J. Phys. A: Math. Gen. **35** (2002) L661–L668 PII: S0305-4470(02)53396-5

**LETTER TO THE EDITOR**

# Temperley–Lieb stochastic processes

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Received 11 September 2002
Published 29 October 2002
Online at stacks.iop.org/JPhysA/35/L661

**Abstract**
We discuss one-dimensional stochastic processes defined through the Temperley–Lieb algebra related to the $Q = 1$ Potts model. For various boundary conditions, we formulate a conjecture relating the probability distribution which describes the stationary state, to the enumeration of a symmetry class of alternating sign matrices, objects that have received much attention in combinatorics.

PACS numbers: 02.50.Ey, 11.25.Hf, 05.50.+q, 75.10.Hk

## 1. Introduction

In recent papers some intriguing connections have been found between the ground-state wavefunctions of the XXZ quantum spin chain at $\Delta = -1/2$, the dense $O(n = 1)$ or Temperley–Lieb loop model on the square lattice and alternating sign matrices (ASMs) [1–5]. In particular, different boundary conditions in the spin chain and the loop model correspond to different symmetry classes of ASMs. It is well known that the lattice version of the quantum spin chain, the six-vertex model and the loop model are closely related [6]. The underlying structure accounting for this equivalence is the Temperley–Lieb (TL) algebra. In this letter we use the semigroup structure of this algebra to show that the loop model has an interpretation as a stochastic process. The ground-state wavefunction therefore gives the stationary probability distribution. While we are primarily concerned here with algebraic properties, the physical interpretation of this stochastic process is that of a fluctuating interface and is presented in [12].

To unify the algebraic formulation we introduce quotients of the TL algebra on a ring and on the line. We also propose new conjectures relating the stationary state with ASMs, or more precisely, their interpretation as fully packed loop (FPL) configurations [13]. While the FPL model is quite different from the $O(n)$ loop model, we will see that the loop connectivities of both models play a crucial role in these conjectures. To complete the picture, we give the finite size scaling spectra of the loop model with closed boundaries. These spectra can be expressed in terms of generic characters of a $c = 0$ logarithmic conformal field theory.

## 2. Temperley–Lieb stochastic processes

Given an arbitrary semigroup $G$, an abstract stochastic process can be defined as follows. Let $\{w_a\}$ be the words of $G$ and consider

$$H = \sum_a c_a(1 - w_a) \qquad c_a \geqslant 0. \tag{2.1}$$

In the regular representation, i.e. on the basis of all independent words in $G$, $H$ is a matrix satisfying $H_{ab} \leqslant 0$ for $a \neq b$ and $\sum_b H_{ab} = 0$. Such a matrix is called an intensity matrix and defines a stochastic process in continuum time given by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P_a(t) = -\sum_b H_{ab} P_b(t) \tag{2.2}$$

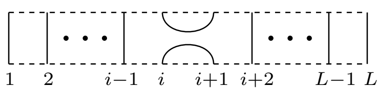
where $P_a(t)$ is the (unnormalized) probability of finding the system in the state $|a\rangle$ at time $t$, and the rate for the transition $|b\rangle \mapsto |a\rangle$ is given by $-H_{ab}$, which is non-negative. In a similar way, a stochastic process can be defined on any ideal of $G$. Since $H$ is an intensity matrix, it has at least one zero eigenvalue and its corresponding right eigenvector $|0\rangle$ gives the probabilities in the stationary state

$$\begin{aligned} &\langle 0|H = 0 \qquad 0 = (1, 1, \ldots, 1) \\ &H|0\rangle = 0 \qquad |0\rangle = \sum_a P_a|a\rangle \qquad P_a = \lim_{t\to\infty} P_a(t). \end{aligned} \tag{2.3}$$

In the rest of the letter, we concentrate on a particular semigroup of which there is a natural interpretation of such a stochastic process [12] and which is solvable. We consider the particular case in which the words $w_a$ are expressed in terms of the generators $e_i$ of the TL algebra $T$ [7],

$$e_i^2 = (q + q^{-1})e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad [e_i, e_j] = 0 \qquad \text{for} \quad |i - j| > 1 \tag{2.4}$$

with $1 \leqslant i \leqslant L - 1$. Restricting ourselves to the case $q = \mathrm{e}^{\mathrm{i}\pi/3}$, we find that the words of the TL algebra form a semigroup. The generators $e_i$ have the following graphical representation as monoids:

$e_i =$ 

. (2.5)

The action of a generator on a word of the algebra is obtained by placing the graph of the generator (2.5) under the graphical representation of the word and erasing the intermediate dashed line. In the combined graph, the loop segments either form closed loops or pairwise connect sites on the upper and lower part of a strip. The TL algebra thus can be represented by loop diagrams. Due to relations (2.4), closed contractible loops may be removed at the cost of a factor $q + q^{-1} = 1$.

In the following we will consider the Hamiltonian $H$ defined by

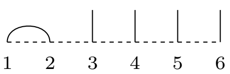
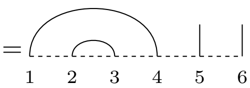
$$H = \sum_{j=1}^{L-1} (1 - e_j). \tag{2.6}$$

This Hamiltonian is closely related to the critical $Q = 1$ Potts model (dense $O(n = 1)$ or Temperley–Lieb loop model) [6, 10]. Because $H$ is of the form (2.1), it is an intensity matrix. Besides being an intensity matrix, $H$ has a rich Jordan cell structure. This can be explained using the graphical representation (2.5) from which it is seen that the terms in the Hamiltonian may connect disconnected lines, but it is not possible to have the reverse process (see [10] for the appearance of Jordan cell structures in the representations of TL algebras). Depending on the representation, the stationary state $|0\rangle$ may not be unique and because of the Jordan cell structure we lack good quantum numbers to label sectors of $H$. In this letter, we will use the Temperley–Lieb loop (TLL) representation as well as appropriate left ideals of the regular representation to define sectors of $H$ that have the same unique stationary state.

The TLL representation is obtained by the action of the generators in the vector space spanned by the distinct right ideals $w_a T$, $w_a \in T$. Because of the semigroup property, the generators $e_i$ map right ideals onto right ideals

$$e_i(w_a T) = w_b T \qquad \text{for some word} \quad w_b \in T. \tag{2.7}$$

Graphically, the right ideals are represented by link diagrams obtained from monoid diagrams by ignoring the upper parts, for $L = 6$ we, for example, have

$e_1 T =$  $\qquad e_2 e_1 e_3 T =$  . (2.8)

The number of such link diagrams with $m$ defects (unpaired links) is

$$C_{L,m} = \binom{L}{\lfloor (L-m+1)/2 \rfloor} - \binom{L}{\lfloor (L-m-1)/2 \rfloor} \tag{2.9}$$

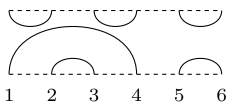
and the dimension of the vector space of right ideals is given by

$$\sum_{s=0}^{\lfloor L/2 \rfloor} C_{L,2s+(L \bmod 2)} = \binom{L}{\lfloor L/2 \rfloor}. \tag{2.10}$$

The construction of using right ideals gives a minimal faithful representation of $T$. In the regular representation of $T$, one has to filter the algebra by fixing appropriate quotients of left ideals [11] and consider the action of $H$ in each of them. In the 0 or 1 defect sector, for example, one may consider the left ideal $T I_0$, generated by the action of $T$ on

$$I_0 = \prod_{i=1}^{\lfloor L/2 \rfloor} e_{2i-1}. \tag{2.11}$$

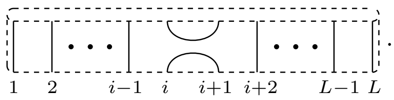
Note that $I_0 T I_0 = I_0$ which immediately implies that $I_0 H I_0 = 0$. In terms of monoid diagrams, the elements of the left ideal have elementary half-loops in the upper half of the diagram and general, non-intersecting half-loops in the lower half of the diagram. An example of a word belonging to the left ideal for $L = 6$ is

$e_2 I_0 =$  . (2.12)

For odd $L$ there will be a defect, i.e. a loop segment joining site $L$ in the upper part to one of the odd sites in the lower part of the diagram. The upper half of the diagram does not change under the action described below (2.5), so it can be ignored, as in the description in terms of right ideals. The dimension of $T I_0$ is given by $C_{L,L \bmod 2} = C_{\lfloor (L+1)/2 \rfloor}$, where $C_n$ is the Catalan number,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = 1, 2, 5, 14, \ldots. \tag{2.13}$$

Levy, Martin and Saleur [8, 9] extended the TL algebra to the cylindrical TL (CTL) algebra by adding a generator $e_L$ and identifying $e_{L+i} = e_i$. In this case the generators $e_i$ have the following graphical representation as monoids:

$e_i =$ 

. (2.14)

The generator $e_L$ connects sites $L$ and 1 via the back of the cylinder. In the standard TL algebra, to which we also refer as closed boundary conditions, the loops are drawn on a strip. In contrast, the CTL algebra is represented by diagrams on a cylinder. In both cases, the $2L$ sites on the top and bottom of the diagram are pairwise connected by lines. A similar analysis as in the case of closed boundaries applies for the Hamiltonian

$$H = \sum_{i=1}^{L} (1 - e_i). \tag{2.15}$$

Consider the case when $L$ is even in the fixed ideal description. The left ideal $TI_0$ is no longer finite dimensional because non-contractible loops can wind around the cylinder [8, 9]. In terms of algebraic relations, $I_0TI_0 = I_0$ is no longer automatically satisfied in the CTL algebra. One way to obtain a non-trivial finite-dimensional quotient is to put an extra relation on the generators that allows for the removal of pairs of non-contractible loops. This can be achieved by taking the following quotient:

$$\text{periodic (DC):} \quad I_0J_0I_0 = I_0 \qquad J_0 = \prod_{i=1}^{\lfloor L/2 \rfloor} e_{2i} \tag{2.16}$$

so that one is left with diagrams having at most one non-contractible loop. The topology of the loop diagrams in this case is such that half-loops connecting $i$ and $j$ via the front and the back of the cylinder are distinct. We therefore call this case periodic boundary conditions with distinct connectivities (DC). The vector space spanned by the words of $TI_0$ for this quotient has dimension $(1 + L/2)C_{L/2}$. One may go a step further taking one more quotient changing the topology. The cylinder can be considered as closed at the top and becomes a disc, so that half-loops connecting $i$ and $j$ via the front and the back are isotopic. We call this case periodic boundaries with identified connectivities (IC). Because we have taken a quotient of a quotient, the spectrum of the periodic IC Hamiltonian is contained in that of the periodic DC one. This quotient is isomorphic to the one induced by the braid translation [8, 9]

$$\text{periodic (IC):} \quad e_L = \left( \prod_{i=1}^{L-1} g_i \right)^{-1} e_1 \left( \prod_{i=1}^{L-1} g_i \right) \qquad g_i^{\pm 1} = 1 - q^{\pm 1} e_i. \tag{2.17}$$

The vector space spanned by the words of $TI_0$ for this quotient is isomorphic to that of $TI_0$ obtained from the standard TL algebra, and its dimension is reduced to $C_{L/2}$.

For odd systems there is a defect present, i.e. a loop segment that connects one site with the top of the cylinder. Since the loops are non-crossing, one cannot have non-contractible loops closing around the cylinder. However, the left ideal $TI_0$ is still infinite dimensional because the defect line may wind around the cylinder. A non-trivial finite-dimensional ideal is obtained by the following quotient:

$$J_0 e_L I_0 = J_0 I_0. \tag{2.18}$$

In this case, there is no distinction between DC and IC and the dimension of the vector space spanned by $TI_0$ is, therefore, $LC_{(L+1)/2}$, i.e. $L$ times that for closed boundary conditions.

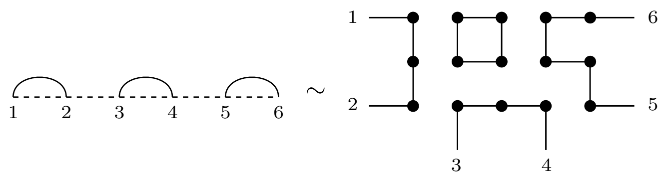
In the spin-1/2 representation of the TL and CTL algebras, Hamiltonians (2.6) and (2.15) become those of the XXZ quantum spin chain at $\Delta = -1/2$. The various quotients give rise to different boundary conditions, as they do for the loop model. In this representation, Hamiltonian (2.6) corresponds to the XXZ chain with diagonal open boundary conditions. For even $L$, Hamiltonian (2.15) in the quotient (2.16) corresponds to the XXZ chain with twisted boundary conditions, while in the quotient (2.17) it corresponds to the XXZ quantum spin chain with non-local boundary conditions [14]. For odd $L$, Hamiltonian (2.15) in the quotient (2.18) corresponds to the XXZ spin chain with periodic boundary conditions.

## 3. Stationary states, combinatorics and fully packed loop conjectures

In this section we consider the connection of the stationary states $|0\rangle = \sum_a P_a|a\rangle$ to combinatorics [1, 2]. In the present context, each ground state assumes a new meaning as the stationary state of a stochastic process. The normalized probability distributions are $p_a = P_a/S(L)$ where $S(L) = \langle 0|0\rangle$.

For periodic IC boundary conditions and $L$ even, Razumov and Stroganov [3] stated a conjecture for the ground state $|0\rangle$ of $H$ in terms of configurations of the FPL model on a suitably defined grid. We state similar conjectures [4] for other boundary conditions, where the entries $P_a$ of $|0\rangle$ are related to various symmetry classes of alternating sign matrices ASMs [15, 16]. While the conjecture for periodic IC boundary conditions was stated first, in this letter it would logically fit in at the end of this section.

For closed boundary conditions with $L$ even, we conjecture that the state $|0\rangle = \sum_a P_a|a\rangle$ is obtained by counting FPL configurations on an $(L-1) \times L/2$ rectangular grid. Loops either connect designated sites on the boundary of the grid or form closed internal loops. For each half-loop configuration $|a\rangle$, $P_a$ is equal to the number of FPL configurations for which the connectivity of the boundary sites is as specified by $|a\rangle$. This same conjecture was independently stated in [5]. We have checked this conjecture out to $L = 10$. For example, for $L = 6$ there are 11 FPL configurations of the type



. (3.1)

There is only one FPL configuration for the state in which site $i$ is connected to site $L-i+1$. It follows [2] that $S(2n) = A_{\mathrm{V}}(2n+1)$ is the number of vertically symmetric $(2n+1) \times (2n+1)$ ASMs

$$A_{\mathrm{V}}(2n+1) = \prod_{j=0}^{n-1}(3j+2)\frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!} = 1, 3, 26, 646, \ldots. \tag{3.2}$$

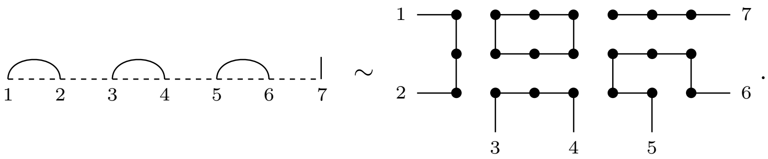
Similarly, the largest entry in $|0\rangle$ corresponding to the state where $2i-1$ is connected to $2i$ is given [2] by the number of cyclically symmetric transposed complement partitions[4]

$$N_8(2n) = \prod_{j=1}^{n-1}(3j+1)\frac{(2j)!(6j)!}{(4j)!(4j+1)!} = 1, 2, 11, 170, \ldots. \tag{3.3}$$

For closed boundary conditions with odd $L$, the left ideal can be written in terms of half-loops and a single defect, a loop segment that is connected to only one site. As for even $L$,

[4] This number happens to be equal to the number $\tilde{A}^{(2)}_{\mathrm{UU}}(4n; 1)$ in the notation of Kuperberg [16].

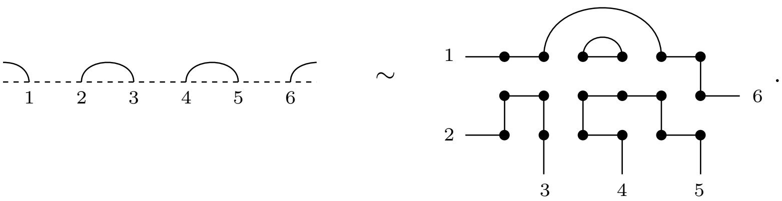
the action of $H$ on the left ideal defines a stochastic process, and one finds that under the action of the monoids the defect may hop. Again the stationary probability distribution has a nice combinatorial expression. We have checked (up to $L = 7$) that the stationary state $|0\rangle = \sum_a P_a|a\rangle$ is given by the number of FPL configurations on an $L \times (L-1)/2$ rectangle, where $P_a$ is equal to the number of FPL configurations for which the connectivity of the boundary sites is as specified by $|a\rangle$. For example, for $L = 7$ there are 26 configurations of the type




(3.4)

The defect line in the FPL configuration above is allowed to end anywhere on the upper boundary. One finds that the normalization factor is $S(2n-1) = N_8(2n)$ [2] and that the largest entry, the weight of the configuration where $2i-1$ is connected to $2i$, is $A_{\mathrm{V}}(2n-1)$. Note that the normalization factor for $L$ sites is equal to the largest entry for $L+1$ sites.

We make a similar FPL conjecture for periodic (DC) boundary conditions which we have checked out to $L = 6$. We conjecture that the state $|0\rangle = \sum_a P_a|a\rangle$ is obtained by counting FPL configurations on an $L \times L/2$ rectangular grid. As before, $P_a$ is equal to the number of FPL configurations for which the connectivity of the boundary sites is as specified by $|a\rangle$ and half-loops connecting sites via the back of the cylinder correspond in the FPL diagrams to connections via up to $L/2$ arcs on the top side of the diagram. For example, for $L = 6$ there are 25 FPL configurations of the type




(3.5)

The total number of such FPL configurations for $L = 2n$ is given by $S(2n) = A_{\mathrm{HT}}(2n)$, where $A_{\mathrm{HT}}(n)$ is the number of $n \times n$ half turn symmetric ASMs,

$$A_{\mathrm{HT}}(2n) = \prod_{k=0}^{n-1} \frac{3k+2}{3k+1} \left( \frac{(3k+1)!}{(n+k)!} \right)^2 = 2, 10, 140, 5544, \ldots \tag{3.6}$$

and the largest entry of $|0\rangle$ is

$$A_{\mathrm{HT}}(2n-1) = \prod_{k=0}^{n-1} \frac{4}{3} \left( \frac{(3k)!(k!)}{(2k)!^2} \right)^2 = 1, 3, 25, 588, \ldots. \tag{3.7}$$

As noted in section 2, for the case of periodic boundary conditions and odd $L$, there is a loop line running along the length of the cylinder so that sites can only be connected in one way, rendering IC and DC periodic boundaries equivalent. For this case, Razumov and Stroganov [5] conjectured that the ground state is obtained by counting FPL configurations corresponding to $(2n+1) \times (2n+1)$ half-turn symmetric ASMs. For $L = 2n+1$, the total number of such FPL configurations is given by $S(2n+1) = A_{\mathrm{HT}}(2n+1)$ and the largest entry of $|0\rangle$ is given by $A(n)^2$, where $A(n)$ is the number of $n \times n$ ASMs,

$$A(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = 1, 2, 7, 42, \ldots. \tag{3.8}$$

Lastly, we mention that the original analogous conjecture for the $n \times n$ grids stated by Razumov and Stroganov [3] applies to the periodic (IC) boundary conditions with $L = 2n$. The total number of these configurations is $A(n)$. It is interesting to note that while there is a duality for closed boundaries between odd and even systems concerning the norm and largest element of the ground state, this is not the case for the periodic boundary conditions considered here.

We see that the stationary distributions are superpositions of equally weighted FPL configurations. Note that the stochastic process is formulated in terms of half-loop patterns and not in terms of the FPL model, and it is a challenge to find the related stochastic process in the space of FPL configurations.

## 4. Conformally invariant spectra and $c = 0$ logarithmic CFT

The spectra of the intensity matrices $H$ are described by a logarithmic conformal field theory (LCFT). As is typical of logarithmic theories, the $c = 0$ CFT admits an infinite number of conformal boundary conditions. At present, these boundary conditions have not been classified and the associated operator content, fusion rules and Verlinde formulae are not well understood [18]. Here we do not consider periodic boundary conditions but just consider the link Hamiltonian $H$ with $2s$ defects constructed by the action on right ideals. In this case, we find that the conformal partition function for $2s$ even (odd) or less defects can be expressed in terms of generic Virasoro characters [18] and is given by

$$Z_s(\tilde{q}) = \sum_{\substack{j=0,1,2,\ldots,s \\ (j=1/2,3/2,5/2,\ldots,s)}} \chi_{2j+1}(\tilde{q}) - \chi_{-2j-1}(\tilde{q}) \tag{4.1}$$

where $\tilde{q}$ is the modular parameter. The Virasoro characters $\chi_{2s+1}(\tilde{q})$ are given by

$$\chi_{2s+1}(\tilde{q}) = \tilde{q}^{\Delta_{2s+1}} \prod_{n=1}^{\infty} (1 - \tilde{q}^n)^{-1} \tag{4.2}$$

with conformal weights

$$\Delta_{2s+1} = \frac{s(2s-1)}{3} = 0, 0, \frac{1}{3}, 1, \ldots \qquad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots. \tag{4.3}$$

In particular, the finite-size corrections [20] to the energy levels $E_n$, $H|n\rangle = E_n|n\rangle$, for large $L$ are of the form

$$\frac{LE_n}{\pi v} = \Delta_{2s+1} + k_n + o(1) \qquad n = 0, 1, 2, \ldots \tag{4.4}$$

where $v = 3\sqrt{3}/2$ [19] is the sound velocity and $k_n = 0, 1, 2, \ldots$ labels descendents.

Expression (4.1) allows for the fact that the defects can annihilate in pairs and is consistent with the observed Jordan cell structure. Recent developments indicate that the spectrum of $H$ could be described by a finite set of characters instead of the infinite set given in (4.2) [18, 21].

## Acknowledgments

This research is supported by the Australian Research Council and by the Foundation ‘Fundamenteel Onderzoek der Materie’ (FOM). VR is an ARC IREX Fellow. We thank Mikhail Flohr and Paul Martin for discussions.

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