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Bethe ansatz for the Temperley–Lieb loop model with open boundaries

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Abstract. We diagonalize the Hamiltonian of the Temperley–Lieb loop model with open boundaries using a coordinate Bethe ansatz calculation. We find that in the ground-state sector of the loop Hamiltonian, but not in other sectors, a certain constraint on the parameters has to be satisfied. This constraint has a natural interpretation in the Temperley–Lieb algebra with boundary generators. The spectrum of the loop model contains that of the quantum spin-1/2 XXZ chain with non-diagonal boundary conditions. We hence derive a recently conjectured solution of the complete spectrum of this XXZ chain. We furthermore point out a connection with recent results for the two-boundary sine–Gordon model.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz), solvable lattice models

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Contents

1. Introduction

It has been known for some time that the quantum spin-1/2 XXZ chain with non-diagonal boundary fields is integrable [1]. However, conventional methods to calculate its spectrum, such as the Bethe ansatz, have been found difficult to apply due to a lack of spin conservation or other good quantum numbers. Recently however, Nepomechie made progress using functional equations. In root of unity cases he obtained Bethe ansatz equations of non-conventional form [2], while in [3] he derived conventional equations for some special cases of the boundary parameters. Cao *et al* [4] succeeded in formulating a more general Bethe ansatz solution of conventional form using the algebraic Bethe ansatz, provided that the parameters appearing in the Hamiltonian satisfy a certain constraint. The same equations were subsequently also found by Nepomechie [5].

This solution was studied numerically and analytically in [6, 7], and it was noted that it did not describe the full spectrum for all parameter values satisfying the constraint. In an addendum to [6] a simple extension of the original Bethe ansatz solution was conjectured on numerical grounds that does describe the complete spectrum, again provided the constraint is satisfied. In this paper we will give a derivation of the complete spectrum of the XXZ chain, also with the condition that a particular constraint has to be satisfied. For even system sizes this constraint is a special case of the constraint considered by the authors above, but it is different for odd L.

We use the fact that the XXZ spin chain can be described in terms of the Temperley–Lieb algebra with boundary generators. In the diagrammatic representation of this algebra the Hamiltonian of the spin chain is that of a loop model. We show that there exists a simple eigenstate in this loop model that serves as a pseudo-vacuum to set up a coordinate Bethe ansatz calculation, as originally used by Bethe [8]. We are thus able to derive the Bethe ansatz equations, some of which were conjectured in [6]. The approach followed in this paper is a generalization of the method described in [9]–[11].

For most sectors of the loop Hamiltonian, whose spectrum is larger than that of the spin chain, there is no additional constraint on the parameters. However, for the sector containing the spin chain spectrum, we do need to restrict ourselves with a constraint similar to the one found in [4] and [5]. This constraint is very natural in terms of the loop model since it preserves a $\mathbb{Z}_2$ symmetry (see section 3). When the constraint is not satisfied, the symmetry is broken and one loses a conservation law which introduces additional difficulties in formulating a Bethe ansatz. We hope however that our formulation offers new insights for finding the general solution of the spectrum.

In section 2 we introduce the Temperley–Lieb loop model with boundary generators, and show its relation to the quantum spin-1/2 XXZ chain with non-diagonal boundary conditions. In section 3 we show that the loop model has a simple eigenstate which we can use as a pseudo-vacuum to set up a Bethe ansatz calculation. In this section we also identify the pseudo-particles that label the sectors of the Hamiltonian. Before presenting a detailed derivation in sections 5, 6 and 7, we state our main result in section 4.

2. Algebra and representations

2.1. The Temperley–Lieb algebra with boundaries

The open Temperley–Lieb (OTL) algebra T_L^{O} with generators e_x $(x = 1, \ldots, L-1)$ and boundary generators f_+ and f_- is defined as follows [12]:

$$
\begin{aligned}
&e_x^2 = te_x && f_-^2 = s_- f_- \\
&e_x e_{x\pm1} e_x = e_x && f_+^2 = s_+ f_+ \\
&e_x e_y = e_y e_x \qquad \text{for } |x-y| \le 2 && f_- f_+ = f_+ f_- \\
&e_1 f_- e_1 = e_1 && e_x f_- = f_- e_x \qquad \text{for } x > 1 \\
&e_{L-1} f_+ e_{L-1} = e_{L-1} && e_x f_+ = f_+ e_x \qquad \text{for } x < L-1.
\end{aligned}
\tag{2.1}
$$

This algebra is infinite dimensional. It can be made finite dimensional by introducing

$$
\begin{aligned}
I_{2n} &= \prod_{x=0}^{n-1} e_{2x+1} \qquad & J_{2n} &= f_- \prod_{x=1}^{n-1} e_{2x} f_+ \\
I_{2n+1} &= f_- \prod_{x=1}^{n} e_{2x} \qquad & J_{2n+1} &= \prod_{x=0}^{n-1} e_{2x+1} f_+
\end{aligned}
\tag{2.2}
$$

and imposing the following relations:

$$
I_L J_L I_L = bI_L, \qquad J_L I_L J_L = bJ_L, \tag{2.3}
$$

both for L even and L odd. For later convenience, we introduce here two sub-algebras. The first one is the original Temperley–Lieb algebra, introduced in [13], and formed by the generators e_x $(x = 1, \ldots, L-1)$. As it does not contain boundary generators we will call it the closed TL algebra and denote it here by T_L^{C}. The second sub-algebra we consider is the mixed TL algebra containing only the boundary generator f_-. We denote it by T_L^{M}, and it is formed by the generators e_x $(x = 1, \ldots, L-1)$ and f_-. The algebra T_L^{M} is the blob algebra introduced in [10, 14, 15]. We further note that the relation (2.3) does not apply to the algebras T_L^{C} and T_L^{M}, but only to the full algebra T_L^{O}.

In this paper we will consider the operators

$$
H_L^{\mathrm{C}} = \sum_{x=1}^{L-1} e_x, \tag{2.4}
$$

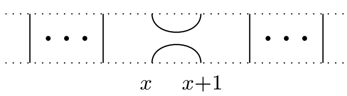
$$
H_L^{\mathrm{M}} = a_- f_- + \sum_{x=1}^{L-1} e_x, \tag{2.5}
$$

$$
H_L^{\mathrm{O}} = a_- f_- + a_+ f_+ + \sum_{x=1}^{L-1} e_x. \tag{2.6}
$$

These operators correspond in certain representations to the quantum spin-1/2 XXZ Hamiltonian with various types of open boundary condition; see section 2.3. The main result of this paper is the diagonalization of these operators in a different representation, namely the loop representation which we will explain now.

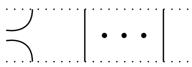
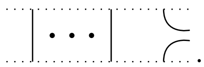
2.2. The loop representation

Instead of the algebra it is convenient to use the graphical representation of T_L^{O},

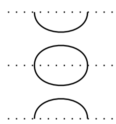
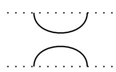
$e_x =$ 

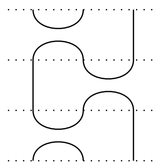
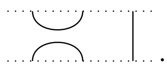
, (2.7)

and

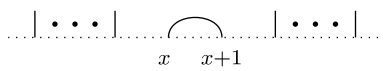
$$f_- = $$  $$f_+ = $$  (2.8)

Multiplication of two words in the algebra corresponds to putting one word below the other and merging the loop lines. For example, the relations $e_x^2 = te_x$ and $e_xe_{x+1}e_x = e_x$ graphically read

 $$= t$$  (2.9)

 $$=$$  (2.10)

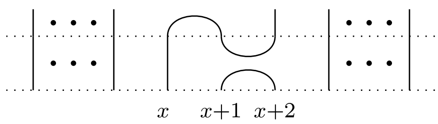
The loop representation is obtained by filtering the algebra by quotients of the left ideals generated by the (un-normalized) idempotents I_n and J_n. This procedure is described in detail in [16]; here we will explain this representation pictorially. A crucial property of the loop representation is that it has a highest weight state, which is represented graphically by L vertical lines, i.e. it is the state in which no sites are connected by loop lines. We will take the view that in this case the sites are 'connected to infinity'. Let us denote this highest weight state by $|\rangle$. All other states in the loop representation can be obtained by applying the words of the algebra to this highest weight state. The pictures corresponding to these other states are obtained by placing the picture of the word underneath that of the highest weight state and removing disconnected parts from the top of the combined picture. For example, the state corresponding to $e_x|\rangle$ will be represented by the bottom half of the picture in (2.7),

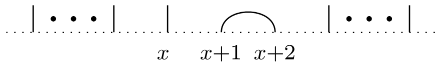
$$e_x|\rangle = |x\rangle = $$ 

, (2.11)

where we have denoted by $|x\rangle$ the state in which sites x and $x+1$ are connected to each other while all other sites are connected to infinity.

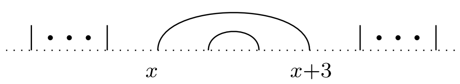
It is helpful to understand the action of e_{x+1} on $|x\rangle$, which is given by the following pictures:

$$e_{x+1}e_x|\rangle = $$ 

$$=$$ 

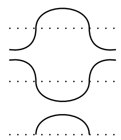
$$= |x+1\rangle.$$ (2.12)

A slightly more complicated example is given by the picture representing $e_{x+1}e_{x+2}e_x|\rangle$,

$$e_{x+1}e_{x+2}e_x|\rangle = $$ 

. (2.13)

Analogous pictures are obtained when the boundary generators $f_\pm$ are involved. In particular, the first relation in (2.3) for $L=2$ reads graphically

$e_1 f_- f_+ e_1|\rangle =$  $= b$  $= b\, e_1|\rangle.$ (2.14)

Parameter b in (2.3) thus is the weight of the two removed horizontal loop lines. Note that the state $e_1 f_- f_+|\rangle$ has the same lower part of the picture as $e_1|\rangle$, but contains only one horizontal loop line. This cannot be removed using the algebraic relations, but in analogy with the above we do have for $L=2$ that

$$f_- f_+ e_1 f_- f_+|\rangle = b f_- f_+|\rangle. \tag{2.15}$$

In general we find two isomorphic invariant subspaces for $b \neq 0$, namely the ones generated by $I_L|\rangle$ and $J_L|\rangle$. Because the spectrum of H_L^{O} is identical on them, even when $b=0$, we will discard the latter subspace. Hence, when considering cases where all loop lines are connected we will always work exclusively in the one generated by $I_L|\rangle$. In this way we construct the loop representations for the algebras T_L^{O}, T_L^{M} and T_L^{C} and we denote the representation spaces by V_L^{C}, V_L^{M} and V_L^{O} respectively. It is furthermore important to note that the loop lines are not allowed to cross.

Instead of this graphical loop representation we will use a more convenient typographical notation in the following. If a site x is connected to a site $y<x$ or to the left-hand boundary, we write a closing parenthesis ')' at x. If a site x is connected to a site $y>x$ or to the right-hand boundary we write an opening parenthesis '(' at x. If a site is not connected (or connected to infinity) we write a vertical bar '|' at x. For the sub-algebra T_2^{M} we thus have the state space

$$V_2^{\mathrm{M}} = \mathrm{Span}\{||,)|, (),))\}. \tag{2.16}$$

There is a similar identification for T_2^{O} where now loops can also connect to the right-hand boundary. For example, |(denotes a state where the loop segment on the second site is connected to the right-hand boundary. The seven basis states of V_2^{O} are

$$||,\ )|,\ |(,\ (),\ )(,\ )),\ ((. \tag{2.17}$$

We will call a sequence of the three symbols ')', '(' and '|' a connectivity.

The dimensions of the representation spaces V_L^{C}, V_L^{M} and V_L^{O} can be calculated by counting the number of sequences of ')', '(' and '|' subject to the non-crossing constraint (for example, '|)' cannot occur). In the appendix we find the generating functions for these dimensions. Asymptotically they are given by

$$\dim V_L^{\mathrm{C}} \approx L^{-1/2} 2^L, \qquad \dim V_L^{\mathrm{M}} = 2^L, \qquad \dim V_L^{\mathrm{O}} \approx L^{1/2} 2^L. \tag{2.18}$$

2.3. XXZ representation

The following representation of the TL algebra makes Hamiltonian (2.6) up to a constant equal to that of the XXZ spin chain with non-diagonal boundary terms [5]:

$$e_x = -\tfrac{1}{2}\left(\sigma_x^1\sigma_{x+1}^1 + \sigma_x^2\sigma_{x+1}^2 + \mathrm{ch}\,\eta\sigma_x^3\sigma_{x+1}^3 - \mathrm{sh}\,\eta(\sigma_x^3 - \sigma_{x+1}^3) - \mathrm{ch}\,\eta\right) \tag{2.19}$$

$$f_- = \frac{1}{2\rho_-}\left(\mathrm{sh}(\alpha_- + \beta_-)\sigma_1^3 + \mathrm{ch}\,\theta_-\sigma_1^1 + \mathrm{i}\,\mathrm{sh}\,\theta_-\sigma_1^2 + \mathrm{ch}(\alpha_- + \beta_-)\right), \tag{2.20}$$

$$f_+ = \frac{1}{2\rho_+}\left(-\mathrm{sh}(\alpha_+ + \beta_+)\sigma_L^3 + \mathrm{ch}\,\theta_+\sigma_L^1 + \mathrm{i}\,\mathrm{sh}\,\theta_+\sigma_L^2 + \mathrm{ch}(\alpha_+ + \beta_+)\right), \tag{2.21}$$

where

$$\rho_\mp = \mathrm{ch}(\alpha_\mp + \beta_\mp + \eta), \tag{2.22}$$

and σ_x^i represents the ith Pauli matrix at site x. The parameters t, $a_\mp$ and $s_\mp$ are given by

$$\begin{aligned} t &= 2\mathrm{ch}\,\eta, \\ a_\mp &= -\mathrm{sh}\,\eta\frac{\mathrm{ch}(\alpha_\mp + \beta_\mp + \eta)}{\mathrm{sh}\,\alpha_\mp\mathrm{ch}\,\beta_\mp}, \\ s_\mp &= \frac{\mathrm{ch}(\alpha_\mp + \beta_\mp)}{\mathrm{ch}(\alpha_\mp + \beta_\mp + \eta)}, \end{aligned} \tag{2.23}$$

while the expression for b depends on the parity of the system size L,

$$b = \begin{cases} \dfrac{\mathrm{ch}(\eta + \alpha_- + \alpha_+ + \beta_- + \beta_+) - \mathrm{ch}(\theta_- - \theta_+)}{2\mathrm{ch}(\alpha_- + \beta_- + \eta)\,\mathrm{ch}(\alpha_+ + \beta_+ + \eta)} & \text{for even } L, \\ \dfrac{\mathrm{ch}(\alpha_- - \alpha_+ + \beta_- - \beta_+) + \mathrm{ch}(\theta_- - \theta_+)}{2\mathrm{ch}(\alpha_- + \beta_- + \eta)\,\mathrm{ch}(\alpha_+ + \beta_+ + \eta)} & \text{for odd } L. \end{cases} \tag{2.24}$$

The dimension of this spin-1/2 representation is 2^L. It is important to note that the dimension of V_L^{O} is larger than this. However, $\dim V_L^{\mathrm{M}} = 2^L$ and this representation seems to be equivalent to XXZ.

The Hamiltonian H_L^{O}, given in (2.6), is related to the Hamiltonian $\mathcal{H}$ given in [6] by

$$\begin{aligned} H_L^{\mathrm{O}} &= -\mathcal{H} + \tfrac{1}{4}(L-1)t + \tfrac{1}{2}(a_-s_- + a_+s_+) \\ &= -\mathcal{H} + \tfrac{1}{2}(L-1)\mathrm{ch}\,\eta - \tfrac{1}{2}\mathrm{sh}\,\eta(\mathrm{cth}\,\alpha_- + \mathrm{th}\,\beta_- + \mathrm{cth}\,\alpha_+ + \mathrm{th}\,\beta_+). \end{aligned} \tag{2.25}$$

3. Pseudo-particles

In the representations on the spaces V_L^{C}, V_L^{M} and V_L^{O} the Hamiltonians (2.4)–(2.6) have a block triangular structure: connections can be created but cannot be undone. This implies that for each of the Hamiltonians (2.4)–(2.6) the state $\langle|$, dual to $|\rangle$, is a *left* eigenvector with eigenvalue 0 (since there are no nonzero matrix elements giving transitions *to* this state). More generally, although the Hamiltonians are not block diagonal, the number of connected sites (either to another site or to the boundary) is still a good quantum number to characterize the spectrum and its corresponding set of *left* eigenvectors, which are elements of the dual spaces $V_L^{\mathrm{C}*}$, $V_L^{\mathrm{M}*}$ and $V_L^{\mathrm{O}*}$. We will denote the basis states in the dual space V_L^* by the same connectivities as their dual states in V_L (recall that a connectivity is a sequence of the three symbols ')', '(' and '|'). Let $\langle c|$ denote the basis state corresponding to the connectivity c. A vector $\langle\psi| \in V_L^*$ then is decomposed on the dual basis as

$$\langle\psi| = \sum_c \psi(c)\langle c|, \tag{3.1}$$

and the left eigenvalue equation we want to solve reads

$$\Lambda\langle\psi| = \langle\psi|H, \Leftrightarrow \Lambda\psi(c) = \sum_{c'} \psi(c')H_{c'c}. \tag{3.2}$$

In the following sections we will calculate the spectra by formulating a coordinate Bethe ansatz for the left eigenvectors of each of the Hamiltonians. Here we want to briefly discuss the main ingredients. The trivial eigenstate will be used as a pseudo-vacuum for the Bethe ansatz. In setting up the Bethe ansatz calculation we need to identify 'pseudo-particles' or 'elementary excitations' on this pseudo-vacuum. We will introduce the notion of pseudo-particle for the space V_L. It is then easy to translate this notion to the space V_L^* by duality.

A link between x and $x+1$, i.e. the graphical representation of $|x\rangle$ in (2.11), will be such a pseudo-particle. Since in this representation $e_{x\pm 1}|x\rangle = |x\pm 1\rangle$, the bulk Hamiltonian (2.4) describes hopping of the pseudo-particle. It furthermore contains a diagonal term ($e_x|x\rangle = t|x\rangle$) and creates pseudo-particles at other sites through the action of e_y with $y \neq x-1, x$ or $x+1$. For example, a state with two pseudo-particles at x and y respectively is given by

$$e_x|y\rangle = |x,y\rangle = \ldots|\cdots|\ldots\underset{x\quad x+1}{\frown}\ldots|\cdots|\ldots\underset{y\quad y+1}{\frown}\ldots|\cdots|\ldots. \tag{3.3}$$

The states with well separated pseudo-particles form just a small subset of all possible states. All the other states however will be considered as bound states of pseudo-particles. For example, we consider a state of the form (2.13) as a bound state of two pseudo-particles.

In V_L^{C} we only have states where sites are connected pairwise. When boundaries are present, sites do not have to be paired to be connected; they can be connected to the left- or right-hand boundary. The number of connected sites can therefore take on odd values. In the following we will take the number of connected sites as our quantum number labelling the different sectors of the Hamiltonians.

Anticipating our results, we note here that when $b=0$ in (2.3) the Hamiltonian H_L^{O} possesses an additional $\mathbb{Z}_2$ symmetry. From (2.14), and more generally from

$$I_L J_L I_L|\rangle = bI_L|\rangle, \tag{3.4}$$

it follows that H_L^{O} is block triangular when $b=0$ with respect to the subspaces with an odd or even number of sites connected to the left-hand boundary. It is clear that this symmetry is always present in sectors where the number of connected sites is less than the system size.

4. Results

Before giving a detailed derivation of the Bethe ansatz for the Temperley–Lieb loop model, we would like to state our main result first. For convenience, we define the following two functions:

$$\phi_\pm(u,\epsilon) = \frac{\operatorname{sh}(\eta/2+\epsilon\alpha_\pm-u)\operatorname{ch}(\eta/2+\epsilon\beta_\pm-u)}{\operatorname{sh}(\eta/2+\epsilon\alpha_\pm+u)\operatorname{ch}(\eta/2+\epsilon\beta_\pm+u)}, \tag{4.1}$$

where $\epsilon^2=1$. The eigenvalues of the loop model H_L^{O} for sectors labelled by n are given by

- n even, $n<L$:
  For these sectors, the spectrum is described by the expressions

$$\begin{aligned}\Lambda_0 &= \sum_{j=1}^{n/2} \frac{\operatorname{sh}^2\eta}{\operatorname{sh}(\eta/2-u_j)\operatorname{sh}(\eta/2+u_j)},\\ \Lambda_1 &= \sum_{j=1}^{n/2-1} \frac{\operatorname{sh}^2\eta}{\operatorname{sh}(\eta/2-v_j)\operatorname{sh}(\eta/2+v_j)} - \operatorname{sh}\eta(\operatorname{cth}\alpha_- + \operatorname{th}\beta_- + \operatorname{cth}\alpha_+ + \operatorname{th}\beta_+),\end{aligned} \tag{4.2}$$

  where the complex numbers u_i and v_i are solutions of the equations

$$\left(\frac{\operatorname{sh}(\eta/2-u_i)}{\operatorname{sh}(\eta/2+u_i)}\right)^{2L} = \phi_-(u_i,+)\phi_+(u_i,+)\prod_{\substack{j=1\\ j\neq i}}^{n/2}\frac{\operatorname{sh}(u_i-u_j+\eta)\operatorname{sh}(u_i+u_j+\eta)}{\operatorname{sh}(u_i-u_j-\eta)\operatorname{sh}(u_i+u_j-\eta)}, \tag{4.3}$$

  and

$$\left(\frac{\operatorname{sh}(\eta/2-v_i)}{\operatorname{sh}(\eta/2+v_i)}\right)^{2L} = \phi_-(v_i,-)\phi_+(v_i,-)\prod_{\substack{j=1\\ j\neq i}}^{n/2-1}\frac{\operatorname{sh}(v_i-v_j+\eta)\operatorname{sh}(v_i+v_j+\eta)}{\operatorname{sh}(v_i-v_j-\eta)\operatorname{sh}(v_i+v_j-\eta)}. \tag{4.4}$$

  These are the Bethe ansatz equations conjectured in the addendum of [6] for the special case of $k=1$. A similar division into two different functional forms was found recently in the ground-state energy for the two-boundary sine–Gordon model [17].

- n odd, $n<L$:
  For these sectors, the spectrum is described by the expressions

$$\begin{aligned}\Lambda_0 &= \sum_{j=1}^{(n-1)/2} \frac{\operatorname{sh}^2\eta}{\operatorname{sh}(\eta/2-u_j)\operatorname{sh}(\eta/2+u_j)} - \operatorname{sh}\eta(\operatorname{cth}\alpha_- + \operatorname{th}\beta_-),\\ \Lambda_1 &= \sum_{j=1}^{(n-1)/2} \frac{\operatorname{sh}^2\eta}{\operatorname{sh}(\eta/2-v_j)\operatorname{sh}(\eta/2+v_j)} - \operatorname{sh}\eta(\operatorname{cth}\alpha_+ + \operatorname{th}\beta_+),\end{aligned} \tag{4.5}$$

  where the complex numbers u_i and v_i are solutions of the equations

$$\left(\frac{\operatorname{sh}(\eta/2-u_i)}{\operatorname{sh}(\eta/2+u_i)}\right)^{2L} = \phi_-(u_i,-)\phi_+(u_i,+)\prod_{\substack{j=1\\ j\neq i}}^{(n-1)/2}\frac{\operatorname{sh}(u_i-u_j+\eta)\operatorname{sh}(u_i+u_j+\eta)}{\operatorname{sh}(u_i-u_j-\eta)\operatorname{sh}(u_i+u_j-\eta)}, \tag{4.6}$$

and

$$\left(\frac{\mathrm{sh}(\eta/2-v_i)}{\mathrm{sh}(\eta/2+v_i)}\right)^{2L}=\phi_-(v_i,+)\phi_+(v_i,-)\prod_{\substack{j=1\\ j\neq i}}^{(n-1)/2}\frac{\mathrm{sh}(v_i-v_j+\eta)\,\mathrm{sh}(v_i+v_j+\eta)}{\mathrm{sh}(v_i-v_j-\eta)\,\mathrm{sh}(v_i+v_j-\eta)}. \tag{4.7}$$

- $n=L$ even:
  In this sector, the spectrum of H_L^{O} is still given by (4.2)–(4.4) provided that
  $$b=0\Leftrightarrow \mathrm{ch}(\eta+\alpha_-+\beta_-+\alpha_++\beta_+)-\mathrm{ch}(\theta_--\theta_+)=0, \tag{4.8}$$
  where b is the parameter appearing in (2.3). The spectrum of H_L^{O} in this sector with the additional constraint is identical to that of the XXZ spin chain with an even number of spins, as conjectured in the addendum of [6].
- $n=L$ odd:
  In this case, the spectrum of H_L^{O} is still given by (4.5)–(4.7) provided that
  $$b=0\Leftrightarrow \mathrm{ch}(\eta+\alpha_-+\beta_--\alpha_+-\beta_+)+\mathrm{ch}(\theta_--\theta_+)=0. \tag{4.9}$$
  The spectrum of H_L^{O} in this sector is identical to that of the XXZ spin chain with an odd number of spins. Note that while the constraint (4.8) for even system sizes is the same as that obtained in [5] for $k=1$, the constraint (4.9) is similar but not identical to the one in [5] with $k=0$. That constraint corresponds to $b=s_-s_+$, a case that is not discussed in this paper (see however section 7.6).

In the following sections we give a detailed derivation of these results. As an intermediate result, we also obtain the spectrum of H_L^{M}, which is closely related to a Hamiltonian diagonalized by the Bethe ansatz in [18].

5. Closed boundaries

In this section we will diagonalize Hamiltonian (2.4) on the space $V_L^{\mathrm{C}*}$. In the XXZ representation this Hamiltonian corresponds to the quantum group symmetric Hamiltonian, which has diagonal, spin conserving, boundary fields. The Bethe ansatz solution of the spectrum of this Hamiltonian is well known [1, 19]. We rederive it here in terms of the loop model which will be a useful exercise for when we introduce boundary fields in later sections.

As explained above, a sector of H_L^{C} is labelled by an integer n, denoting the number of connected sites. For closed boundaries n will be even since sites cannot be connected to the boundary, only to each other.

5.1. $n=0$

Consider the configuration with all sites disconnected, i.e. $\langle|=|\cdots|$. We will try to solve (3.2) with the trial function $\langle\psi_0|$ given by

$$\langle\psi_0|=\langle|. \tag{5.1}$$

Plugging $\langle\psi_0|$ into (3.2) we obtain

$$\Lambda\langle|=0, \tag{5.2}$$

and immediately obtain that $\langle\psi_0|$ is a left eigenvector with eigenvalue

$$\Lambda = 0, \tag{5.3}$$

as asserted in section 3. Hence we can use $\langle|$ as a pseudo-vacuum.

5.2. $n = 2$

Here and in the following we will consider a trial eigenvector $\langle\psi_n|$, which is a linear combination of states whose coefficients for states with more than n connected sites are zero. The coefficients of $\langle\psi_n|$ are denoted by $\psi_n(\mathrm{c})$, where c labels the connectivity. For $n = 2$, we thus take a linear combination of the pseudo-vacuum, with coefficient $\psi_2()$, and states with a pseudo-particle at site x, i.e. states of the form $\langle x| = |\cdots|()|\cdots|$, with nonzero coefficients $\psi_2(x)$. Now we will try to solve (3.2) with the trial function $\langle\psi_2|$ given by

$$\langle\psi_2| = \psi_2()\langle| + \sum_{x=1}^{L}\psi_2(x)\langle x|. \tag{5.4}$$

Plugging $\langle\psi_2|$ into (3.2) and equating each component to zero the eigenvalue equation is equivalent to

$$\begin{aligned}\Lambda\psi_2() &= \sum_{x=1}^{L-1}\psi_2(x),\\ \Lambda\psi_2(x) &= t\psi_2(x) + \psi_2(x+1) + \psi_2(x-1) \qquad \text{for } 1 < x < L-1,\end{aligned} \tag{5.5}$$

together with the following equations at the boundaries $x = 1$ and $L - 1$:

$$\begin{aligned}\Lambda\psi_2(1) &= t\psi_2(1) + \psi_2(2),\\ \Lambda\psi_2(L-1) &= t\psi_2(L-1) + \psi_2(L-2).\end{aligned} \tag{5.6}$$

We first note that, due to the block triangular structure of the Hamiltonian, the coefficient $\psi_2()$ only appears in the first line of (5.5), which we therefore regard as defining $\psi_2()$.[3] The remaining equations can be satisfied by the ansatz

$$\psi_2(x) = f(x), \tag{5.7}$$

where f is the function

$$f(x) = A^+z^x + A^-z^{-x}. \tag{5.8}$$

The parameters $A^\pm$ and z are complex numbers to be determined later. We now find from (5.5) that

$$\Lambda = t + z + z^{-1}, \tag{5.9}$$

and in addition the consistency relations arising from the boundary equations (5.6),

$$f(0) = 0, \qquad f(L) = 0. \tag{5.10}$$

These consistency relations determine the ratio of the amplitudes $A^\pm$ and the complex number z. In this sector therefore the spectrum is given by the solutions of

$$\Lambda = \lambda(z), \qquad z^{2L} = 1, \tag{5.11}$$

where we have defined λ for future convenience,

$$\lambda(z) = t + z + z^{-1}. \tag{5.12}$$

[3] We need here that $\Lambda \neq 0$. Here and in the following we will not digress into such isolated singularities.

5.3. $n = 4$

Let us denote by $\psi_4(x_1, x_2)$ the coefficient for states of the form $|\cdots|()|\cdots|()|\cdots|$, with 'well separated' pseudo-particles at positions x_1 and x_2, i.e. $x_2 > x_1 + 1$. We furthermore denote the coefficient for states of the form $|\cdots|(())|\cdots|$ with 'merged' pseudo-particles by $\varphi_4(x)$. As a trial vector to solve the eigenvalue equation (3.2) we consider a vector with nonzero components $\psi_4(c)$ in all sectors m of V_L^{C*} with less than four connected sites. Aside from the states with $m = 4$ mentioned above, this trial vector also contains the states $|x\rangle$ and $|\rangle$ (with coefficients $\psi_4(x)$ and $\psi_4()$) in the $m = 2$ and 0 sectors respectively. In analogy with the case $n = 2$ we can write down the bulk eigenvalue equation componentwise. However, we will only look at the equations relating the eigenvector elements $\psi_4(c)$ in the $m = 4$ sector to each other. The equations relating elements $\psi_4(c)$ for $m < 4$ to those of $m = 4$ are similar to (5.5) and can be regarded as definitions of $\psi_4(c)$ for $m < 4$. We will therefore not write them down explicitly. The remaining equations are

$$\begin{aligned}\Lambda\psi_4(x_1, x_2) &= \psi_4(x_1 - 1, x_2) + \psi_4(x_1 + 1, x_2) + \psi_4(x_1, x_2 - 1)\\ &\quad+ \psi_4(x_1, x_2 + 1) + 2t\psi_4(x_1, x_2) \qquad \text{for } x_2 > x_1 + 2,\\ \Lambda\psi_4(x, x + 2) &= \psi_4(x - 1, x + 2) + \varphi_4(x) + \psi_4(x, x + 3) + 2t\psi_4(x, x + 2),\\ \Lambda\varphi_4(x) &= 2\psi_4(x, x + 2) + \psi_4(x - 1, x + 1) + \psi_4(x + 1, x + 3) + t\varphi_4(x)\\ &\quad \text{for } 1 < x < L - 3.\end{aligned} \tag{5.13}$$

As before, these are modified at the boundary to

$$\begin{aligned}&\Lambda\psi_4(1, x) = \psi_4(2, x) + \psi_4(1, x - 1) + \psi_4(1, x + 1) + 2t\psi_4(1, x),\\ &\Lambda\psi_4(x, L - 1)\\ &\qquad = \psi_4(x - 1, L - 1) + \psi_4(x + 1, L - 1) + \psi_4(x, L - 2) + 2t\psi_4(x, L - 1),\\ &\Lambda\psi_4(1, 3) = \psi_4(1, 4) + \varphi_4(1) + 2t\psi_4(1, 3),\\ &\Lambda\psi_4(L - 3, L - 1) = \psi_4(L - 4, L - 1) + \varphi_4(L - 3) + 2t\psi_4(L - 3, L - 1),\\ &\Lambda\varphi_4(1) = 2\psi_4(1, 3) + \psi_4(2, 4) + t\varphi_4(1),\\ &\Lambda\varphi_4(L - 3) = 2\psi_4(L - 3, L - 1) + \psi_4(L - 4, L - 2) + t\varphi_4(L - 3).\end{aligned} \tag{5.14}$$

Equations (5.13) and (5.14) can be solved with the following ansatz:

$$\psi_4(x_1, x_2) = f(x_1, x_2), \tag{5.15}$$

where

$$f(x_1, x_2) = \sum_{\pi\in S_2} \sum_{\sigma} A^{\sigma_1\sigma_2}_{\pi_1\pi_2} z^{\sigma_1 x_1}_{\pi_1} z^{\sigma_2 x_2}_{\pi_2}. \tag{5.16}$$

Here S_n is the group of permutations of n integers $\{1, 2, \ldots, n\}$ and $\pi : \{1, 2, \ldots, n\} \mapsto \{\pi_1, \pi_2, \ldots, \pi_n\}$ is a particular permutation. The sum over σ denotes a sum over all signs $\sigma_1 = \pm 1$ and $\sigma_2 = \pm 1$. The four amplitudes A^{++}_{12}, A^{++}_{21}, A^{--}_{12} and A^{--}_{21} as well as the two complex numbers z_1 and z_2 are to be determined later. The form of $\varphi_4(x)$ will be also derived from the eigenvalue equations.

From the bulk relations (5.13) we find

$$\Lambda = \sum_{j=1}^{2} \lambda(z_j), \tag{5.17}$$

where λ is defined in (5.12), and we also find

$$\varphi_4(x) = f(x, x+1) + f(x+1, x+2), \tag{5.18}$$

$$0 = f(x,x) + 2f(x+1, x+1) + f(x+2, x+2) + t(f(x, x+1) + f(x+1, x+2)). \tag{5.19}$$

Equation (5.18) defines the coefficient $\varphi_4(x)$ and (5.19) is satisfied if

$$\frac{A^{\sigma_1\sigma_2}_{\pi_1\pi_2}}{A^{\sigma_2\sigma_1}_{\pi_2\pi_1}} = -\frac{S(z^{\sigma_2}_{\pi_2}, z^{\sigma_1}_{\pi_1})}{S(z^{\sigma_1}_{\pi_1}, z^{\sigma_2}_{\pi_2})}, \tag{5.20}$$

where

$$S(z, w) = 1 + tw + zw. \tag{5.21}$$

In addition, it can also be shown that the first two of the boundary relations (5.14) are equivalent to

$$f(0, x) = 0, \qquad f(x, L) = 0, \tag{5.22}$$

which can be satisfied by demanding that

$$z^{2\sigma_1 L}_{\pi_1} = -\frac{A^{\sigma_2,-\sigma_1}_{\pi_2\pi_1}}{A^{\sigma_2,\sigma_1}_{\pi_2\pi_1}}, \qquad 1 = -\frac{A^{-\sigma_1,\sigma_2}_{\pi_1\pi_2}}{A^{\sigma_1,\sigma_2}_{\pi_1\pi_2}}, \tag{5.23}$$

for all choices of the signs $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1$ and both permutations of $\pi = (1, 2)$. It turns out that the remaining equations do not impose any further constraints and are now automatically satisfied. Putting everything together we find that

$$z^{2L}_{\pi_1} = -\frac{A^{+-}_{\pi_2\pi_1}}{A^{-+}_{\pi_1\pi_2}}\frac{A^{-+}_{\pi_1\pi_2}}{A^{++}_{\pi_1\pi_2}}\frac{A^{++}_{\pi_1\pi_2}}{A^{++}_{\pi_2\pi_1}} = \frac{S(z^{-1}_{\pi_1}, z_{\pi_2})S(z_{\pi_2}, z_{\pi_1})}{S(z_{\pi_2}, z^{-1}_{\pi_1})S(z_{\pi_1}, z_{\pi_2})}, \tag{5.24}$$

for both permutations of $\pi = (1, 2)$. We have found the two-particle Bethe ansatz equation of the XXZ spin chain.

5.4. General $n = 2k$

Because of the integrability of this model, the eigenvalue equations for sectors containing more than two pseudo-particles all reduce to those for two particles. The programme of the previous sections can therefore be carried out for larger values of n. As a trial function for a given n we take a vector in which all coefficients in sectors higher than n are zero. We also note that coefficients of states in sectors lower than n are defined by the eigenvalue equation in terms of those for n. To find the latter, we assume that the wavefunction elements $\psi_{2k}(x_1, \ldots, x_k)$, corresponding to well separated pseudo-particles, have the following form:

$$\psi_{2k}(x_1, \ldots, x_k) = \sum_{\pi\in S_k}\sum_{\sigma} A^{\sigma_1\cdots\sigma_k}_{\pi_1\cdots\pi_k}\prod_{i=1}^{k} z^{\sigma_i x_i}_{\pi_i}, \tag{5.25}$$

where, as in (5.16), S_k is the group of permutations of k integers $\{1, 2, \ldots, k\}$ and the sum over σ denotes a sum over all signs $\sigma_i = \pm 1$ $(i = 1, \ldots, k)$. We also assume that the other wavefunction elements in the same sector can be consistently calculated from the eigenvalue equation if the amplitudes satisfy

$$\frac{A^{\cdots\sigma_i\sigma_{i+1}\cdots}_{\cdots\pi_i\pi_{i+1}\cdots}}{A^{\cdots\sigma_{i+1}\sigma_i\cdots}_{\cdots\pi_{i+1}\pi_i\cdots}} = -\frac{S(z^{\sigma_{i+1}}_{\pi_{i+1}}, z^{\sigma_i}_{\pi_i})}{S(z^{\sigma_i}_{\pi_i}, z^{\sigma_{i+1}}_{\pi_{i+1}})}, \tag{5.26}$$

and

$$z^{2\sigma_n L}_{\pi_n} = -\frac{A^{\cdots -\sigma_n}_{\cdots\pi_n}}{A^{\cdots\sigma_n}_{\cdots\pi_n}}, \qquad 1 = -\frac{A^{-\sigma_1,\cdots}_{\pi_1\cdots}}{A^{\sigma_1,\cdots}_{\pi_1\cdots}}. \tag{5.27}$$

The complex numbers z_i satisfy the obvious generalization of (5.24) to arbitrary but even values of n,

$$z_i^{2L} = \prod_{\substack{j=1\\ j\neq i}}^{n/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}. \tag{5.28}$$

Furthermore, the spectrum is given by

$$\Lambda = \sum_{j=1}^{n/2} \lambda(z_j), \tag{5.29}$$

where we recall that λ and S are defined in (5.12) and (5.21) respectively. Equation (5.28) is of course the well known Bethe ansatz equation for the XXZ spin chain with diagonal boundary conditions [1, 19]. It is reproduced here from a Bethe ansatz in the loop representation of the Temperley–Lieb algebra. Using the well known transformation

$$z_i = \frac{\operatorname{sh}(\eta/2 + u_i)}{\operatorname{sh}(\eta/2 - u_i)}, \tag{5.30}$$

the eigenvalue takes on the familiar form,

$$\Lambda = \sum_{j=1}^{n/2} \frac{\operatorname{sh}^2\eta}{\operatorname{sh}(\eta/2 - u_j)\operatorname{sh}(\eta/2 + u_j)}, \tag{5.31}$$

where the complex numbers u_i are solutions of the equations

$$\left(\frac{\operatorname{sh}(\eta/2 - u_i)}{\operatorname{sh}(\eta/2 + u_i)}\right)^{2L} = \prod_{\substack{j=1\\ j\neq i}}^{n/2} \frac{\operatorname{sh}(u_i - u_j + \eta)\operatorname{sh}(u_i + u_j + \eta)}{\operatorname{sh}(u_i - u_j - \eta)\operatorname{sh}(u_i + u_j - \eta)}. \tag{5.32}$$

6. Mixed boundaries

A similar program as for closed boundaries can be carried out for mixed boundaries, i.e. we can diagonalize H_L^{M} on the space $V_L^{\mathrm{M}*}$. However, now n can take on odd values as well, representing sectors where an odd number of loop segments are connected to the left-hand boundary. We will denote by $\psi_{2k+l}(x_1, \ldots, x_k; y_1, \ldots, y_l)$ a coefficient in the eigenvector of a state with k well separated pseudo-particles at positions x_i $(i = 1, \ldots, k)$ and l sites at positions y_j $(j = 1, \ldots, l)$ connected to the left-hand boundary.

6.1. $n = 0$

We again start with the state with all sites disconnected, i.e. $\langle| = |\cdots|$, and find that it is a left eigenstate of H_L^{M} with eigenvalue $\Lambda = 0$.

6.2. $n = 1$

Next we consider eigenvectors in which the coefficient $\psi_1(;)$ of the pseudo-vacuum is nonzero as well as $\psi_1(;1)$ of the state $)|\cdots|$ with one site connected to the left-hand boundary. The eigenvalue equations are

$$\begin{aligned} \Lambda\psi_1(;) &= a_-\psi_1(;1), \\ \Lambda\psi_1(;1) &= a_-s_-\psi_1(;1). \end{aligned} \tag{6.1}$$

As before we regard the first line in (6.1) as defining $\psi_1(;)$. The remaining equation is solved by $\Lambda = a_-s_-$.

6.3. $n = 2$

Now we also consider nonzero coefficients of states with two connected sites, i.e. states of the form $|\cdots|()|\cdots|$ and the state $))|\cdots|$ whose coefficients in the eigenvector are denoted by $\psi_2(x;)$ and $\psi_2(;1,2)$ respectively. The resulting eigenvalue equations read

$$\begin{aligned} \Lambda\psi_2(;) &= a_-\psi_2(;1) + \sum_{y=1}^{L-1}\psi_2(y;), \\ \Lambda\psi_2(;1) &= a_-s_-\psi_2(;1) + \psi_2(1;), \\ \Lambda\psi_2(x;) &= t\psi_2(x;) + \psi_2(x+1;) + \psi_2(x-1;), \qquad \text{for } 1 < x < L-1. \end{aligned} \tag{6.2}$$

We also find the boundary relations,

$$\begin{aligned} \Lambda\psi_2(1;) &= t\psi_2(1;) + \psi_2(2;) + a_-\psi_2(;1,2), \\ \Lambda\psi_2(;1,2) &= a_-s_-\psi_2(;1,2) + \psi_2(1;), \\ \Lambda\psi_2(L-1;) &= t\psi_2(L-1;) + \psi_2(L-2;). \end{aligned} \tag{6.3}$$

We take the first two lines in (6.2) as definitions for $\psi_2(;)$ and $\psi_2(;1)$, and solve the remaining equations by making the ansatz

$$\psi_2(x;) = f(x), \tag{6.4}$$

where

$$f(x) = A^+z^x + A^-z^{-x}. \tag{6.5}$$

We then find from (6.2) and the first line in (6.3) that

$$\begin{aligned} \Lambda &= t + z + z^{-1}, \\ \psi_2(;1,2) &= a_-^{-1}f(0). \end{aligned} \tag{6.6}$$

The other two boundary equations in (6.3) imply the consistency equations,

$$\begin{aligned} (\lambda(z) - a_-s_-)f(0) &= a_-f(1), \\ f(L) &= 0. \end{aligned} \tag{6.7}$$

From these we find equations for the amplitudes $A^{\pm}$ and the complex number z. In this sector we thus find that the spectrum is given by

$$\begin{aligned}
&\Lambda = \lambda(z) = t + z + z^{-1},\\
&z^{2L} = -\frac{K_+(z)}{K_+(z^{-1})}\frac{A^-}{A^+} = \frac{K_+(z)K_-(z)}{K_+(z^{-1})K_-(z^{-1})},
\end{aligned} \tag{6.8}$$

where

$$K_+(z) = 1, \qquad K_-(z) = \lambda(z) - a_-(s_- + z). \tag{6.9}$$

6.4. $n = 3$

From now on we will only look at the equations relating elements of the eigenvector in the sector n to each other. The equations involving the elements of sectors $m < n$ can be regarded as definitions for those elements. The elements in sectors $m > n$ are taken to be zero.

The eigenvalue equation for the sector $n = 3$ then reads

$$\Lambda\psi_3(x;1) = (a_-s_- + t)\psi_3(x;1) + \psi_3(x+1;1) + \psi_3(x-1;1) \qquad \text{for } 2 < x < L, \tag{6.10}$$

with boundary relations

$$\begin{aligned}
&\Lambda\psi_3(2;1) = (a_-s_- + t)\psi_3(2;1) + \psi_3(3;1) + \psi_3(1;3),\\
&\Lambda\psi_3(1;3) = t\psi_3(1;3) + \psi_3(2;1) + a_-\psi_3(;1,2,3),\\
&\Lambda\psi_3(;1,2,3) = a_-s_-\psi_3(;1,2,3) + \psi_3(1;3) + s_-\psi_3(2;1),\\
&\Lambda\psi_3(L-1;1) = (a_-s_- + t)\psi_3(L-1;1) + \psi_3(L-2;1).
\end{aligned} \tag{6.11}$$

Equations (6.10) and (6.11) can be satisfied by making the ansatz

$$\psi_3(x;1) = f(x), \qquad f(x) = B^+z^x + B^-z^{-x}. \tag{6.12}$$

We then find

$$\begin{aligned}
&\Lambda = a_-s_- + t + z + z^{-1},\\
&\psi_3(1;3) = f(1),\\
&\psi_3(;1,2,3) = a_-^{-1}\left(f(0) + a_-s_-f(1)\right),
\end{aligned} \tag{6.13}$$

with the consistency equations

$$\begin{aligned}
&(\lambda(z) + a_-s_-)f(0) = a_-(1 - s_-t)f(1),\\
&f(L) = 0.
\end{aligned} \tag{6.14}$$

Solving for $B^{\pm}$ and z we find that in this sector the spectrum is given by

$$\Lambda = a_-s_- + \lambda(z) \tag{6.15}$$

where z is a complex number satisfying

$$z^{2L} = -\frac{\tilde{K}_+(z)}{\tilde{K}_+(z^{-1})}\frac{B^-}{B^+} = \frac{\tilde{K}_+(z)\tilde{K}_-(z)}{\tilde{K}_+(z^{-1})\tilde{K}_-(z^{-1})} \tag{6.16}$$

and

$$\tilde{K}_+(z) = 1, \qquad \tilde{K}_-(z) = \lambda(z) + a_-(s_- + z(s_-t - 1)). \tag{6.17}$$

6.5. General n

From the previous sections we conclude that both sectors $n = 2$ and 3 contain only one pseudo-particle. There is therefore an additional quantum number, namely the parity of the number of loop lines connected to the left-hand boundary, that distinguishes between these two sectors. As for closed boundaries in section 5.3, we have checked the sectors containing two pseudo-particles (here $n = 4$ and 5). In both these sectors, the eigenvalue equation can be solved by an ansatz similar to the one used for closed boundaries; see (5.16). The Bethe ansatz equations (6.8) for the even sector and (6.16) for the odd sector are now multiplied by the two particle scattering factors $S(z_i, z_j)$, in much the same way as (5.24). Furthermore, we have checked that the Bethe ansatz equations remain valid in the case when all sites are connected, i.e. when $n = L$.

Because the Hamiltonian H_L^{M} is integrable, the Bethe ansatz programme can be consistently carried out for arbitrary values of n, and the higher sector equations factorize into those for two particles. Integrability assures us that we can find consistent expressions for the more complicated 'merged pseudo-particles' (for the spectrum we do not need to know these explicitly).

Before giving the results for general n we first recall the definitions of S and λ,

$$S(z, w) = 1 + tw + zw, \qquad \lambda(z) = t + z + z^{-1}. \tag{6.18}$$

6.5.1. Even n. Having derived the equations for $n = 2$ and 4, we postulate the generalization of (5.24) for arbitrary but even values of n to the case of mixed boundary conditions,

$$z_i^{2L} = \frac{K_+(z)K_-(z)}{K_+(z^{-1})K_-(z^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{n/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}, \tag{6.19}$$

and

$$K_+(z) = 1, \qquad K_-(z) = \lambda(z) - a_-(s_- + z). \tag{6.20}$$

In terms of the solutions of (6.19), the eigenvalues of H_L^{M} are given by

$$\Lambda = \sum_{j=1}^{n/2} \lambda(z_j). \tag{6.21}$$

Using the parametrization of section 2.3 and the transformation (5.30), these equations are equivalent to the first line of (4.2) together with (4.3) in the limit $\beta_+, \alpha_+ \to \infty$.

6.5.2. Odd n. For n an odd number we postulate the solution from the investigations of $n = 1$, 3 and 5. The eigenvalue for odd n is given by

$$\Lambda = a_- s_- + \sum_{j=1}^{(n-1)/2} \lambda(z_j), \tag{6.22}$$

and the numbers z_i satisfy

$$z_i^{2L} = \frac{\tilde{K}_+(z)\tilde{K}_-(z)}{\tilde{K}_+(z^{-1})\tilde{K}_-(z^{-1})} \prod_{\substack{j=1 \\ j\neq i}}^{n/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}, \tag{6.23}$$

where

$$\tilde{K}_+(z) = 1, \qquad \tilde{K}_-(z) = \lambda(z) + a_-(s_- + z(s_-t - 1)). \tag{6.24}$$

Using the parametrization of section 2.3 and the transformation (5.30), these equations are equivalent to the first line of (4.5) together with (4.6) in the limit $\beta_+, \alpha_+ \to \infty$.

7. Open boundaries

To diagonalize the Hamiltonian H_L^{O} on the space $V_L^{\mathrm{O}*}$ a similar programme can be carried out as for H_L^{C} and H_L^{M}. When the number of connected loops is less than the system size, i.e. $n < L$, there is no fundamental difference from the two previous cases. The main difference from the case of mixed boundaries is that now $K_+(z)$ and $\tilde{K}_+(z)$ become nontrivial functions. However, when $n = L$ there arises a difficulty which we will only partially resolve in this paper.

In the following we will first consider the cases when $n < L$ and we will turn our attention to $n = L$ later. We will denote by $\psi_{2k+l+m}(x_1, \ldots, x_k; y_1, \ldots, y_l; \tilde{y}_1, \ldots, \tilde{y}_m)$ a coefficient in the eigenvector of a state with k well separated pseudo-particles at positions x_i $(i = 1, \ldots, k)$, l sites at positions y_i $(i = 1, \ldots, l)$ connected to the left-hand boundary, and m sites at positions $\tilde{y}_i$ $(i = 1, \ldots, m)$ connected to the right-hand boundary.

7.1. $n = 0$

As before, we start with the state with all sites disconnected, i.e. $\langle| = |\cdots|$, and find that it is a left eigenstate of H_L^{O} with eigenvalue $\Lambda = 0$.

7.2. $n = 1$

For $n = 1$ we consider eigenvectors in which the coefficient $\psi_1(;;)$ of the pseudo-vacuum is nonzero as well as $\psi_1(;1;)$ of the state $)|\cdots|$ with one site connected to the left-hand boundary and $\psi_1(;;L)$ of the state $|\cdots|($ with one site connected to the right-hand boundary. The eigenvalue equations are

$$\begin{aligned}
&\Lambda\psi_1(;;) = a_-\psi_1(;1;) + a_+\psi_1(;;L), \\
&\Lambda\psi_1(;1;) = a_-s_-\psi_1(;1;), \\
&\Lambda\psi_1(;;L) = a_+s_+\psi_1(;;L).
\end{aligned} \tag{7.1}$$

As before we regard the first line in (7.1) as the definition of $\psi_1(;;)$. In the generic case of $a_-s_- \neq a_+s_+$, the remaining equations are solved by either

$$\Lambda = a_-s_-, \qquad \psi_1(;;L) = 0, \tag{7.2}$$

or

$$\Lambda = a_+s_+, \qquad \psi_1(1;;) = 0. \tag{7.3}$$

7.3. $n = 2$

In this sector all states with two connected sites, i.e. states of the form $|\cdots|()|\cdots|$, $))|\cdots|$ and $|\cdots|(($, may have nonzero coefficients in the eigenvector. These coefficients are denoted by $\psi_2(x;;)$, $\psi_2(;1,2;)$ and $\psi_2(;;L-1,L)$ respectively. The resulting eigenvalue equations read

$$\begin{aligned}
&\Lambda\psi_2(;;) = a_-\psi_2(;1;) + a_+\psi_2(;;L) + \sum_{y=1}^{L-1}\psi_2(y;;),\\
&\Lambda\psi_2(;1;) = a_-s_-\psi_2(;1;) + a_+\psi_2(;1;L) + \psi_2(1;;),\\
&\Lambda\psi_2(;;L) = a_+s_+\psi_2(;;L) + a_-\psi_2(;1;L) + \psi_2(L-1;;),\\
&\Lambda\psi_2(x;;) = t\psi_2(x;;) + \psi_2(x+1;;) + \psi_2(x-1;;), \qquad \text{for } 1 < x < L-1.
\end{aligned} \tag{7.4}$$

In addition, we also find the boundary relations,

$$\begin{aligned}
&\Lambda\psi_2(1;;) = t\psi_2(1;;) + \psi_2(2;;) + a_-\psi_2(;1,2;),\\
&\Lambda\psi_2(L-1;;) = t\psi_2(L-1;;) + \psi_2(L-2;;) + a_+\psi_2(;;L-1,L),\\
&\Lambda\psi_2(;1,2;) = a_-s_-\psi_2(;1,2;) + \psi_2(1;;),\\
&\Lambda\psi_2(;;L-1,L) = a_+s_+\psi_2(;;L-1,L) + \psi_2(L-1;;),\\
&\Lambda\psi_2(;1;L) = (a_-s_- + a_+s_+)\psi_2(;1;L).
\end{aligned} \tag{7.5}$$

As before, we take the first three lines in (7.4) as definitions for $\psi_2(;;)$, $\psi_2(;1;)$ and $\psi_2(;;L)$, and solve the remaining equations by making the ansatz

$$\psi_2(x;;) = f(x), \tag{7.6}$$

where

$$f(x) = A^+z^x + A^-z^{-x}. \tag{7.7}$$

We then find from the last line in (7.4) and the first two lines in (7.5) that

$$\begin{aligned}
&\Lambda = t + z + z^{-1},\\
&\psi_2(;1,2;) = a_-^{-1}f(0),\\
&\psi_2(;;L-1,L) = a_+^{-1}f(L).
\end{aligned} \tag{7.8}$$

The third and fourth boundary equations in (7.5) imply the consistency equations,

$$\begin{aligned}
&(\lambda(z) - a_-s_-)f(0) = a_-f(1),\\
&(\lambda(z) - a_+s_+)f(L) = a_+f(L-1).
\end{aligned} \tag{7.9}$$

In the case of mixed boundaries we saw that we needed to distinguish the case with an odd number from that with an even number of loops connected to the left-hand boundary. The same holds true here, as can be seen by the last equation of (7.5), which implies that $\psi_2(;1;L) = 0$ unless $\Lambda = a_-s_- + a_+s_+$, which will generically not be the case with Λ given by (7.8). The complete spectrum in this sector is therefore given by

$$\Lambda = a_-s_- + a_+s_+, \tag{7.10}$$

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in which case $A^- = A^+ = 0$ and $\psi_2(;1;L) \neq 0$, or

$$\Lambda = \lambda(z) = t + z + z^{-1}, \tag{7.11}$$

in which case $\psi_2(;1;L) = 0$. The complex number z in (7.11) satisfies the Bethe ansatz equation

$$z^{2L} = -\frac{K_+(z)}{K_+(z^{-1})}\frac{A^-}{A^+} = \frac{K_+(z)K_-(z)}{K_+(z^{-1})K_-(z^{-1})}, \tag{7.12}$$

where

$$K_\pm(z) = \lambda(z) - a_\pm(s_\pm + z). \tag{7.13}$$

7.4. $n = 3$

In analogy with the case of mixed boundaries, we no longer write explicitly the equations relating eigenvector elements of sectors smaller than n to those of n, but only the equations relating the latter to each other. The eigenvalue equation then reads

$$\begin{aligned} \Lambda\psi_3(x;1;) &= (a_-s_- + t)\psi_3(x;1;) + \psi_3(x+1;1;) + \psi_3(x-1;1;), \\ &\quad \text{for } 2 < x < L-1, \\ \Lambda\psi_3(x;;L) &= (a_+s_+ + t)\psi_3(x;;L) + \psi_3(x+1;;L) + \psi_3(x-1;;L), \\ &\quad \text{for } 1 < x < L-2 \end{aligned} \tag{7.14}$$

with the additional relations when an odd number of loops is connected to the left-hand boundary

$$\begin{aligned} \Lambda\psi_3(2;1;) &= (a_-s_- + t)\psi_3(2;1;) + \psi_3(3;1;) + \psi_3(1;3;), \\ \Lambda\psi_3(1;3;) &= t\psi_3(1;3;) + \psi_3(2;1;) + a_-\psi_3(;1,2,3;), \\ \Lambda\psi_3(;1,2,3;) &= a_-s_-\psi_3(;1,2,3;) + \psi_3(1;3;) + s_-\psi_3(2;1;), \\ \Lambda\psi_3(L-1;1;) &= (a_-s_- + t)\psi_3(L-1;1;) + \psi_3(L-2;1;) + a_+\psi_3(;1;L-1,L), \\ \Lambda\psi_3(;1;L-1,L) &= (a_-s_- + a_+s_+)\psi_3(;1;L-1,L) + \psi_3(L-1;1;) \end{aligned} \tag{7.15}$$

and similar equations when an even number of loops is connected to the left-hand boundary. These equations can be satisfied by making the ansatz

$$\begin{aligned} \psi_3(x;1;) &= f(x), \qquad f(x) = C^+z^x + C^-z^{-x}, \\ \psi_3(x;;L) &= g(x), \qquad g(x) = D^+z^x + D^-z^{-x}. \end{aligned} \tag{7.16}$$

Since the equations do not mix states with an even and odd numbers of loops connected to the left-hand boundary, we find they can be satisfied if either $D^\pm = 0$ or $C^\pm = 0$. In the first case we have

$$\begin{aligned} \Lambda &= a_-s_- + t + z + z^{-1}, \\ \psi_3(1;3;) &= f(1), \\ \psi_3(;1,2,3;) &= a_-^{-1}\left(f(0) + a_-s_-f(1)\right), \\ \psi_3(;1;L-1,L) &= a_+^{-1}f(L), \end{aligned} \tag{7.17}$$

with the consistency equations

$$\begin{aligned} &(\lambda(z) + a_- s_-) f(0) = a_-(1 - s_- t) f(1), \\ &(\lambda(z) - a_+ s_+) f(L) = a_+ f(L-1). \end{aligned} \tag{7.18}$$

Solving for $C^\pm$ and z we find that in this sector one part of the spectrum is given by the first line of (7.17) where z is a complex number satisfying

$$z^{2L} = -\frac{K_+(z)}{K_+(z^{-1})} \frac{C^-}{C^+} = \frac{K_+(z) \tilde{K}_-(z)}{K_+(z^{-1}) \tilde{K}_-(z^{-1})} \tag{7.19}$$

and

$$\begin{aligned} &K_+(z) = \lambda(z) - a_+(s_+ + z), \\ &\tilde{K}_-(z) = \lambda(z) + a_-(s_- + z(s_- t - 1)). \end{aligned} \tag{7.20}$$

Solving for $D^\pm$ and z in the case where $C^\pm = 0$, we find that the other part of the spectrum is given by

$$\Lambda = a_+ s_+ + \lambda(z) \tag{7.21}$$

where z is a complex number now satisfying

$$z^{2L} = -\frac{\tilde{K}_+(z)}{\tilde{K}_+(z^{-1})} \frac{D^-}{D^+} = \frac{\tilde{K}_+(z) K_-(z)}{\tilde{K}_+(z^{-1}) K_-(z^{-1})} \tag{7.22}$$

with

$$\begin{aligned} &\tilde{K}_+(z) = \lambda(z) + a_+(s_+ + z(s_+ t - 1)), \\ &K_-(z) = \lambda(z) - a_-(s_- + z). \end{aligned} \tag{7.23}$$

7.5. General n

Sectors higher than $n = 3$ introduce multi-particle scattering. The spectrum in these sectors can be calculated using a similar ansatz as for mixed boundaries. While the equations become more cumbersome, the final answer is the obvious generalization of the previous sectors if we keep the results for mixed boundaries in mind. The results below are valid for $n < L$.

Before giving the final result, we first recall the definitions

$$\begin{aligned} &\lambda(z) = t + z + z^{-1}, \\ &S(z, w) = 1 + tw + zw, \\ &K_\pm(z) = \lambda(z) - a_\pm(s_\pm + z), \\ &\tilde{K}_\pm(z) = \lambda(z) + a_\pm(s_\pm + z(s_\pm t - 1)). \end{aligned} \tag{7.24}$$

7.5.1. Even n. The spectrum in these sectors is described by the following two cases.

- Odd number of loops to both boundaries.

$$\Lambda = a_- s_- + a_+ s_+ + \sum_{j=1}^{n/2-1} \lambda(z_j), \tag{7.25}$$

where the numbers z_i satisfy

$$z_i^{2L} = \frac{\tilde{K}_+(z)\tilde{K}_-(z)}{\tilde{K}_+(z^{-1})\tilde{K}_-(z^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{n/2-1} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}. \tag{7.26}$$

- Even number of loops to both boundaries.

$$\Lambda = \sum_{j=1}^{n/2} \lambda(z_j), \tag{7.27}$$

where the numbers z_i satisfy

$$z_i^{2L} = \frac{K_+(z)K_-(z)}{K_+(z^{-1})K_-(z^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{n/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}. \tag{7.28}$$

Using the parametrization of section 2.3 and transformation (5.30), these equations are equivalent to (4.2)–(4.4).

7.5.2. Odd n. For these sectors the spectrum also falls into two classes.

- Odd number of loops connected to the left- and even number of loops to the right-hand boundary.

$$\Lambda = a_- s_- + \sum_{j=1}^{(n-1)/2} \lambda(z_j), \tag{7.29}$$

where the numbers z_i satisfy

$$z_i^{2L} = \frac{K_+(z)\tilde{K}_-(z)}{K_+(z^{-1})\tilde{K}_-(z^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{(n-1)/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}. \tag{7.30}$$

- Even number of loops connected to the left- and odd to the right-hand boundary.

$$\Lambda = a_+ s_+ + \sum_{j=1}^{(n-1)/2} \lambda(z_j), \tag{7.31}$$

where the numbers z_i satisfy

$$z_i^{2L} = \frac{\tilde{K}_+(z)K_-(z)}{\tilde{K}_+(z^{-1})K_-(z^{-1})} \prod_{\substack{j=1\\ j\neq i}}^{(n-1)/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}. \tag{7.32}$$

Using the parametrization of section 2.3 and transformation (5.30), these equations are equivalent to (4.5)–(4.7).

7.6. $n = L = 1$

We now consider the cases where $n = L$. These sectors are quite different from the previous cases since the nonlocal additional relations (2.3) involving the parameter b come into play; see also sections 2.2 and 3. In this sense they are similar to the $n = L$ sectors in the periodic TL algebra; see e.g. [20].

Recall that for $n = L$ we only consider the subspace generated by $I_L|\rangle$; see section 2.2. The eigenvalue equations in the $n = L = 1$ sector are

$$\begin{aligned} \Lambda\psi_1(;1;) &= a_-s_-\psi_1(;1;) + a_+\psi_1(;;1), \\ \Lambda\psi_1(;;1) &= a_+s_+\psi_1(;;1) + a_-b\psi_1(;1;). \end{aligned} \tag{7.33}$$

These equations are easily solved, and the spectrum is given by the solutions of the characteristic equation

$$(\Lambda - a_-s_-)(\Lambda - a_+s_+) = a_-a_+b. \tag{7.34}$$

Hence we see that if $b = 0$ we have the same solutions (7.2) and (7.3) as in the case $n = 1 < L$. We also note that $b = s_-s_+$ is another special value where the solutions do not contain a radical (see also the remarks at the end of section 4).

7.7. $n = L = 2$

In this sector, the resulting eigenvalue equations are

$$\begin{aligned} \Lambda\psi_2(1;;) &= t\psi_2(1;;) + a_-\psi_2(;1,2;) + a_+\psi_2(;;1,2), \\ \Lambda\psi_2(;1,2;) &= a_-s_-\psi_2(;1,2;) + \psi_2(1;;) + a_+\psi_2(;1;2), \\ \Lambda\psi_2(;;1,2) &= a_+s_+\psi_2(;;1,2) + \psi_2(1;;) + a_-\psi_2(;1;2), \\ \Lambda\psi_2(;1;2) &= (a_-s_- + a_+s_+)\psi_2(;1;2) + b\psi_2(1;;). \end{aligned} \tag{7.35}$$

For $b = 0$, these equations can be solved in the same way as for $n = 2 < L$ by making the ansatz

$$\psi_2(x;;) = f(x), \tag{7.36}$$

where

$$f(x) = A^+z^x + A^-z^{-x}. \tag{7.37}$$

We then find that the complete spectrum in this sector is given by

$$\Lambda = a_-s_- + a_+s_+, \tag{7.38}$$

in which case one can take $\psi_2(;1;2) \neq 0$, or

$$\Lambda = \lambda(z) = t + z + z^{-1} \neq a_-s_- + a_+s_+, \tag{7.39}$$

in which case $\psi_2(;1;2) = 0$. The complex number z in (7.39) satisfies the Bethe ansatz equation

$$z^{2L} = -\frac{K_+(z)}{K_+(z^{-1})}\frac{A^-}{A^+} = \frac{K_+(z)K_-(z)}{K_+(z^{-1})K_-(z^{-1})}, \tag{7.40}$$

where

$$K_\pm(z) = \lambda(z) - a_\pm(s_\pm + z). \tag{7.41}$$

The eigenvalue is a zero of a quartic polynomial. For $b = 0$ this polynomial factorizes into a cubic and a linear polynomial. The zero of the linear polynomial gives the solution (7.38). If $b \neq 0$, the eigenvalue is a solution of a quartic equation that does not factorize in general. We have not found a systematic way of solving the case $b \neq 0$ and $n = L$.

7.8. $n = L = 3$

In this sector there are eight equations resulting from the eigenvalue equation,

$$\begin{aligned}
\Lambda\psi_3(2;1;) &= (a_-s_- + t)\psi_3(2;1;) + \psi_3(1;3;) + a_+\psi_3(;1;2,3),\\
\Lambda\psi_3(1;3;) &= t\psi_3(1;3;) + \psi_3(2;1;) + a_-\psi_3(;1,2,3;) + a_+\psi_3(1;;3),\\
\Lambda\psi_3(;1;2,3) &= (a_-s_- + a_+s_+)\psi_3(;1;2,3) + \psi_3(2;1;) + \psi_3(1;;3),\\
\Lambda\psi_3(;1,2,3;) &= a_-s_-\psi_3(;1,2,3;) + \psi_3(1;3;) + s_-\psi_3(2;1;) + a_+\psi_3(;1,2;3),
\end{aligned} \tag{7.42}$$

and

$$\begin{aligned}
\Lambda\psi_3(1;;3) &= (a_+s_+ + t)\psi_3(1;;3) + \psi_3(2;;1) + a_-\psi_3(;1,2;3)\\
\Lambda\psi_3(2;;1) &= t\psi_3(2;;1) + \psi_3(1;;3) + a_-b\psi_3(2;1;) + a_+\psi_3(;;1,2,3),\\
\Lambda\psi_3(;1,2;3) &= (a_-s_- + a_+s_+)\psi_3(;1,2;3) + \psi_3(1;;3) + b\psi_3(2;1;),\\
\Lambda\psi_3(;;1,2,3) &= a_+s_+\psi_3(;;1,2,3) + \psi_3(2;;1) + s_+\psi_3(1;;3) + a_-b\psi_3(;1;2,3).
\end{aligned} \tag{7.43}$$

We will try to solve them in a similar way as for $n = 3 < L$. First write

$$\begin{aligned}
\psi_3(2;1;) &= f(2), \qquad \psi_3(1;3;) = f(1), \qquad f(x) = C^+z^x + C^-z^{-x},\\
\psi_3(1;;3) &= g(1), \qquad \psi_3(2;;1) = g(2), \qquad g(x) = D^+z^x + D^-z^{-x}.
\end{aligned} \tag{7.44}$$

If $b = 0$, the equations (7.43) only contain coefficients of states with an even number of loops connected to the left-hand boundary. Hence (7.43) will be trivially satisfied if we set these coefficients all to zero, i.e. $D^+ = D^- = 0$. Equations (7.42) can then be solved in a similar way as for the case $n = 3 < L$, and we find

$$\Lambda = a_-s_- + \lambda(z), \tag{7.45}$$

where z is a complex number satisfying

$$z^{2L} = -\frac{K_+(z)}{K_+(z^{-1})}\frac{C^-}{C^+} = \frac{K_+(z)\tilde{K}_-(z)}{K_+(z^{-1})\tilde{K}_-(z^{-1})}, \tag{7.46}$$

and

$$K_+(z) = \lambda(z) - a_+(s_+ + z), \qquad \tilde{K}_-(z) = \lambda(z) + a_-(s_- + z(s_-t - 1)). \tag{7.47}$$

Another solution for $b = 0$ is obtained in the following way. We find a non-trivial solution of the eigenvalue problem (7.43), which exists provided one satisfies

$$\Lambda = a_+s_+ + \lambda(z), \tag{7.48}$$

where z is a complex number satisfying

$$z^{2L} = -\frac{\tilde{K}_+(z)}{\tilde{K}_+(z^{-1})}\frac{D^-}{D^+} = \frac{\tilde{K}_+(z)K_-(z)}{\tilde{K}_+(z^{-1})K_-(z^{-1})}, \tag{7.49}$$

and

$$\begin{aligned} \tilde{K}_+(z) &= \lambda(z) + a_+(s_+ + z(s_+t - 1)),\\ K_-(z) &= \lambda(z) - a_-(s_- + z). \end{aligned} \tag{7.50}$$

Given such a solution, the four equations (7.42) then contain four unknowns, the four coefficients with an even number of loops connected to the left-hand boundary. Apart from accidental degeneracies, (7.42) thus has a solution, that we do not need to know explicitly here. Hence for $b = 0$, the solution for $n = L = 3$ is the same as that for $n = 3 < L$.

7.9. General $n = L$

We hope to have convinced the reader that for $b = 0$ the Bethe ansatz solution for the spectrum for the case $n < L$ in section 7.5 also remains valid for $n = L$.

8. Conclusion

We have derived equations for the spectrum of the Temperley–Lieb loop model with closed, mixed and open boundaries using a coordinate Bethe ansatz. The spectra of the quantum XXZ spin chain with diagonal, one non-diagonal and two non-diagonal open boundaries respectively are contained in those of the loop model. In the case of two non-diagonal boundaries, we thus derive a recent numerically conjectured result, and obtain the spectrum of the spin chain in a part of the parameter space where it was previously not known.

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Appendix A. Dimension of representation spaces

We use a functional equation method [21] to find the generating functions of the various dimensions of the representation spaces. Let us first consider all sequences of well nested parentheses that have the same number of opening and closing parentheses. Such a sequence may be the empty sequence, or it has the form •(•) where • represents any well nested sequence of parentheses, including the empty one. The generating function for such sequences hence satisfies the functional equation

$$D(z) = 1 + z^2 D(z)^2, \tag{A.1}$$

where the 1 corresponds to the empty sequence, z^2 to the opening and closing parenthesis and $D(z)^2$ to the two well nested sequences. Equation (A.1) is easily solved with the initial condition $D(0) = 1$. The solution is the Catalan generating function

$$D(z) = \frac{2}{1+\sqrt{1-4z^2}}. \tag{A.2}$$

Let us now consider all sequences of well nested parenthesis with an arbitrary number of vertical bars '|' in between them. Each such sequence with n vertical bars is of the form

$$\bullet | \bullet | \cdots \bullet | \bullet .$$

The generating function for these sequences is therefore given by

$$G_{\mathrm{C}}(z) = D(z) + D(z)zD(z) + D(z)zD(z)zD(z) = \cdots \tag{A.3}$$

$$= D(z)\sum_{n=0}^{\infty}(zD(z))^n = \frac{D(z)}{1-zD(z)}. \tag{A.4}$$

The sequences corresponding to states in V_L^{M} are of the form

$$\bullet)\bullet)\cdots\bullet)\bullet|\bullet|\cdots\bullet|\bullet,$$

and hence the generating function is given by

$$G_{\mathrm{M}}(z) = D(z)\left(\sum_{n=0}^{\infty}(zD(z))^n\right)^2 = \frac{D(z)}{(1-zD(z))^2} = \frac{1}{1-2z}. \tag{A.5}$$

Finally, the sequences corresponding to states in V_L^{O} are of the form

$$\bullet)\bullet)\cdots\bullet)\bullet|\bullet|\cdots\bullet|\bullet(\bullet\cdots(\bullet(\bullet,$$

and hence the generating function is given by

$$G_{\mathrm{O}}(z) = D(z)\left(\sum_{n=0}^{\infty}(zD(z))^n\right)^3 = \frac{D(z)}{(1-zD(z))^3}. \tag{A.6}$$

The generating functions for the subspaces generated by $I_L|\rangle$ and $J_L|\rangle$, see section 2.2, are equal to $G_{\mathrm{M}}(z)$, which follows from replacing '|' by '(' in the sequences above (A.5).

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