

Brauer loops and the commuting variety

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Abstract. We observe that the degree of the commuting variety and other related varieties occur as coefficients in the leading eigenvector of an integrable loop model based on the Brauer algebra.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), loop models and polymers, solvable lattice models

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1. Introduction

In [6] Knutson introduces the upper–upper scheme E which is closely related to the commuting variety, the variety of pairs of commuting matrices. The scheme E is defined as the variety of pairs of $n \times n$ matrices (X, Y) such that $\{(X, Y) : XY \text{ and } YX \text{ upper triangular}\}$. E is a reduced complete intersection with $n!$ components E_π labelled by permutations of n . The degrees of these components provide interesting invariants associated with permutations. The variety corresponding to the long permutation is a degree preserving degeneration of the commuting variety.

In unrelated research, we have encountered the degrees of E_π as elements of the leading eigenvector of an integrable lattice model associated with the Brauer algebra. In this note we report on this unexpected observation, extending previous work relating the combinatorics of alternating sign matrices to a lattice model associated with the Temperley–Lieb algebra.

This paper is organized as follows. In the next section we introduce the statistical mechanical Brauer loop model and formulate some observations of its leading eigenvector in conjecture 1. In section 3 we briefly review a result by Knutson and formulate our main observation in conjecture 3.

2. Brauer algebra and Hamiltonian**2.1. Introduction**

The Temperley–Lieb algebra plays a major role in the study of solvable models in statistical mechanics and integrable quantum chains. In particular the XXZ quantum chain and the bond percolation model are based on a representation of this algebra. These and some related models have attracted renewed interest in the last few years [12, 1, 13, 10] from an unexpected relation to the problem of counting alternating sign matrices [3]. This relation, which so far remains unproven, we try to generalize in this paper.

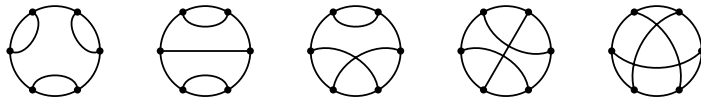


Figure 1. Chord diagrams for $L = 6$.

The existing results concern the ground state of the Hamiltonian given by

$$H = \sum_{i=1}^L (1 - e_i) \tag{1}$$

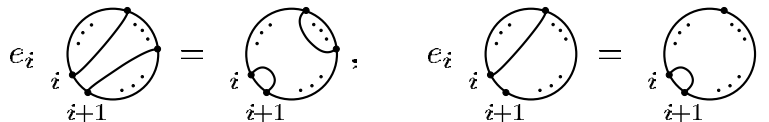
where the e_i are the generators of the Temperley–Lieb algebra satisfying

$$e_i^2 = e_i, \quad e_i e_j e_i = e_i \quad \text{for } |i - j| = 1, \quad \text{and} \quad [e_i, e_j] = 0 \quad \text{for } |i - j| > 1.$$

While rather general boundary conditions can be considered, in this paper we concentrate on periodic boundary conditions, requiring that $e_{L+i} \equiv e_i$.

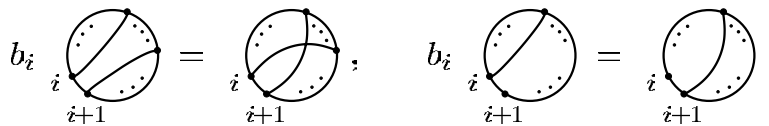
In a convenient finite dimensional representation [8, 11] the e_i act on (a linear combination of) chord diagrams in which L points around a circle are pairwise connected. The five distinct chord diagrams modulo rotations and reflections for $L = 6$ are given in figure 1.

The action of e_i on a chord diagram results in a new chord diagram in which the sites i and $i + 1$ are connected and also the two sites which were previously connected to i and $i + 1$ respectively. For an odd number of points, site i or $i + 1$ could be unpaired. In the case where $i + 1$ is unpaired e_i again connects i with $i + 1$ but unconnects the site paired to i , and vice versa if i is an unpaired site. Pictorially,



If i and $i + 1$ are already connected to each other, e_i simply acts as the identity.

The Brauer algebra [2] is obtained by introducing besides the monoid e_i also the braid b_i which acts on the chord diagram by interchanging i and $i + 1$. In terms of a picture,



If i and $i + 1$ are already paired to each other, b_i leaves the chord diagram unchanged.

The b_i satisfy the usual braid relations

$$b_i b_j b_i = b_j b_i b_j \quad \text{for } |i - j| = 1, \quad [b_i, b_j] = 0 \quad \text{for } |i - j| > 1,$$

and the additional relation

$$b_i^2 = 1,$$

which are just the defining relations of the symmetric group. The braids and monoids satisfy the mixed relations

$$b_i e_i = e_i b_i = e_i, \quad b_i b_j e_i = e_j b_i b_j = e_j e_i \quad \text{for } |i - j| = 1, \\ [e_i, b_j] = 0 \quad \text{for } |i - j| > 1.$$

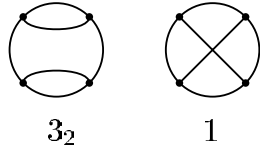


Figure 2. Ground-state wavefunction for $L = 4$.

This algebra admits integrable models which have been studied in various contexts, e.g. in an $\text{Osp}(3|2)$ symmetric representation [9] and to describe the low-temperature phase of $O(n)$ models [5]. We will study the ground state of the following operator which we will call the Brauer loop Hamiltonian:

$$H = \sum_{i=1}^L (3 - 2e_i - b_i).$$

This operator has two properties in common with the one in (1): it is integrable and it has a smallest eigenvalue equal to zero. We believe that it is this combination of properties that lead to numerous special properties of the corresponding eigenvector.

2.2. Ground state

Let us consider the action of the Brauer loop Hamiltonian on the chord diagrams. For example, there are three distinct chord diagrams for $L = 4$, and we find

$$\begin{aligned} H \ominus &= 6 \ominus - 4 \circ - 2 \otimes, \\ H \circ &= 6 \circ - 4 \ominus - 2 \otimes, \\ H \otimes &= 12 \otimes - 6 \ominus - 6 \circ. \end{aligned}$$

From this we infer that

$$\psi_0 = 3 \ominus + 3 \circ + \otimes$$

is annihilated by H ,

$$H\psi_0 = 0.$$

In fact, because the generators of the Brauer algebra $B_n(1)$ also define a semi-group, the Hamiltonian H is an intensity matrix (i.e. $H_{ij} \leq 0$ for $i \neq j$ and $\sum_i H_{ij} = 0$). It therefore always has an eigenvalue zero with an otherwise positive spectrum. In the following we will study the properties of the corresponding eigenvector which we call the ground state.

We will denote states that are related by a rotation or a reflection by the same chord diagram, and will depict the elements of the ground state ψ_0 as in figure 2, where we use a subscript to denote multiplicity. Let us calculate the ground state for $L = 6$, in which case there are five distinct symmetry classes of chord diagrams (see appendix B). The result is given in figure 3. We see that the smallest two elements have the same value as those for $L = 4$, and are related to them by having an additional link that crosses each of the other links once. In particular we note that the smallest element can be chosen to be unity while leaving the other elements integers (instead of rational numbers).

For $L = 8$ there are 17 distinct symmetry classes, and we find the ground state given in figure 4. As before, we note that a chord diagram for $L = 8$ has the same weight as a

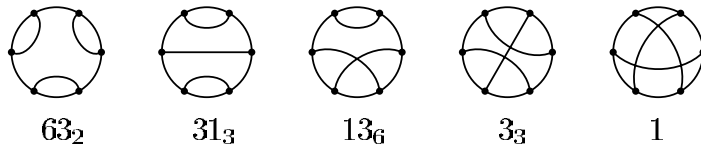


Figure 3. Ground-state wavefunction for $L = 6$.

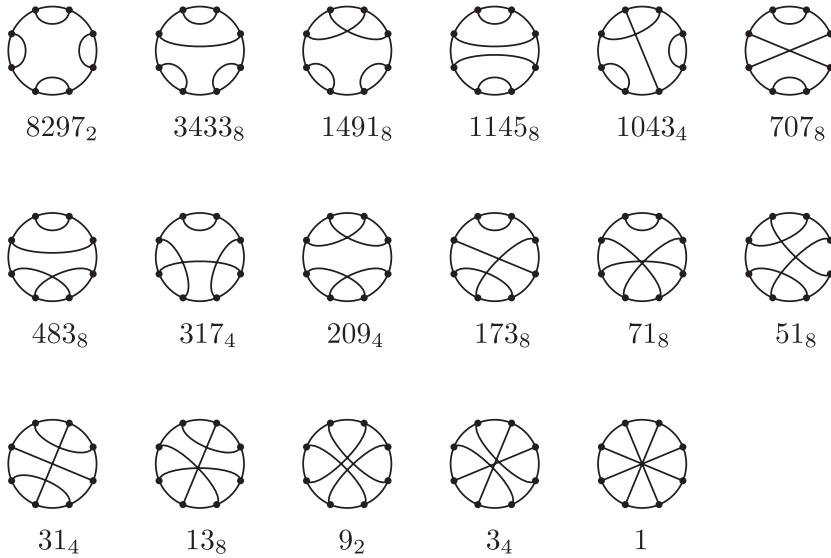


Figure 4. Ground-state wavefunction for $L = 8$.

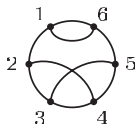


Figure 5. Chord diagram corresponding to $\pi = (312)$.

chord diagram of $L = 6$ if it can be obtained by the addition of a link that crosses each of the other links. Moreover, the weight of a chord diagram for $L = 8$ is three times the weight of a chord diagram for $L = 4$, if the first is obtained from the latter by the addition of two links that cross each of the other links, but not each other. In the next section, after we have introduced some more notation, we will formulate a precise conjecture that includes these observations.

2.3. Chord diagrams related to (partial) permutations

The last four pictures in figure 3 each connect the three sites on the left-hand side to those on the right-hand side. We can therefore interpret these pictures as representing elements of S_3 , the permutation group on three elements. For each permutation $\pi \in S_3$, the corresponding chord diagram is the one where site i is connected to site $3 + \pi(i)$. For example, $\pi = (312)$ corresponds to the picture in figure 5.

For odd systems the situation is slightly more complicated since we cannot divide the system into two halves with equal numbers of sites. However, for $L = 2n + 1$ we

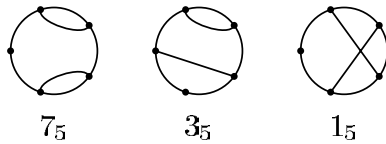


Figure 6. Ground-state wavefunction for $L = 5$.

can define partial permutations of rank n for those chord diagrams where n of the $n + 1$ sites on the left-hand side are connected to the n sites on the right-hand side. Consider for example the ground state for $L = 5$ in figure 6. We define for each diagram two partial permutations π and π' . The partial permutation π describes the connectivity of site i to $3 + \pi(i)$, and π' describes the reverse connectivity, site $3 + i$ to $\pi'(i)$ (if π were a permutation, π' would be the inverse of π). For the diagrams in figure 6 we thus find

$$\begin{aligned} \pi &= (2 \cdot 1) & \pi &= (21 \cdot) & \pi &= (1 \cdot 2) \\ \pi' &= (31) & \pi' &= (21) & \pi' &= (13), \end{aligned}$$

or in terms of matrices with a one at $(i, \pi(i))$ or $(3 + i, \pi'(i))$ respectively, and zeros everywhere else (hence $P_{\pi'} = P_{\pi}^T$),

$$\begin{aligned} P_{\pi} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ P_{\pi'} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Note that each chord diagram for $L = 5$ corresponds to a partial permutation. This is no longer true for larger systems.

We will now formulate a conjecture, based on numerical evidence, about certain elements of the ground state ψ_0 .

Conjecture 1 (i) For each system size, the overall normalization can be chosen such that the smallest element of ψ_0 is equal to unity and all other elements are integers.

(ii) Let π_n and π_m be two permutations (one of them may also be a partial permutation), and let π_{n+m} be the permutation obtained by their concatenation, i.e., $\pi_{n+m}(i) = \pi_n(i)$ for $1 \leq i \leq n$ and $\pi_{n+m}(n + i) = n + \pi_m(i)$ for $1 \leq i \leq m$. Then, with the normalization as in (i), the weight of a chord diagram corresponding to the (partial) permutation π_{n+m} is equal to the product of the weights of the chord diagrams corresponding to π_n and π_m .

(iii) Among the weights of chord diagrams that correspond to a permutation, the one related to the long permutation $w = (n, n - 1, \dots, 2, 1)$ is largest.

The weights of the chord diagrams corresponding to the long permutation form the sequence 1, 3, 31, 1145, 154 881, 77 899 563, 147 226 330 175, 1053 765 855 157 617, ... (indexed as A094579 in Sloane's database [14]). Surprisingly, the first four numbers of this sequence form the first entries of sequence A029729, the degree of the variety of pairs of commuting $n \times n$ matrices. The computation of these degrees is difficult and at the time of this writing the four terms in A029729 are the only ones known [15]. Assuming

the connection, the Brauer loop model provides us with a way of obtaining these degrees for relatively large n . This motivated us to find an interpretation for other entries of the ground state as well, and in the next section we will formulate our main result, interpreting many more entries of ψ_0 as degrees of certain algebraic varieties related to the commuting variety.

3. The scheme \overline{E}_π

The commuting variety is the variety of pairs of $n \times n$ matrices (X, Y) such that $XY = YX$. In [6] Knutson introduces generalizations of the commuting variety: the diagonal commutator scheme and its flat degeneration, the upper–upper scheme. The diagonal commutator scheme is the variety of pairs of matrices (X, Y) such that $XY - YX$ is diagonal and the upper–upper scheme E is $\{(X, Y) : XY \text{ and } YX \text{ upper triangular}\}$. E can be generalized to rectangular $m \times n$ and $n \times m$ matrices, and can be naturally decomposed into subsets E_π , labelled by partial permutations π of rank m if $m < n$. The degrees of these varieties provide interesting invariants. In [6] Knutson conjectures that the variety \overline{E}_π for $\pi \in S_n$ is defined as a scheme by the following three sets of equations.

Conjecture 2 (Knutson) *The variety \overline{E}_π is defined as a scheme by three sets of equations,*

- XY and YX upper triangular
- $\text{diag}(XY) = \text{diag}(P_\pi Y X P_\pi^T)$
- those defining the P_π, P_π^T matrix Schubert varieties: for each pair i, j the rank of the lower left $i \times j$ rectangle in X (in Y) is bounded above by the number of 1s in that rectangle in P_π (in P_π^T).

Knutson furthermore calculates the degrees of \overline{E}_π for $\pi \in S_3$ [6, proposition 3], and finds

$$\begin{aligned} d_{(123)} &= 1, & d_{(132)} &= d_{(213)} = 3, \\ d_{(231)} &= d_{(312)} = 13, & d_{(321)} &= 31. \end{aligned}$$

These degrees are exactly the same as the ground-state wavefunction elements for $L = 6$ corresponding to the permutation π , as given in figure 3. We have calculated the degrees of \overline{E}_π for $\pi \in S_4$ using Macaulay2, see the appendix, and indeed found a correspondence with the ground state for $L = 8$. We have furthermore done similar calculations for odd system sizes, and are led to the following intriguing conjecture.

Conjecture 3 *The ground-state element of the Brauer Hamiltonian for $L = 2n$ corresponding to a permutation $\pi \in S_n$ is equal to the degree of the subset \overline{E}_π of the upper–upper scheme for pairs of $n \times n$ matrices. For $L = 2n + 1$ we find that the ground-state elements corresponding to the partial permutation matrix π of rank n correspond to the degree of the subset \overline{E}_π of the upper–upper scheme for an $n \times (n + 1)$ and an $(n + 1) \times n$ matrix.*

For the long permutation $w = (n, n - 1, \dots, 2, 1)$, the last of the three sets of defining equations in conjecture 2 is empty and the remaining two are those defining a degeneration of the commuting variety [6], consistent with our observation in the previous section. It

was furthermore proved by Knutson, [6, proposition 3], that the sum over all degrees of \overline{E}_π has a simple expression,

$$\sum_{\pi \in S_n} \deg \overline{E}_\pi = 2^{n^2-n}.$$

Assuming conjecture 3, the same holds for the corresponding sum over ground-state elements for $L = 2n$, an observation made by independent numerical calculations by Zuber [16]. For $L = 2n + 1$ we find that the sum over all ground-state elements corresponding to a partial permutation of rank n is equal to 2^{n^2} , which is indeed what one would expect for the total degree of E for an $n \times (n + 1)$ and an $(n + 1) \times n$ matrix following Knutson's argument.

It remains an open question whether there also exists a combinatorial interpretation of the numbers appearing in the ground state of the Brauer Hamiltonian, as is the case for the pure Temperley–Lieb model [12, 1, 13]. More specifically it would be very interesting if a direct connection between the degrees of the upper–upper scheme and combinatorics could be found.

Acknowledgments

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Appendix A. A Macaulay2 session

In this section we describe a Macaulay2 [4] session calculating the degree of the variety $\overline{E}_{(2,4,3,1)}$. We start by defining a polynomial ring of 32 variables, and define two matrices X and Y and their products.

```
i1 : R = ZZ[x_1..x_16, y_1..y_16];
i2 : X = genericMatrix(R, x_1, 4, 4)
o2 = | x_1 x_5 x_9 x_13 |
      | x_2 x_6 x_10 x_14 |
      | x_3 x_7 x_11 x_15 |
      | x_4 x_8 x_12 x_16 |
o2 : Matrix R^4 <--- R^4
i3 : Y = genericMatrix(R, y_1, 4, 4)
o3 = | y_1 y_5 y_9 y_13 |
      | y_2 y_6 y_10 y_14 |
      | y_3 y_7 y_11 y_15 |
      | y_4 y_8 y_12 y_16 |
o3 : Matrix R^4 <--- R^4
i4 : XY=X*Y;
o4 : Matrix R^4 <--- R^4
i5 : YX=Y*X;
o5 : Matrix R^4 <--- R^4
```

Next, we take the elements of XY and YX below the diagonal and concatenate them.

```
i6 : XYupperTri = (flatten XY)_{1,2,3,6,7,11};
o6 : Matrix R^1 <--- R^6
i7 : YXupperTri = (flatten YX)_{1,2,3,6,7,11};
o7 : Matrix R^1 <--- R^6
i8 : upperTri = XYupperTri|YXupperTri;
o8 : Matrix R^1 <--- R^12
```

The second set of polynomials that are part of the ideal is obtained by taking the diagonal elements of $XY - P_\pi YX P_\pi^\top$, where P_π is the permutation matrix corresponding to $\pi = (2, 4, 3, 1)$ with a one at $(i, \pi(i))$ and zeros everywhere else.

```
i9 : Perm = {2,4,3,1};
i10 : diagXY = map((ring XY)^(numgens source XY), source XY,
  (i,j) -> if i === j then XY_(i,i) else 0);
o10 : Matrix R^4 <--- R^4
i11 : diagPYX = map((ring YX)^(numgens source YX), source YX,
  (i,j) -> if i === j then YX_(Perm#i-1,Perm#i-1) else 0);
o11 : Matrix R^4 <--- R^4
i12 : diagXYPYX = compress flatten (diagXY-diagPYX)
o12 : Matrix R^1 <--- R^4
```

Lastly, we create the set of polynomials forming the ideals of the Schubert varieties of P_π and P_π^\top . All equations are concatenated and the ideal I corresponding to \overline{E}_π is defined.

```
i13 : Schubert = matrix {{y_4, y_3, det(submatrix(X,(2,3),(0,1))),
  det(submatrix(X,(1,3),(0,1))), det(submatrix(X,(1,2),(0,1)))}};
o13 : Matrix R^1 <--- R^5
i14 : total = upperTri|diagXYPYX;
o14 : Matrix R^1 <--- R^16
i15 : total = total|Schubert;
o15 : Matrix R^1 <--- R^21
i16 : I = ideal total;
o16 : Ideal of R
i17 : degree I
o17 : 173
```

Appendix B. Symmetry classes of chord diagrams

The total number $c_n = 1, 2, 5, 17, 79, \dots$, of symmetry classes of chord diagrams, or pairings on a bracelet, is given by [7, 14]

$$c_n = \frac{1}{4} \left(\frac{1}{n} \sum_{pq=2n} \alpha(p, q) \phi(q) + d_n + d_{n-1} \right),$$

where

$$\alpha(p, q) = \begin{cases} \sum_{k=0}^{\lfloor p/2 \rfloor} \binom{p}{2k} q^k (2k-1)!! & q \text{ even} \\ q^{p/2} (p-1)!! & q \text{ odd,} \end{cases}$$

$$d_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!},$$

and $\phi(n)$ is Euler's totient function.

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