Sheet 1: Solutions.

(a) Firstly we need to set up the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & -2.49 \\ -1 & -3 & -2 & 3.48 \\ 3 & 1 & 1 & 3.73 \end{array}\right] \quad R_1+R_2 \to R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0.99 \\ -1 & -3 & -2 & 3.48 \\ 3 & 1 & 1 & 3.73 \end{array}\right] \quad \begin{array}{l} R_2+R_1 \to R_2 \\ R_3-3R_1 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0.99 \\ 0 & -2 & -1 & 4.47 \\ 0 & -2 & -2 & 0.76 \end{array}\right] \quad \begin{array}{l} -R_2 \to R_2 \\ R_3-R_2 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0.99 \\ 0 & 2 & 1 & -4.47 \\ 0 & 0 & -1 & -3.71 \end{array}\right] \quad \begin{array}{l} R_1+R_3 \to R_1 \\ R_2+R_3 \to R_2 \\ -R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.37 \\ 0 & 1 & 0 & -4.09 \\ 0 & 0 & 1 & 3.71 \end{array}\right] \quad \begin{array}{l} R_1-\frac{1}{2}R_2 \to R_1 \\ \frac{1}{2}R_2 \to R_2 \end{array}$$

So $x_1 = 1.37$, $x_2 = -4.09$, $x_3 = 3.71$

Don't forget to check your answer in the original equations.

$$2(1.37) + 4(-4.09) + 3(3.71) = -2.49$$
$$-1.37 - 3(-4.09) - 2(3.71) = 3.48$$
$$3(1.37) + (-4.09) + (3.71) = 3.73.$$

The solution checks out OK!

(b)

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & -1 \\ -1 & -3 & -2 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right] \quad R_1 + R_2 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & -3 & -2 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right] \quad R_2 + R_1 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right] \quad \begin{array}{l} -\frac{1}{2}R_2 \to R_2 \\ R_2 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t$, $t \in \mathbb{R}$.

Then $x_2 = -\frac{1}{2}(1 + t)$

$x_1 = -\frac{1}{2}(t - 1)$.

Check your answer in <u>all</u> equations.

① LHS $= 2(-\frac{1}{2}t + \frac{1}{2}) + 4(-\frac{1}{2} - \frac{1}{2}t) + 3t$

$= -t + 1 - 2 - 2t + 3t$

$= -1 = $ RHS.

② LHS $= -1(-\frac{1}{2}t + \frac{1}{2}) - 3(-\frac{1}{2} - \frac{1}{2}t) - 2t$

$= \frac{1}{2}t - \frac{1}{2} + \frac{3}{2} + \frac{3}{2}t - 2t$

$= 1 = $ RHS.

③ LHS $= 2(-\frac{1}{2} - \frac{1}{2}t) + t$

$= -1 - t + t$

$= -1 = $ RHS.

So our solution is

$$\{ (-\tfrac{1}{2}(t-1), -\tfrac{1}{2}(t+1), t) : t \in \mathbb{R} \}$$

Don't forget there are many alternative ways of solving these problems – but you should get the same answer.

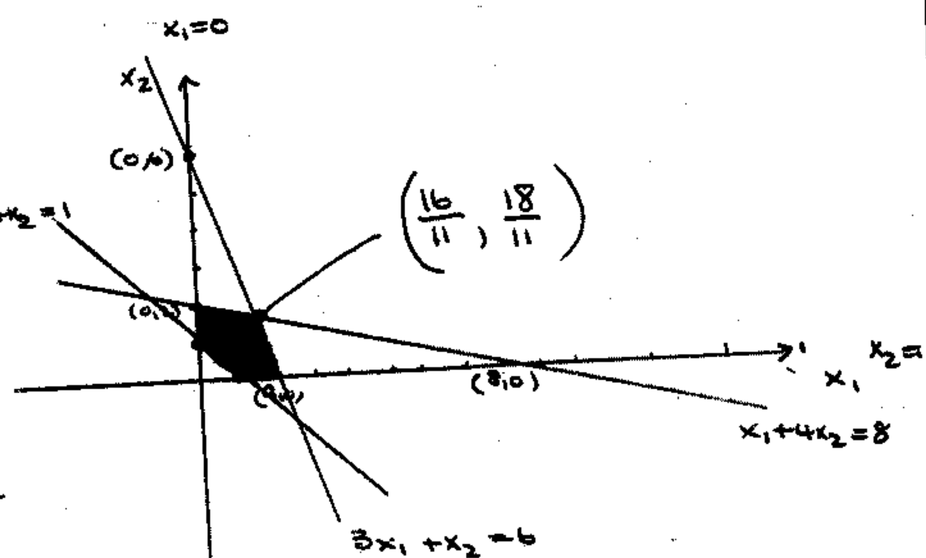
Sheet 2: Solutions.

(a) maximize $C = x_1 + x_2$

subject to
$$\begin{aligned} 3x_1 + x_2 &\le 6 \\ x_1 + 4x_2 &\le 8 \\ x_1 + x_2 &\geq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

The objective function has graph parallel to $x_1 + x_2 = 1$.

It will take its maximum value at the intersection of $x_1 + 4x_2 = 8$ and $3x_1 + x_2 = 6$.



To find this point we solve these equations simultaneously.

$$\left[\begin{array}{cc|c} 1 & 4 & 8 \\ 3 & 1 & 6 \end{array}\right] \quad R_2 - 3R_1 \rightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 4 & 8 \\ 0 & -11 & -18 \end{array}\right] \quad -\tfrac{1}{11} R_2 \rightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 4 & 8 \\ 0 & 1 & \frac{18}{11} \end{array}\right] \quad R_1 - 4R_2 \rightarrow R_1 \qquad \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{16}{11} \\ 0 & 1 & \frac{18}{11} \end{array}\right]$$

so $x_1 = \frac{16}{11}$ $\quad x_2 = \frac{18}{11}$

So $C = \frac{16}{11} + \frac{18}{11} = \frac{34}{11}$ is the max value

So the solution to this problem is

$C = 3\frac{3}{11}$ at the point $\frac{1}{11}(16, 18)$.

(b) Let x be the number of minutes that the band plays and y be the number of minutes that the compare talks.

We have $x + y \leq 120$

$x \geq 100$

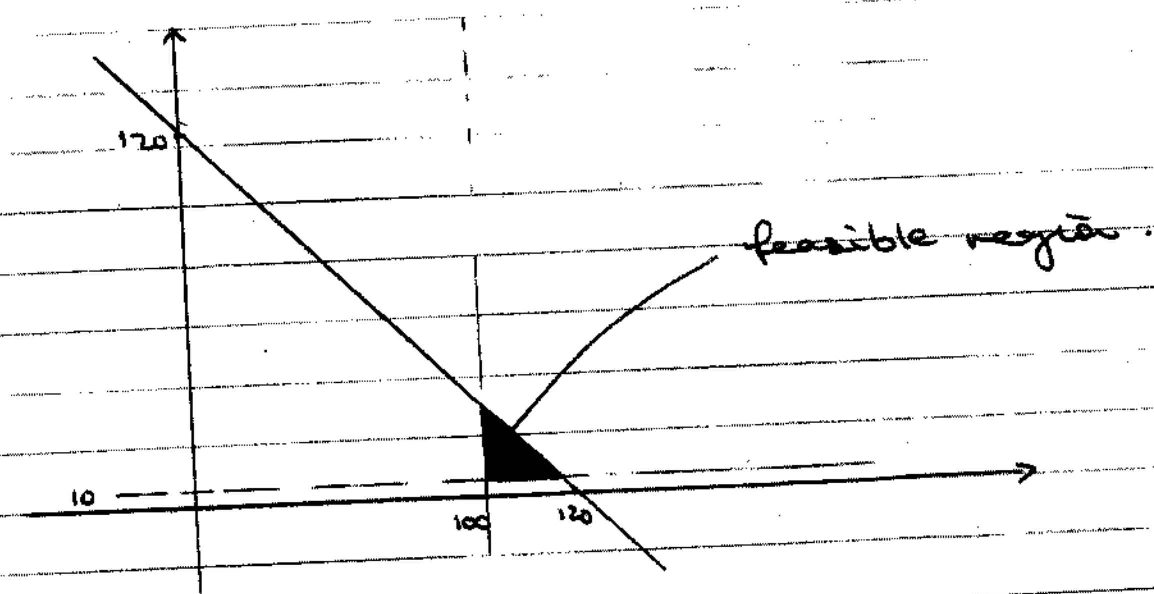
$y \geq 10$

$x \geq 0, y \geq 0$

$100 \leq x \leq 120$

$10 \leq y \leq 20$

and we wish to maximize $P = 500(10x + y)$.



The corner points are

(a) (100, 10)
(b) (100, 20)
(c) (110, 10)

For these points we have

$$P(100,10) = 500(10\times 100 + 10) = \$505{,}000$$
$$P(100,20) = 500(10\times 100 + 20) = \$510{,}000$$
$$P(110,10) = 500(10\times 110 + 10) = \$555{,}000$$

The the optimal solution is $P = \$555{,}000$ at the point (110, 10).

Sheet 3: Solutions

Maximize $f = -x_1 + x_2 + 2x_3$

subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 &\le 20 \\ -2x_1 + 4x_2 + 2x_3 &\le 60 \\ x_1 + 3x_2 + x_3 &\le 50 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \end{aligned}$$

First we rewrite the equations in an appropriate form.

max. $f + x_1 - x_2 - 2x_3 = 0$

sub. to

$$\begin{aligned} x_1 + 2x_2 - x_3 \quad &= 20 \\ -2x_1 + 4x_2 + 2x_3 \quad &= 60 \\ x_1 + 3x_2 + x_3 \quad &= 50 \end{aligned}$$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	f	RHS	
s_1	1	2	-1	1	0	0	0	20	
s_2	-2	4	$\boxed{2}$	0	1	0	0	60	$R_2/2 \to R_2$
s_3	1	3	1	0	0	1	0	50	
	1	-1	-2	0	0	0	1	0	

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	f	RHS	
s_1	1	2	−1	1	0	0	0	20	$R_1+R_2 \to R_1$
x_3	−1	2	$\boxed{1}$	0	½	0	0	30	
s_3	1	3	1	0	0	1	0	50	$R_3-R_2 \to R_3$
	1	−1	−2	0	0	0	1	0	$R_4+2R_2 \to R_4$

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	f	RHS	
s_1	0	4	0	1	½	0	0	50	
x_3	−1	2	1	0	½	0	0	30	
s_3	$\boxed{2}$	1	0	0	−½	1	0	20	$R_3/2 \to R_3$
	−1	3	0	0	1	0	1	60	

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	f	RHS	
s_1	0	4	0	1	½	0	0	50	
x_3	−1	2	1	0	½	0	0	30	$R_2+R_3 \to R_2$
x_1	$\boxed{1}$	½	0	0	−¼	½	0	10	
	−1	3	0	0	1	0	1	60	$R_4+R_2 \to R_4$

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	f	RHS
s_1	0	4	0	1	$\frac{1}{2}$	0	0	50
x_3	0	$2\frac{1}{2}$	1	0	$\frac{1}{4}$	$\frac{1}{2}$	0	40
x_1	1	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	$\frac{1}{2}$	0	10
	0	$3\frac{1}{2}$	0	0	$\frac{3}{4}$	$\frac{1}{2}$	1	70

Optimal solution $f = 70$

at $(10, 0, 40)$

Check $f = -(10) + 0 + 2 \cdot 40 = 70$ ✓✓

Sheet 4 : Solutions

(a) Minimize $f = x_1 + x_2 + 12x_3$

subject to
$$x_1 - x_2 + x_3 \geq 3$$
$$-x_1 + 2x_2 + 3x_3 \geq 4$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ -1 & 2 & 3 & 4 \\ \hline 1 & 1 & 12 & 1 \end{array}\right]$$

$$A^t = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 3 & 12 \\ \hline -3 & 4 & 1 \end{array}\right]$$

maximize $g = 3y_1 + 4y_2$

subject to
$$y_1 - y_2 \leq 1$$
$$-y_1 + 2y_2 \leq 1$$
$$y_1 + 3y_2 \leq 12$$
$$y_1, y_2 \geq 0$$

optimal solution $f = 17$ $(x_1, x_2, x_3) = (10, 7, 0)$

Check L.H.S $= 17$.

R.H.S $= 10 + 7 + 12 \times 0 = 17$. = L.H.S.

(b) $6 + \log_e(x^3) = \log_e(64)$

$\Rightarrow 6 = \log_e(64) - \log_e(x^3)$

$\Rightarrow 6 = \log_e\left(\frac{64}{x^3}\right)$

$\Rightarrow e^6 = \frac{64}{x^3}$

$\Rightarrow x^3 = \frac{64}{e^6}$

$\Rightarrow x = \frac{4}{e^2}$ (OR $4e^{-2}$).

Sheet 5: Solutions

(a) (i) $$\frac{d}{dx}\left(x^5 e^{4x}\right) = e^{4x}\frac{d}{dx}(x^5) + x^5\frac{d}{dx}(e^{4x})$$

$$= e^{4x}.\,5x^4 + x^5.\,4e^{4x}$$

$$= e^{4x}x^4(5+4x)$$

(ii) $$\frac{d}{dx}\left(2x^5 + e^{3x}\right)^{13}$$

$$= 13\left(2x^5 + e^{3x}\right)^{12}.\,\frac{d}{dx}\left(2x^5+e^{3x}\right)$$

$$= 13\left(2x^5+e^{3x}\right)^{12}\left(10x^4 + 3e^{3x}\right)$$

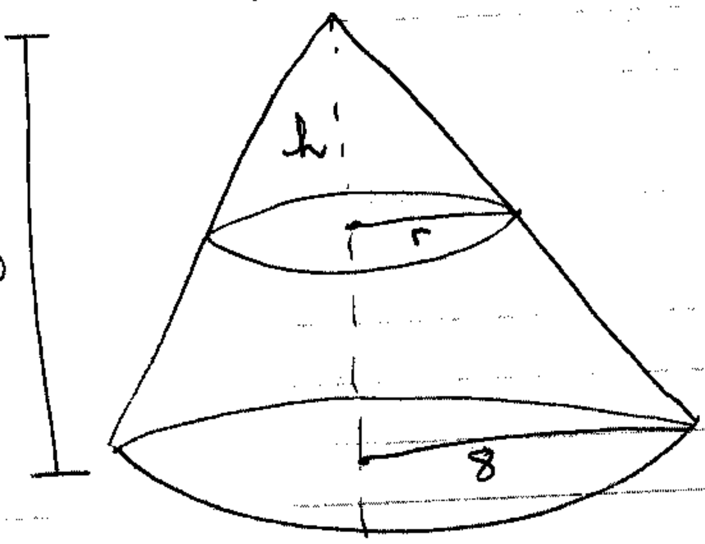
(iii) $$\frac{d}{dx}\left(\frac{\log_e(3x)}{2x+5x^2+8x^3}\right)$$

$$= \frac{\frac{d}{dx}\left(\log_e(3x)\right)\left(2x+5x^2+8x^3\right) - \frac{d}{dx}\left(2x+5x^2+8x^3\right)\log_e(3x))}{\left(2x+5x^2+8x^3\right)^2}$$

$$= \frac{\frac{1}{x}\left(2x+5x^2+8x^3\right) - \left(2+10x+24x^2\right)\log_e(3x)}{\left(2x+5x^2+8x^3\right)^2}$$

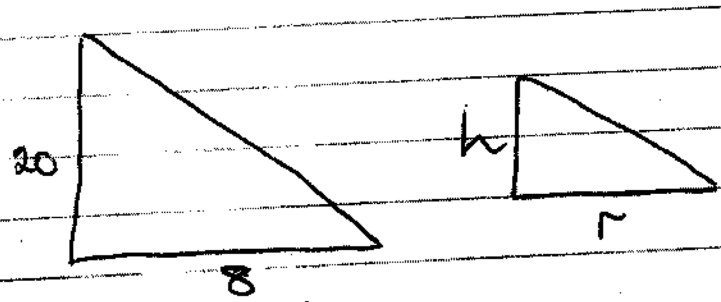
$$= \frac{\left(2+5x+8x^2\right) - \log_e(3x)\left(2+10x+24x^2\right)}{\left(2x+5x^2+8x^3\right)^2}$$

(b)



Rate of "h" decreasing = Rate of water height increasing

$\frac{dV}{dt} = 12$ $\qquad \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$



$\frac{r}{h} = \frac{8}{20} \qquad r = \frac{2}{5}h$

$V = \frac{1}{3}\pi r^2 h$

$\Rightarrow V = \frac{1}{3}\pi \frac{4}{25} h^2 . h$ (since $r = \frac{2}{5}h$)

$\Rightarrow V = \frac{4}{75}\pi h^3$

So $\frac{dV}{dh} = \frac{12}{75}\pi h^2$

When the water is 9cm deep, $h = 20 - 9$
$= 11$

So $\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{75}{12 . \pi . 11^2} . 12$

$= \frac{75}{\pi . 121}$ cm/hour

$\approx .192$ cm/hour.

Sheet 6

8(a) Find the slope of the curve

$x+y = (y-2)^4$ at any point (x,y)

on the curve.

Ans. $$\frac{d}{dx}(x+y) = \frac{d}{dx}\left((y-2)^4\right)$$

$$\Rightarrow 1 + \frac{dy}{dx} = 4(y-2)^3 \cdot \frac{dy}{dx}$$

$$\Rightarrow 1 = 4(y-2)^3 . \frac{dy}{dx} - \frac{dy}{dx}$$

$$\Rightarrow 1 = \frac{dy}{dx}\left(4(y-2)^3 - 1\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4(y-2)^3 - 1}$$

When $\left\{ \begin{array}{l} x = -2 \\ y = 3 \end{array} \right.$ the curve $x+y = (y-2)^4$

is LHS $-2+3 = 1$

RHS $(3-2)^4 = 1$ so the point

$(-2,3)$ is indeed on the curve.

We find the slope of the tangent line by evaluating $\frac{dy}{dx}$ at $(-2,3)$.

$$\frac{dy}{dx} = \frac{1}{4(3-2)^3 - 1} = \frac{1}{3}$$

So our tangent line is $y = \frac{1}{3}x + c$

To determine c, we know that the line passes through the point $(-2,3)$

So $\quad 3 = \frac{1}{3}(-2) + c$

$\Rightarrow \quad c = 3\frac{2}{3}$

So $\quad y = \frac{1}{3}x + 3\frac{2}{3}$.

(b) $f(x) = (4+x)^{1/2}$ $\quad f(0) = 2$

$f'(x) = \frac{1}{2}(4+x)^{-1/2}$ $\quad f'(0) = 1/4$

$f''(x) = -\frac{1}{4}(4+x)^{-3/2}$ $\quad f''(0) = -\frac{1}{32}$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$$

$$= 2 + \frac{1}{4}x - \frac{1}{32}\left(\frac{x^2}{2!}\right)$$

$$= 2 + \frac{x}{4} - \frac{x^2}{64}$$

$$\text{Error} = \frac{f^{n+1}(c)x^{n+1}}{(n+1)!}$$

Now $f'''(x) = \frac{3}{8}(4+x)^{-5/2}$

So error term is

$$\mathcal{E}_f(x) = \frac{\frac{3}{8}(4+c)^{-5/2}\, x^3}{3!}$$

$$= \frac{x^3}{16(4+c)^{5/2}}$$

The value of the error term decreases over the range of values $c \in [0, 0.4]$.

So to find the maximum error, we choose $c = 0$. To find the error for $(4.4)^{1/2}$ ~~x must take~~ the value .4 $(f(x) = (4+x)^{1/2})$.

$$\text{So } E_{max} = \frac{(0.4)^3}{16(2)^5} = .000125.$$

$$P(x) = 2 + \frac{(0.4)}{4} - \frac{(0.4)^2}{64} = 2.0975$$

Using the calculator, we have

$$(4.4)^{1/2} = 2.0976176$$

Actual error $= 2.0976176 - 2.0975$

$= .0001176$

$\therefore$ actual error < possible error.

Sheet 7 : Solutions

(a) $f(x) = 2x^5 - 5x^4 + 1 \qquad 1 \leq x \leq 3$

$f'(x) = 10x^4 - 20x^3$

To find max/min values, we set $f'(x) = 0$.

So $10x^4 - 20x^3 = x^3(10x - 20) = 0$

This means $x = 0$ OR $x = 2$.

We do not consider the point $x = 0$ since it is not in the domain.

For $x = 2$ $\qquad f(2) = 2.2^5 - 5.2^4 + 1$

$= 64 - 80 + 1$

$= -15.$

For absolute max/min values we need to check the endpoints.

$f(1) = 2.1^5 - 5.1^4 + 1$

$= 2 - 5 + 1 = -2$

$f(3) = 2.3^5 - 5.3^4 + 1$

$= 82.$

So the absolute maximum value is $f(3) = 82$ and the absolute minimum is $f(2) = -15$.

Sheet 8: Solutions

(a) $z = x^3y^4 + 7x^4 + 3y^2 + y + x$

$$\frac{\partial z}{\partial x} = 3x^2y^4 + 28x^3 + 1$$

$$\frac{\partial z}{\partial y} = 4x^3y^3 + 6y + 1$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2y^4 + 28x^3 + 1\right)$$

$$= 6xy^4 + 84x^2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(4x^3y^3 + 6y + 1\right)$$

$$= 12x^3y^2 + 6$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(4x^3y^3 + 6y + 1\right)$$

$$= 12x^2y^3$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(3x^2y^4 + 28x^3 + 1\right)$$

$$= 12x^2y^3.$$

(b) $P(x,y) = 14x + 16y - 0.002x^2 - 0.003y^2$

$x = 1000 \quad y = 800$

$P = 14 \times 1000 + 16 \times 800 - 0.002 \times 10^6 - 0.003 \times 640000$

$= 14000 + 12800 - 2000 - 1920$

$= 26800 - 3920$

$= 22880$

$P_x = 14 - 0.004x$

$P_x(1000, 800) = 14 - 4$

$= 10$

$P_y = 16 - 0.006y$

$P_y(1000, 800) = 16 - 4.8$

$= 11.2$

$\Delta P = P_x \Delta x + P_y \Delta y$

$= 10 \times 20 + 11.2 \times 25$

$= 200 + 280$

$= 480$

Sheet 9: Solutions

a) $f(x,y) = 3x^2 - 2xy + y^2 - 24y + 2$

$f_x(x,y) = 6x - 2y$

$f_y(x,y) = -2x + 2y - 24$

$f_{xx}(x,y) = 6$ $\qquad = -2.$

$f_{yy}(x,y) = 2$

We look for points where $f_x(x,y) = f_y(x,y) = 0$

So $6x - 2y = 0$

$\Rightarrow y = 3x$ and $f_y(x, 3x) = -2x + 2(3x) - 24$

$= 4x - 24 = 0$

$\Rightarrow x = 6$

and $y = 18$.

So stationary point $(6,18)$

$f_{xx}(6,18) = 6$ $\quad f_{yy}(6,18) = 2$ $\quad f_{xy}(6,18) = -2$.

Now $f_{xx} f_{yy} - (f_{xy})^2 = 6.2. - 4 = 8 > 0$

So $(6,18)$ is a minimum.

Sheet 9 : Solutions

(b) $F(x,y,z,\lambda) = 2x+y+2z+\lambda(x^2+y^2+z^2-9)$

$F_x = 2+2x\lambda = 0 \quad \Rightarrow \lambda = -2/2x = -1/x$

$F_y = 1+2y\lambda = 0 \quad \Rightarrow \lambda = -1/2y$

$F_z = 2+2z\lambda = 0 \quad \Rightarrow \lambda = -2/2z = -1/z$.

So $-\frac{1}{x} = -\frac{1}{z} \Rightarrow x = z$.

and $-1/2y = \frac{-1}{x} \Rightarrow y = \frac{x}{2}$

So $x^2 + \frac{x^2}{4} + x^2 = 9$

$\Rightarrow \frac{9x^2}{4} = 9$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2,\ y = \pm 1,\ z = \pm 2$.

Max. $2x+y+2z$ — we take positive values.

$(x,y,z) = (2,1,2)$.

Sheet 10: Solutions

$y = mx + d.$ data points $\left\{ \begin{array}{ll} (-1, 2.6) & (1, 2.0) \\ (0, 2.1) & (2, 1.3) \\ & (3, 1.0) \end{array} \right.$

normal equation for m and d.

$$m = \frac{n\left(\sum_{k=1}^{n} x_k y_k\right) - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right)}{n\left(\sum_{k=1}^{n} x_k^2\right) - \left(\sum_{k=1}^{n} x_k\right)^2}$$

$$d = \frac{\sum_{k=1}^{n} y_k - m\left(\sum_{k=1}^{n} x_k\right)}{n}$$

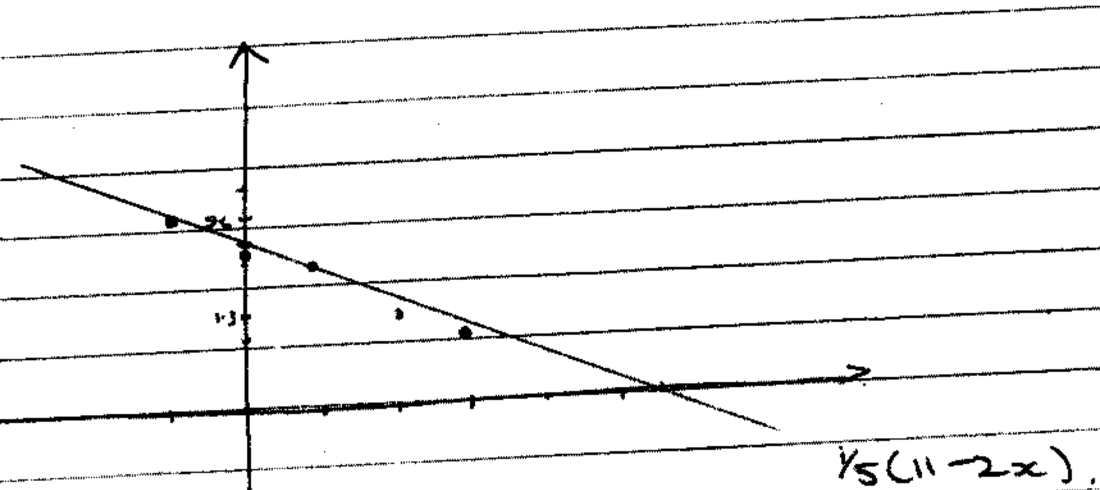
So $m = 5\left(-2.6 + 2 + 2.6 + 3\right) - (-1+0+1+2+3)(2.6+2.1+2.0+1.3+1)$

$$= \frac{5(5) - 5 \times 9}{5.(15) - 25} = \frac{-20}{50} = -2/5$$

$$d = \frac{9 - \left(-\frac{2}{5}\right)5}{5} = \frac{11}{5}$$

so the line which best fits the above points is

$$y = -\tfrac{2}{5}x + \tfrac{11}{5} = \tfrac{1}{5}(11 - 2x).$$



x	y	$ax + b$	Residual	$a = -\frac{2}{5}$ $b = \frac{11}{5}$
-1	2·6	$-a + b$	$2.6 + a - b$	0
1	2·0	$a + b$	$2 - a - b$	·2
0	2·1	b	$2.1 - b$	$-$·1
2	1·3	$2a + b$	$1.3 - 2a - b$	$-$·1
3	1·0	$3a + b$	$1.0 - 3a - b$	0

(So (1, 2·0) is furthest from the line
(−1, 2·6) and (3, 1·0) are on it.)