620-161 SOLUTIONS TO HOMEWORK SHEET 9
Semester 1 2005

Out of 8 - Parts (a) and (b) each out of 4.

(a) We have $f(x,y) = x^3 + y^3 - 3xy$.
Stationary points occur where $f_x(x,y) = f_y(x,y) = 0$. So we want

$$\begin{aligned} & f_x(x,y) = 3x^2 - 3y = 0 \qquad \Rightarrow \quad x^2 = y & (1) \\ \text{and} \qquad & f_y(x,y) = 3y^2 - 3x = 0 \qquad \Rightarrow \quad y^2 = x & (2) \end{aligned}$$

Substituting (1) into (2) we get

$$x^4 = x \ ,$$

which has solutions $x = 0$ and $x = 1$, and thus $y = 0$ and $y = 1$ accordingly. Hence there are two stationary points: $(0,0)$ and $(1,1)$.

To classify these stationary points we need the second partial derivatives:

$$\begin{aligned} A &= f_{xx}(x,y) = 6x \\ B &= f_{xy} = -3 \\ C &= f_{yy} = 6y \end{aligned}$$

and so

$$AC - B^2 = (6x)(6y) - (-3)^2 = 36xy - 9 \ .$$

At $(0,0)$, we have

$$AC - B^2 = 0 - 9 = -9 < 0$$

so $(0,0)$ is a saddle point.
At $(1,1)$, we have

$$AC - B^2 = 36 - 9 = 27 > 0 \qquad \text{and} \quad A = 6 > 0$$

so $(1,1)$ is a local minimum point.

(b) (i) We want to minimise $f(x,y) = -6x^2 + 2y^2$ subject to the constraint $2x + y = 4$, or $2x + y - 4 = 0$. Using the method of Lagrange multipliers, define $F(x,y,\lambda) = (-6x^2 + 2y^2) + \lambda(2x + y - 4)$. Then we want

$$\begin{aligned} F_x(x,y,\lambda) &= -12x + 2\lambda = 0 \\ F_y(x,y,\lambda) &= 4y + \lambda = 0 \\ F_\lambda(x,y,\lambda) &= 2x + y - 4 = 0 \end{aligned}$$

The first equation gives $\lambda = 6x$ and the second gives $\lambda = -4y$, so equating these gives $y = -\frac{3}{2}x$. Substituting this into the third equation gives:

$$2x + \big(-\frac{3}{2}\big)x - 4 = 0 \qquad \Rightarrow \quad \frac{1}{2}x = 4 \qquad \Rightarrow \quad x = 8 \ .$$

Then $y = -\frac{3}{2}(8) = -12$, and so $(x,y) = (8,-12)$ is the required minimum point (as it is the only stationary point of F). The minimum value of f is $f(8,-12) = -6(8)^2 + 2(-12)^2 = -96$.

(ii) We want to minimise $f(x,y,z) = x^2+y^2+z^2$ subject to $2x+y-z=3$, or $2x+y-z-3=0$. Define $F(x,y,z,\lambda) = (x^2+y^2+z^2) + \lambda(2x+y-z-3)$. Then we want:

$$\begin{aligned} F_x(x,y,z,\lambda) &= 2x + 2\lambda = 0 \\ F_y(x,y,z,\lambda) &= 2y + \lambda = 0 \\ F_z(x,y,z,\lambda) &= 2z - \lambda = 0 \\ F_\lambda(x,y,z,\lambda) &= 2x + y - z - 3 = 0 \ . \end{aligned}$$

The first equation gives $\lambda = -x$, the second gives $\lambda = -2y$ and the third $\lambda = 2z$. Equating these yields, for example, $x = 2y$ and $z = -y$. Substituting into the final equation, we get:

$$2(2y) + y - (-y) - 3 = 0 \qquad \Rightarrow \quad 6y = 3 \qquad \Rightarrow \quad y = \frac{1}{2} \ .$$

Then $x = 2(\frac{1}{2}) = 1$ and $z = -\frac{1}{2}$, so that $(x,y,z) = (1, \frac{1}{2}, -\frac{1}{2})$ is the required minimum point (as it is the only stationary point of F). The minimum value of f is therefore $f(1,\frac{1}{2},-\frac{1}{2}) = 1^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2 = \frac{3}{2}$.