

THE ROTOR MODEL AND COMBINATORICS

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We examine the groundstate wavefunction of the rotor model for different boundary conditions. Three conjectures are made on the appearance of numbers enumerating alternating sign matrices. In addition to those occurring in the $O(n = 1)$ model we find the number $A_V(2m + 1; 3)$, which 3-enumerates vertically symmetric alternating sign matrices.

1. Introduction

The XXZ Heisenberg spin chain and the related six-vertex model stand as central pillars in the study of exactly solved models in statistical mechanics.^{1,2} It has been known for many years that, with appropriate boundary conditions, their groundstate energy is trivial at the particular anisotropy value $\Delta = -1/2$. Only recently has it been realised that the corresponding groundstate wavefunction possesses some rather remarkable properties.^{3–5} These observations extend to the related $O(n)$ loop model^{6,7} at $n = 1$.^{4,8–10}

Consider first the periodic antiferromagnetic XXZ chain

$$H = -\frac{1}{2} \sum_{j=1}^N (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z), \quad (1)$$

defined on an odd number N of sites. Here $(\sigma_j^x, \sigma_j^y, \sigma_j^z)$ are the Pauli spin matrices acting at site j . Normalize the smallest component of the groundstate wavefunction to be unity. Then at $\Delta = -1/2$ the largest component is conjectured to be given by³

$$A(m) = \prod_{j=0}^{m-1} \frac{(3j+1)!}{(m+j)!}, \quad (2)$$

for size $N = 2m + 1$. The remarkable point being that $A(m)$ is the number of $m \times m$ alternating sign matrices.¹¹ The resulting sequence $A(m) = 1, 2, 7, 42, 429, 7436 \dots$ is also known to count other combinatorial objects.^{12,13} Moreover, these numbers appear in the sum of all the groundstate wavefunction components. These observations remain to be proved.

An even number of sites and other boundary conditions have also been considered, both for the XXZ chain (twisted and closed^a quantum symmetric bc's) and the $O(n = 1)$ loop model (periodic and closed bc's). These see the appearance of other well known numbers counting alternating sign matrices and related objects in different symmetry classes. For example, with the smallest component of the groundstate wavefunction again unity, the $O(n = 1)$ loop model with closed boundary conditions has largest component given by $A_V(2m - 1)$ for $N = 2m - 1$ and $N_8(2m)$ for $N = 2m$. Here

$$A_V(2m + 1) = \prod_{j=0}^{m-1} (3j + 2) \frac{(2j + 1)!(6j + 3)!}{(4j + 2)!(4j + 3)!} \quad (3)$$

is the number of $(2m + 1) \times (2m + 1)$ vertically symmetric alternating sign matrices and

$$N_8(2m) = \prod_{j=0}^{m-1} (3j + 1) \frac{(2j)!(6j)!}{(4j)!(4j + 1)!} \quad (4)$$

is the number of cyclically symmetric transpose complement plane partitions. The number $N_8(2m)$ is conjectured to be $A_{VH}(4m + 1)/A_V(2m + 1)$, where $A_{VH}(4m + 1)$ is the number of $(4m + 1) \times (4m + 1)$ vertically and horizontally symmetric alternating sign matrices.^{14,15} Another quantity, which appears for periodic boundary conditions, is

$$A_{HT}(2m) = A(m)^2 \prod_{j=0}^{m-1} \frac{3j + 2}{3j + 1}, \quad (5)$$

the number of $2m \times 2m$ half turn symmetric alternating sign matrices.

Further developments include the combinatorial interpretation of the elements of the $O(n = 1)$ loop model wavefunction in terms of link patterns⁸⁻¹⁰ and the relation to a one-dimensional stochastic process.¹⁰ There has been some progress attempting to prove these conjectures using Bethe Ansatz techniques.^{16,17}

In this paper, we examine the groundstate wavefunction of the rotor model¹⁸ discussed by Martins and Nienhuis. The rotor model is based on a variant of the Temperley-Lieb algebra, which underpins the six-vertex model, the $O(n)$ model and the critical Q -state Potts model.^{1,2,19} The rotor model is defined in Section 2, with our results presented in Section 3.

^aThe standard nomenclature for these bc's is open bc, but since these bc's are spin-conserving in the XXZ chain or loop reflecting in the $O(n = 1)$ model we find the term closed bc more appropriate, here reserving open bc for non-conserving boundary conditions.

Here we see the appearance of another number, $A_V(2m + 1; 3)$, which is the 3-enumeration of $(2m + 1) \times (2m + 1)$ vertically symmetric alternating sign matrices, or equivalently, the number of vertically symmetric 6-vertex configurations with domain wall boundary conditions and $\Delta = -1/2$. It is given by

$$A_V(2m + 1; 3) = \frac{3^{m(m-3)/2}}{2^m} \prod_{j=1}^m \frac{(j-1)!(3j)!}{j(2j-1)!^2} = 1, 5, 126, 16038, \dots \tag{6}$$

In general, the x -enumeration of alternating sign matrices in the terminology of Kuperberg,¹⁵ is equivalent to the enumeration of six-vertex configurations with domain wall boundaries with $\Delta = 1 - x/2$ and at the symmetric point with respect to the spectral parameter.

We give some concluding remarks in Section 4.

2. The rotor model

We suggest that the remarkable observations of this $O(n = 1)$ model are related to the combination of two key properties, namely solvability and the absence of finite size corrections to the groundstate energy. Now the $O(n = 1)$ model is not unique in this combination. Recently Martins and Nienhuis¹⁸ introduced a model that shares the same two properties. In this so-called rotor model a set of loops covers all the edges of the square lattice precisely twice. At the vertices all the loops make a turn of $\pi/2$ which permits four types of vertices as displayed in Figure 1.

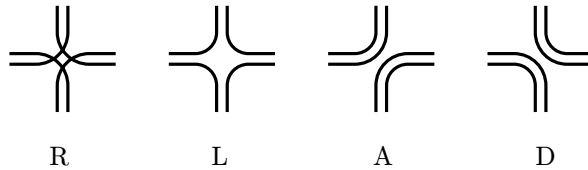


Fig. 1. Vertices of the rotor model.

A natural interpretation is that the loops are trajectories of particles, and that the two loop segments visiting the same edge are traversed in opposite directions. Thus the four kinds of vertices shown in Figure 1 behave as scatterers: right (R) and left (L) rotors, at which the particles always turn right and left respectively, and ascending (A) and descending (D) diagonal mirrors at which the particles get reflected. To clearly display the scatterers we propose that the particles always follow the left hand side of the road, as is customary in Australia where this paper was conceived.

In a different interpretation the two loop segments at the same edge are the trajectories of different kinds of particles, traversed in either direction. Then the scatterers can all be interpreted as double mirrors on each site, each reflecting one kind of particle and transmitting the other. At the R and L sites these mirrors

are placed crosswise, AD and DA respectively, while at the original ascending and descending mirrors, the double mirrors are placed parallel, AA and DD respectively. This alternate interpretation will not affect the distributions of trajectories in an infinite system, but it will result in changes on some finite systems.

Martins and Nienhuis solved this model by means of the Yang-Baxter equation when these scatterers occur with the respective weights

$$\begin{aligned} \omega_R &= \omega_L = \sin u \cos(2\pi/3 - u), \\ \omega_A &= \sin(\pi/3 - u) \cos(2\pi/3 - u), \\ \omega_D &= -\sin u \cos(\pi/3 - u). \end{aligned} \tag{7}$$

independently at each vertex. In this paper we consider this model with periodic boundary conditions (pbc) and with closed boundaries at which the trajectories are reflected. We will be interested in the structure of the groundstate eigenvector. Since the transfer matrix as a function of u forms a commuting family, the groundstate is independent of u . Then it is convenient to consider the Hamiltonian, found (up to a constant) as the logarithmic derivative of the transfer matrix with respect to u at $u = 0$:

$$H = \sum_i 3 - R_i - L_i - E_i. \tag{8}$$

For system size N the operators R , L and E are shown in terms of the loops in Figure 2.

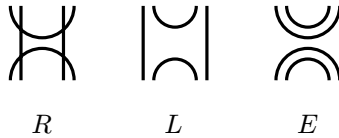


Fig. 2. Generators.

Martins and Nienhuis showed that the operators L_{2i} and R_{2i-1} generate a Temperley-Lieb (TL) algebra, and so do the operators L_{2i-1} and R_{2i} . In periodic systems of even size, and in bounded systems these two TL algebras commute with each other. What changes the physics is the presence in the Hamiltonian of the term $E_i = R_i L_i$. Also the E_i by themselves generate a TL algebra. In odd, periodic systems the odd and even sites cannot be distinguished. In this case the L and the R together form a TL algebra of $2N$ sites.

When the system is odd and periodic, the interpretation of the R and L vertices as rotors or alternatively as crossing mirrors, will naturally result in different pbc. The rotor interpretation permits closed trajectories that wind the cylinder twice. In the alternative interpretation no closed winding trajectories are possible, and the odd system must have two unmatched terminals. In this paper we follow the latter interpretation.

The states of the model are the pairings of those terminals that are connected by a trajectory in the ‘past’ half of the strip or cylinder. When the system is periodic, one may distinguish the side of the cylinder along which the trajectory runs: a connection between site 1 and site N may pass all sites $2, \dots, N - 1$, or it may simply connect site N to site $N + 1$ which is identified to 1. These two states can be distinguished, in which case we speak of pbc *per se*, or they may be identified, for which we reserve the phrase pbc with identified connectivities.

3. Results for the groundstate wavefunction

The groundstate wavefunction of the Hamiltonian (8) satisfies the eigenvalue equation $H\psi_0 = 0$. In this section we formulate three conjectures regarding ψ_0 for the different types of boundary conditions discussed in Section 2.

Conjecture 1: *For closed boundary conditions, if the smallest element of the rotor model groundstate wavefunction for $N = 2m - 1$ is normalized to $A_V(2m - 1; 3)$, then all of its elements are integers and the sum of its elements is given by $S(2m - 1) = 3^{(m-1)^2} N_8(2m)$. For $N = 2m$, normalize the groundstate wavefunction to the smallest integer such that all elements are integers, the sum of the elements is given by $S(2m) = 3^{2\theta_m} A_V(2m + 1)$, where $\theta_m = 0, 1, 3, 6, 9 = \lfloor (m - 1)(m + 2)/3 \rfloor$ for $m = 1, \dots, 5$.*

This conjecture is based on the results presented in Table 1 and was checked up to $N = 10$.

Conjecture 2: *For periodic boundary conditions, normalize the smallest element of the rotor model groundstate wavefunction to the smallest integer such that all elements are integer. The sum of its elements is then given by $S(2m - 1) = 3^{3m} A_V(2m + 1; 3)^2$ for odd system sizes and by $S(2m) = 3^{m^2} A_{HT}(2m)$ for even system sizes.*

This conjecture is based on the results presented in Table 2 and was checked up to $N = 9$.

Conjecture 3: *For periodic boundary conditions and identified connectivities, normalize the smallest element of the rotor model groundstate wavefunction to the smallest integer such that all elements are integer. The sum of its elements is then given by $S(2m) = 3^{\theta_m} A(m)$, where $\theta_m = 0, 1, 3, 6, 9, 13 = \lfloor (m - 1)(m + 2)/3 \rfloor$ for $m = 1, \dots, 6$.*

This conjecture is based on the results presented in Table 3 and was checked up to $N = 12$.

4. Discussion

In this paper we have examined the groundstate wavefunction of the rotor model for three different boundary conditions. As for the $O(n = 1)$ model, numbers known to

Table 1. Groundstate wavefunctions of the rotor model with closed boundaries. Note that by $\psi_0 = (2, 1)$ with multiplicity $(2, 2)$ we mean $\psi_0 = (2, 2, 1, 1)$.

N	m	ψ_0	multiplicity	$S_N^{(1)}$
1	1	(1)	(1)	1
2	1	(1)	(1)	1
3	2	(2,1)	(2,2)	6
4	2	(14,5,4)	(1,1,2)	27
5	3	(113, 111, 55, 31, 25, 21, 19, 11, 5)	(2, 1, 4, 2, 4, 2, 4, 4, 2)	891
6	3	(4760, 1440, 1192, 1028, 601, 565, 326, 310, 126, 121, 86)	(1, 2, 4, 1, 4, 2, 2, 2, 1, 2, 4)	18954

Table 2. Groundstate wavefunctions of the rotor model with periodic boundaries.

N	m	ψ_0	multiplicity	$S_N^{(1)}$
1	1	(1)	(1)	1
2	1	(2,1)	(2,2)	6
3	2	(5,2)	(3,6)	27
4	2	(118, 35, 25, 22, 20, 5, 4)	(2, 2, 8, 4, 8, 8, 4)	810
5	3	(1036, 463, 208, 143, 127, 122, 65, 22, 10)	(5, 10, 10, 20, 5, 10, 20, 10, 10)	18225

Table 3. Groundstate wavefunctions of the rotor model with periodic boundaries and identified connectivities.

N	m	ψ_0	multiplicity	$S_N^{(1)}$
2	1	(1)	(1)	1
4	2	(2,1)	(2,2)	6
6	3	(26, 9, 7, 2)	(2, 3, 14, 6)	189
8	4	(1798, 486, 410, 267, 234, 232, 165, 106, 90, 81, 76, 70, 56, 45, 20, 9, 4)	(2, 8, 16, 2, 16, 16, 8, 16, 4, 16, 8, 8, 16, 32, 16, 8, 4)	30618

enumerate equally weighted alternating sign matrices appear in the normalization of the wavefunction. For the rotor model we also see the number $A_V(2m + 1; 3)$, enumerating alternating sign matrices in which the minus signs have weight 3.¹⁵

We find it quite surprising that the conjectures in Section 3 can be formulated at all. They are a result of the normalizations factoring into relatively small primes and thus enabling their recognition. This property appears to be absent for other boundary conditions, for example, pbc in the rotor interpretation for odd system sizes. It is even more remarkable that these numbers have a well known combinatorial meaning.

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