Questions for Problem Set 2

√Sheet 1

Find all solutions of the systems given below, using Gauss-Jordan elimination. At each step, indicate clearly the row operations which you perform.

(a)

$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &= -2.49 \\ -x_1 - 3x_2 - 2x_3 &= 3.48 \\ 3x_1 + x_2 + x_3 &= 3.73\,. \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &= -1 \\ -x_1 - 3x_2 - 2x_3 &= 1 \\ 2x_2 + x_3 &= -1\,. \end{aligned}$$

√Sheet 2

(a) Using a graphical method, solve the following linear programming problem.

Maximize $C = x_1 + x_2$

subject to

$$\begin{aligned} 3x_1 + x_2 &\le 6 \\ x_1 + 4x_2 &\le 8 \\ x_1 + x_2 &\ge 1 \\ x_1 \ge 0\,, x_2 &\ge 0\,. \end{aligned}$$

(b) A band is hired to perform at a fund-raising dinner. Between songs, a compere makes announcements about the fund-raising.

The band needs at least 100 minutes to perform all their songs, but can insert extra verses to some songs if necessary. The band can be present for a maximum of two hours. (What does this imply about the total possible time that the compere can talk?)

Suppose that the band plays for a total of x minutes ($x \ge 100$) and that the compere talks for a total of y minutes.

The organisers insist that the compere be allowed to speak for at least ten minutes in total ($y \ge 10$), in order to explain the purpose of the fund-raising.

The dinner patrons are known to find the band songs ten times more attractive than talk by the compere, and the total funds raised can be assumed to be \$500 $(10x + y)$.

What is the maximum possible amount of money raised? Write down a linear programming formulation of the problem, and sketch a feasible region in the x, y plane.

Sheet 3

Solve the following problem using the simplex method.

$$\text{Maximize} \quad f = -x_1 + x_2 + 2x_3$$

$$\text{subject to} \quad \begin{aligned} x_1 + 2x_2 - x_3 &\le 20 \\ -2x_1 + 4x_2 + 2x_3 &\le 60 \\ x_1 + 3x_2 + x_3 &\le 50 \end{aligned}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$$

At each step, indicate clearly which element is the pivot.

Sheet 4

(a) Write down the dual of the following problem.

$$\text{Minimize} \quad f = x_1 + x_2 + 12x_3$$

$$\text{subject to} \quad \begin{aligned} x_1 - x_2 + x_3 &\ge 3 \\ -x_1 + 2x_2 + 3x_3 &\ge 4 \end{aligned}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$$

The final simplex tableau for this dual is given below. The decision variables are denoted by y_1 and y_2, the slack variables by s_1, s_2 and s_3, and g is the objective function for the dual.

y_1	y_2	s_1	s_2	s_3	g	
1	0	2	1	0	0	3
0	1	1	1	0	0	2
0	0	–5	–4	1	0	3
0	0	10	7	0	1	17

Use this tableau to write down, without performing any simplex calculations, the optimal solution for the minimization problem above.

(b) Solve for x the equation: $6 + \log_e(x^3) = \log_e(64)$.

Sheet 5

(a) Differentiate:

(i) $x^5 e^{4x}$; (ii) $(2x^5 + e^{3x})^{13}$; (iii) $\dfrac{\log_e(3x)}{2x+5x^2+8x^3}$.

(b) A conical flask has an inside height of 20 cm and an inside base radius of 8 cm. The hole in the top is very small, so the inside of the flask may be taken to be a right circular cone. Water is dripped in through the small hole at a constant rate of 12 cm^3 per hour. How fast is the depth of water in the flask rising when the water is 9 cm deep?

[You may assume that the volume of a right circular cone of base radius r and height h is $\pi r^2 h / 3$.]

Sheet 6

8. (a) Find the slope of the curve

$$x + y = (y - 2)^4$$

at any point (x, y) on the curve.

Check that $(-2, 3)$ is on the curve, and find the equation of the tangent line to the curve at $(-2, 3)$.

(b) Find the Taylor polynomial of order 2 about $x = 0$ for the function

$$f(x) = (4 + x)^{1/2} ,$$

and write down a formula for the error if this polynomial is used to estimate $f(x)$ for small values of x . What is the largest possible error if the polynomial of order 2 is used to estimate $4.4^{1/2}$?

Estimate $4.4^{1/2}$, and check that the actual error is less than the the possible error that you calculated by the error formula.

Sheet 7

(a) Find the absolute maximum and minimum values of

$$f(x) = 2x^5 - 5x^4 + 1,$$

on the interval $1 \le x \le 3$.

(b) A rectangular holding pen of area $108\ m^2$ is to be constructed as follows. One side must be made with very strong material, costing \$150 per metre. The other three sides are to be made with lighter material which costs only \$30 per metre.

Find the dimensions of the holding pen for minimum cost.

Sheet 8

(a) If $z = x^3y^4 + 7x^4 + 3y^2 + y + x$, write down

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y} \text{ and } \frac{\partial^2 z}{\partial y \partial x}.$$

(b) A company distributes two types of shirts, type A and type B, to various shops. The company makes \$14 profit for each type A shirt and \$16 for each type B shirt distributed, except that owing to volume discounting, the total profit is reduced slightly. The total weekly profit when x type A and y type B shirts are distributed in that week is \$P, where

$$P(x, y) = 14x + 16y - 0.002x^2 - 0.003y^2.$$

(i) Calculate the weekly profit when $x = 1000$, $y = 800$.

(ii) Calculate $P_x(1000, 800)$ and $P_y(1000, 800)$.

In a particular week, the owners of one shop change their order, requiring 20 more type A and 25 more type B shirts (so that $x = 1020$ and $y = 825$). Using your answers from (ii) above, estimate the increase in weekly profit caused by this change.

Sheet 9

(a) Show that the function

$$f(x, y) = 3x^2 - 2xy + y^2 - 24y + 2$$

has just one stationary point. Locate it, and determine whether it is a local maximum, local minimum or saddle.

(b) Using a Lagrange multiplier method, find the maximum value of

$2x + y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

State the value of (x, y, z) for which the maximum occurs.

Sheet 10

A straight line $y = mx + d$ is to be fitted to the five data points:

$(-1, 2.6)$, $(0, 2.1)$, $(1, 2.0)$, $(2, 1.3)$ and $(3, 1.0)$.

For the least squares straight line which best fits the above points, write down the normal equations for m and d.

Using your equations above, or otherwise, find the least squares straight line of best fit to the five points.

Sketch the graph of the line of best fit, and plot the data points. State which data point is furthest from the line.