of dimer configurations on such hexagons is given by Theorem 1.6 of [7]. Theorem 1 then follows immediately from that result.

6 Concluding remarks

There is another model known that displays the appearance of alternating sign matrix numbers in its stationary state. This is the rotor model [3] which is based on two Temperley-Lieb algebras. It is conjectured that in that model also 3-enumerations of ASMs [25] play a role.

Many conjectures are stated but remain unproven. One may hope that again the solvability of the six-vertex model and the XXZ spin chain [4] can be used to find proofs. In particular the very special properties at $q = \exp(i\pi/3)$, see e.g. [5], have already led to several interesting results [9, 10, 12, 20, 42, 43].

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