1. Use strict Gauss-Jordon elimination with augmented matrices on each of the systems of linear equations below to find the solution(s), if one or more exist, for \((x_1, x_2, x_3)\).

In the case of a single solution, your final matrix should allow you to answer the question without further algebraic manipulation; that is, your augmented matrix must be in Reduced Row Echelon form. If multiple solutions exist, state them all. If no solution exists, state why such a deduction can be made from your final matrix.

Show the row operations used at each step.

Check your proposed solutions.

(a)  
\[
\begin{align*}
2x_1 + 4x_2 &= 8 \\
x_2 + 6x_3 &= 12 \\
3x_1 + 9x_2 - 6x_3 &= 12 \\
5x_1 + 12x_2 + 12x_3 &= 50.
\end{align*}
\]

(b)  
\[
\begin{align*}
5x_1 + 10x_2 - 25x_3 &= -15 \\
2x_1 + 8x_2 - 6x_3 &= 12 \\
x_1 + 5x_3 &= 10.
\end{align*}
\]

SOLUTION:

(a) Write the augmented matrix

\[
\begin{bmatrix}
2 & 4 & 0 & | & 8 \\
0 & 1 & 6 & | & 12 \\
3 & 9 & -6 & | & 12 \\
5 & 12 & 12 & | & 50
\end{bmatrix}
\]

Divide \(R_1\) by 2

\[
\begin{bmatrix}
1 & 2 & 0 & | & 4 \\
0 & 1 & 6 & | & 12 \\
3 & 9 & -6 & | & 12 \\
5 & 12 & 12 & | & 50
\end{bmatrix}
\]
$R_3 - 3R_1 \rightarrow R'_3$ and $R_4 - 5R_1 \rightarrow R'_4$

\[
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 1 & 6 & 12 \\
0 & 3 & -6 & 0 \\
0 & 2 & 12 & 30 \\
\end{bmatrix}
\]

$R_1 - 2R_2 \rightarrow R'_1$ and $R_3 - 3R_2 \rightarrow R'_3$ and $R_4 - 2R_2 \rightarrow R'_4$

\[
\begin{bmatrix}
1 & 0 & -12 & -20 \\
0 & 1 & 6 & 12 \\
0 & 0 & -24 & -36 \\
0 & 0 & 0 & 6 \\
\end{bmatrix}
\]

There are **no real solutions** since $0x_1 + 0x_2 + 0x_3 \neq 6$ for any $(x_1, x_2, x_3) \in \mathbb{R}^3$.

(b) Write the augmented matrix

\[
\begin{bmatrix}
5 & 10 & -25 & -15 \\
2 & 8 & -6 & 12 \\
1 & 0 & 5 & 10 \\
\end{bmatrix}
\]

$R_1 \rightarrow R_2, \ R_2 \rightarrow R_3, \ R_3 \rightarrow R_1$

\[
\begin{bmatrix}
1 & 0 & 5 & 10 \\
5 & 10 & -25 & -15 \\
2 & 8 & -6 & 12 \\
\end{bmatrix}
\]

$R_2 - 5R_1 \rightarrow R'_2$ and $R_3 - 2R_1 \rightarrow R'_3$

\[
\begin{bmatrix}
1 & 0 & 5 & 10 \\
0 & 10 & -50 & -65 \\
0 & 8 & -16 & -8 \\
\end{bmatrix}
\]

Divide $R_2$ by 10 and $R_3$ by 8.

\[
\begin{bmatrix}
1 & 0 & 5 & 10 \\
0 & 1 & -5 & -6\frac{1}{2} \\
0 & 1 & -2 & -1 \\
\end{bmatrix}
\]

$R_3 - R_2 \rightarrow R'_3$.

\[
\begin{bmatrix}
1 & 0 & 5 & 10 \\
0 & 1 & -5 & -6\frac{1}{2} \\
0 & 0 & 3 & 5\frac{1}{2} \\
\end{bmatrix}
\]

Divide $R_3$ by 3.

\[
\begin{bmatrix}
1 & 0 & 5 & 10 \\
0 & 1 & -5 & -6\frac{1}{2} \\
0 & 0 & 1 & 1\frac{1}{6} \\
\end{bmatrix}
\]
\[ R_1 - 5R_3 \rightarrow R'_1 \text{ and } R_2 + 5R_3 \rightarrow R'_2 \]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{5}{6} \\
\frac{6}{6} \\
\frac{11}{6}
\end{bmatrix}
\]

So the solution is \((x_1, x_2, x_3) = \frac{1}{6}(5, 16, 11)\).

**Check your solution.**

\[
5\left(\frac{5}{6}\right) + 10\left(\frac{16}{6}\right) - 25\left(\frac{11}{6}\right) = \frac{1}{6}(25 + 160 - 275) = -15.
\]

\[
2\left(\frac{5}{6}\right) + 8\left(\frac{16}{6}\right) - 6\left(\frac{11}{6}\right) = \frac{1}{6}(10 + 128 - 66) = 12.
\]

\[
1\left(\frac{5}{6}\right) + 0\left(\frac{16}{6}\right) + 5\left(\frac{11}{6}\right) = \frac{1}{6}(5 + 55) = 10.
\]

2. (a) Solve for \(a, b, c\) and \(d\) in the matrix equation

\[
\begin{bmatrix}
a & -1 \\
6 & -3 \\
4 & d
\end{bmatrix}
\begin{bmatrix}
1 & b & -1 \\
2 & 3 & c
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(b) (i) Find the inverse of the square matrix

\[
\begin{bmatrix}
1 & 2 & 4 \\
2 & 1 & 4 \\
2 & 5 & 10
\end{bmatrix}
\]

using row reduction. At each step show clearly the row operation(s) that you perform.

Check that you have indeed found the inverse.

(ii) **Using the result from part** (b)(i), solve for \((x, y, z)\) in the following system of linear equations:

\[
\begin{align*}
2x + 4y + 8z &= 8 \\
2x + y + 4z &= 6 \\
6x + 15y + 30z &= 9
\end{align*}
\]

Check your proposed solution.

(20 marks)

**SOLUTION:**

(a) The product is

\[
\begin{bmatrix}
\begin{array}{ccc}
a - 2 & ab - 3 & -a - c \\
6 - 6 & 6b - 9 & -6 - 3c \\
4 + 2d & 4b + 3d & -4 + dc
\end{array}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Equating entries, gives \(a - 2 = 0 \Rightarrow a = 2\)

\(ab - 3 = 0 \Rightarrow (2)b - 3 = 0 \Rightarrow b = \frac{3}{2}\)

\(-a - c = 0 \Rightarrow -(2) - c = 0 \Rightarrow c = -2\)
$4b + 3d = 0 \Rightarrow 4\left(\frac{3}{2}\right) + 3d = 0 \Rightarrow d = -2$

So the proposed solution is $(a, b, c, d) = (2, \frac{3}{2}, -2, -2)$.

We can use the other equations to check our solutions.

$6b - 9 = 6\left(\frac{3}{2}\right) - 9 = 0.$
\[\begin{align*}
-6 - 3c &= -6 - 3(-2) = 0. \\
4 + 2d &= 4 + 2(-2) = 0. \\
-4 + dc &= -4 + (-2)(-2) = 0.
\end{align*}\]

So the solution is valid.

(b) Set up the augmented matrix

\[
\begin{pmatrix}
1 & 2 & 4 & 1 & 0 & 0 \\
2 & 1 & 4 & 0 & 1 & 0 \\
2 & 5 & 10 & 0 & 0 & 1
\end{pmatrix}
\]

$R_2 - 2R_1 \rightarrow R_2' \text{ and } R_3 - 2R_1 \rightarrow R_3'$

\[
\begin{pmatrix}
1 & 2 & 4 & 1 & 0 & 0 \\
0 & -3 & -4 & -2 & 1 & 0 \\
0 & 1 & 2 & -2 & 0 & 1
\end{pmatrix}
\]

$R_2 \rightarrow R_3' \text{ and } R_3 \rightarrow R_2'$

\[
\begin{pmatrix}
1 & 2 & 4 & 1 & 0 & 0 \\
0 & 1 & 2 & -2 & 0 & 1 \\
0 & -3 & -4 & -2 & 1 & 0
\end{pmatrix}
\]

$R_1 - 2R_2 \rightarrow R_1' \text{ and } R_3 + 3R_2 \rightarrow R_3'$

\[
\begin{pmatrix}
1 & 0 & 0 & 5 & 0 & -2 \\
0 & 1 & 2 & -2 & 0 & 1 \\
0 & 0 & 2 & -8 & 1 & 3
\end{pmatrix}
\]

$R_3$ divided by 2

\[
\begin{pmatrix}
1 & 0 & 0 & 5 & 0 & -2 \\
0 & 1 & 2 & -2 & 0 & 1 \\
0 & 0 & 1 & -4 & \frac{1}{2} & \frac{3}{2}
\end{pmatrix}
\]

$R_2 - 2R_3 \rightarrow R_2'$

\[
\begin{pmatrix}
1 & 0 & 0 & 5 & 0 & -2 \\
0 & 1 & 0 & 6 & -1 & -2 \\
0 & 0 & 1 & -4 & \frac{1}{2} & \frac{3}{2}
\end{pmatrix}
\]

The inverse is therefore

\[
\begin{pmatrix}
5 & 0 & -2 \\
6 & -1 & -2 \\
-4 & \frac{1}{2} & \frac{3}{2}
\end{pmatrix}
\]

Check solution:

\[
\begin{pmatrix}
5 & 0 & -2 \\
6 & -1 & -2 \\
-4 & \frac{1}{2} & \frac{3}{2}
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 4 \\
2 & 1 & 4 \\
2 & 5 & 10
\end{pmatrix}
= \begin{pmatrix}
5 + 0 - 4 & 10 + 0 - 10 & 20 + 0 - 20 \\
6 - 2 - 4 & 12 - 1 - 10 & 24 - 4 - 20 \\
-4 + 1 + 3 & -8 + \frac{1}{2} + \frac{15}{2} & -16 + 2 + 15
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
(b)(ii) We divide equation 1 by 2 and equation 3 by 3. So

\[ \begin{align*}
    x + 2y + 4z &= 4 \\
    2x + y + 4z &= 6 \\
    2x + 5y + 10z &= 3 .
\end{align*} \]

So the system is \( Ax = b \) where \( A \) is the matrix from part (i).

So \( A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b \). Hence

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & 0 & -2 \\ 6 & -1 & -2 \\ -4 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 12 \\ -8\frac{1}{2} \end{bmatrix}.
\]

They should check their answers:

\[ 14 + 2(12) + 4(-8\frac{1}{2}) = 4 \]
\[ 2(14) + 12 + 4(-8\frac{1}{2}) = 6 \]
\[ 2(14) + 5(12) + 10(-8\frac{1}{2}) = 8 . \]

So \((x, y, z) = (14, 12, -8\frac{1}{2})\).

3. (i) Verify that the point \((2, 2)\) lies on the curve

\[ 2e^{x-y}y + x^2y = 12 \]

and find the slope of the curve at \((2, 2)\).

You may assume that \(x, y \neq 0\).

(ii) Find the equation of the tangent to the curve at \((2, 2)\).

(SOLUTION:

(i) The equation is \(2e^{x-y}y + x^2y = 12\). Sub in \(x = 2, y = 2\) and check LHS = RHS:

\[ 2e^{2-2} \times 2 + 2^2 \times 2 = 2e^0 \times 2 + 8 = 4 + 8 = 12 \checkmark \]

Hence the point lies on the curve.

Find \( \frac{dy}{dx} \): differentiate both sides wrt \(x\):

\[ \frac{d(2e^{x-y}y)}{dx} + \frac{d(x^2y)}{dx} = \frac{d(12)}{dx} \]

\[
\left( e^x(-e^{-y}2y + 2e^{-y}) \frac{dy}{dx} + 2e^{x-y}y \right) + \left( x^2 \frac{dy}{dx} + 2xy \right) = 0
\]

\[
\frac{dy}{dx} (x^2 + 2e^{x-y} - 2ye^{x-y}) = -2xy - e^{x-y}2y
\]

\[
\frac{dy}{dx} = \frac{-2xy + e^{x-y}2y}{(x^2 + 2e^{x-y} - 2ye^{x-y})}
\]
Sub in \( x = 2, y = 2 \) and find \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \frac{-2(2)(2) - e^02(2)}{(2^2 + 2e^0 + 2(2)e^0)}
\]

\[
\frac{dy}{dx} = \frac{-4}{2} = -2
\]

Hence the slope of the curve at \((2, 2)\) is \(-2\).

(ii) The equation of the tangent is: \( y = -2x + c \). Use the point to determine \( c \)

\[
2 = -2 \times 2 + c
\]

so \( c = 6 \) and the equation of the tangent is \( y = 6 - 2x \).

4. Find an antiderivative, \( F(x) \), for each of the following functions:

(a) \[
f(x) = \frac{e^x}{e^{2x} - 1}
\]

(b) \[
f(x) = \frac{x - 5}{x^2 - 6x + 9}
\]

(c) \[
f(x) = 2x^3e^{x^2}
\]

(d) \[
f(x) = \frac{2}{x} \log_e(x)
\]

(20 marks)

SOLUTION:

(a) This is a substitution and partial fractions

\[
F(x) = \int \frac{e^x}{e^{2x} - 1} \, dx
\]

Put \( e^x = u \). Then \( \frac{du}{dx} = e^x \). So

\[
\int \frac{e^x}{e^{2x} - 1} \, dx = \int \frac{1}{u^2 - 1} \, du = \int \frac{1}{u - 1} \, du - \int \frac{1}{u + 1} \, du
\]

\[
\frac{1}{(u - 1)(u + 1)} = \frac{A}{u - 1} + \frac{B}{u + 1}
\]

Hence \( A + B = 0 \) and \( A - B = 1 \) and so \( (A, B) = (\frac{1}{2}, -\frac{1}{2}) \).
So integral is

\[
\frac{1}{2} \left( \log_e |u - 1| - \log_e |u + 1| \right) + c
\]

Either of last two is fine.
(b) This is partial fractions.

\[
\frac{x - 5}{x^2 - 6x + 9} = \frac{x - 5}{(x - 3)^2}
\]

\[
= \frac{A}{x - 3} + \frac{B}{(x - 3)^2}
\]

Therefore \(A(x - 3) + B = x - 5\). So \(A = 1\) and \(B - 3A = -5\) \(\Rightarrow\) \(B = -2\).

Hence

\[
\int \frac{x - 5}{x^2 - 6x + 9} \, dx = \int \frac{1}{x - 3} - \frac{2}{(x - 3)^2} \, dx
\]

\[
= \log_e |x - 3| + \frac{2}{x - 3} + c
\]

(c) This is integration by parts. Put \(u = x^2\) and \(\frac{dv}{dx} = 2xe^{x^2}\). Then \(\frac{du}{dx} = 2x\) and \(v = e^{x^2}\). So

\[
\int 2x^3e^{x^2} \, dx = \int x^2 \cdot 2xe^{x^2} \, dx
\]

\[
= x^2 e^{x^2} - \int e^{x^2} \cdot 2x \, dx
\]

\[
= x^2 e^{x^2} - \int \frac{dv}{dx} \, dx
\]

\[
= x^2 e^{x^2} - e^{x^2} + c
\]

(d) This is a substitution. Put \(u = \log_e (x)\) so that \(\frac{du}{dx} = \frac{1}{x}\). So

\[
\int f(x) \, dx = \int \frac{2}{x} \log_e (x) \, dx
\]

\[
= 2 \int \frac{du}{dx} \, dx
\]

\[
= 2 \left( \frac{u^2}{2} \right) + c
\]

\[
= (\log_e (x))^2 + c
\]

5. Use the simplex method to solve the following standard maximum problem.

Maximise \(P = 3x_1 + 2x_2\)

subject to

\[
\begin{align*}
6x_1 + 8x_2 - 12x_3 &\leq 36 \\
3x_1 - 2x_2 + 2x_3 &\leq 10 \\
3x_1 + 6x_2 - 2x_3 &\leq 15
\end{align*}
\]
with \( x_1 \geq 0, x_2 \geq 0 \) and \( x_3 \geq 0 \).

At each step show clearly the row operation(s) that you perform and clearly circle the pivot element. You should state the maximum possible value of \( P \) and all the values of \((x_1, x_2, x_3)\) for which this maximum occurs. \( \text{(20 marks)} \)

**SOLUTION:**

(i) Introduce slack variables, \( s_1, s_2, s_3 \geq 0 \):

\[
\begin{align*}
6x_1 + 8x_2 - 12x_3 + s_1 &= 36 \\
3x_1 - 2x_2 + 2x_3 + s_2 &= 10 \\
3x_1 + 6x_2 - 2x_3 + s_3 &= 15
\end{align*}
\]

Translate into a tableau:

<table>
<thead>
<tr>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( P )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
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<td>( s_1 )</td>
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<td>10</td>
</tr>
<tr>
<td>( s_3 )</td>
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<td>6</td>
<td>-2</td>
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<td>0</td>
<td>1</td>
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<td>15</td>
</tr>
<tr>
<td>( P )</td>
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<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

(ii) The pivot entry is column 1 row 2.

Divide \( R_2 \) by 3. \( R_1 - 6R_2, R_3 - 3R_2, R_4 + 3R_2 \).

<table>
<thead>
<tr>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( P )</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>6</td>
<td>8</td>
<td>-12</td>
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<td>0</td>
<td>0</td>
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<td>6</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>3</td>
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<td>2</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>( \circ )</td>
</tr>
</tbody>
</table>

The pivot entry is column 2 row 3.

Divide \( R_3 \) by 8. \( R_1 - 12R_3, R_2 + \frac{2}{3}R_2, R_4 + 4R_2 \).

<table>
<thead>
<tr>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( P )</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
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<td>12</td>
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<td>16</td>
<td>( \frac{16}{2} )</td>
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<tr>
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<td>( \frac{1}{3} )</td>
<td>0</td>
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<td>( \frac{10}{3} )</td>
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</tr>
<tr>
<td>( s_3 )</td>
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<td>-4</td>
<td>0</td>
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<td>1</td>
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<td>( \frac{5}{5} )</td>
</tr>
<tr>
<td>( P )</td>
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<td>-4</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>10</td>
<td>( \circ )</td>
</tr>
</tbody>
</table>

There is a non-unique solution since \( x_3 \) is non-basic but it has 0 reduced cost.

<table>
<thead>
<tr>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( P )</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>-10</td>
<td>1</td>
<td>-( \frac{1}{2} )</td>
<td>-( \frac{3}{2} )</td>
<td>0</td>
<td>( \frac{2}{2} )</td>
<td>( \circ )</td>
</tr>
<tr>
<td>( x_1 )</td>
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<td>0</td>
<td>( \frac{1}{3} )</td>
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<td>( \frac{1}{4} )</td>
<td>( \frac{1}{12} )</td>
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<td>( \frac{15}{4} )</td>
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<tr>
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<td>1</td>
<td>-( \frac{1}{2} )</td>
<td>0</td>
<td>-( \frac{1}{5} )</td>
<td>( \frac{1}{5} )</td>
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<td>( \frac{5}{5} )</td>
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</tr>
<tr>
<td>( P )</td>
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<td>0</td>
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<td>( \frac{1}{2} )</td>
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<td>1</td>
<td>( \frac{25}{2} )</td>
<td>( \circ )</td>
<td></td>
</tr>
</tbody>
</table>
METHOD 1

First way of solution is to pivot again.
The pivot entry is column 3 row 2.
Multiply $R_2$ by 3. $R_1 + 10R_2$. $R_3 + \frac{1}{2}R_2$.

<table>
<thead>
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<th>$BV$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$P$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
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<td>7</td>
<td>1</td>
<td>0</td>
<td>121</td>
</tr>
<tr>
<td>$x_3$</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>$\frac{45}{4}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>$\frac{25}{4}$</td>
</tr>
<tr>
<td>$P$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{25}{4}$</td>
</tr>
</tbody>
</table>

We can now write down the solution.
Solution (a): $(x_1, x_2, x_3) = (\frac{15}{4}, \frac{5}{8}, 0)$
Solution (b): $(x_1, x_2, x_3) = (0, \frac{25}{4}, \frac{45}{4})$

So all solutions are given as a convex combination of these two.

$$t\left(\frac{15}{4}, \frac{5}{8}, 0\right) + (1-t)(0, \frac{25}{4}, \frac{45}{4}), \quad t \in [0, 1]$$

Now

$$t\left(\frac{15}{4}, \frac{5}{8}, 0\right) + (1-t)(0, \frac{25}{4}, \frac{45}{4})$$

$$= \left(\frac{15}{4}t, \frac{5}{8} + \frac{25}{4} - \frac{25}{4}t, \frac{45}{4} - \frac{45}{4}t\right)$$

$$= \frac{1}{8}(30t, 50 - 45t, 90 - 90t)$$

So $P^* = 25/2$ for $\left\{\frac{1}{8}(30t, 50 - 45t, 90 - 90t) : t \in [0, 1]\right\}$.

Check

$$P = 3 \left(\frac{15t}{4}\right) + 2 \left(\frac{25}{4} - 45t/8\right) = 45t/4 + 25/2 - 45t/4 = 25/2....OK$$

$$6 \left(\frac{15t}{4}\right) + 8 \left(\frac{25}{4} - 45t/8\right) - 12 \left(\frac{45}{4} - 45t/4\right)$$

$$= 45t + 50/4 - 45t - 135 + 135t = 135t - 245/2 \leq 36....OK$$

$$3 \left(\frac{15t}{4}\right) - 2 \left(\frac{25}{4} - 45t/8\right) + 2 \left(\frac{45}{4} - 45t/4\right)$$

$$= 45t/4 - 25/2 + 45t/4 + 45/2 - 45t/2 = 10....OK$$

$$3 \left(\frac{15t}{4}\right) + 6 \left(\frac{25}{4} - 45t/8\right) - 2 \left(\frac{45}{4} - 45t/4\right)$$

$$= 45t/4 + 75/2 - 135t/4 - 45/2 + 45t/2 = 15....OK$$

METHOD 2
From the tableau we have:

\[ P = \frac{25}{2} - \frac{1}{2} s_2 - \frac{1}{2} s_3 \]

Hence we must have \( s_2 \) and \( s_3 \) equal to zero. Given this the other equations are

\[
\begin{align*}
-10x_1 + s_1 &= \frac{17}{2} \\
x_1 + \frac{1}{3} x_3 &= \frac{15}{4} \\
x_2 - \frac{1}{2} x_3 &= \frac{5}{8}
\end{align*}
\]

Put \( x_3 = t \in \mathbb{R}^+ \). Then

\[
\begin{align*}
x_1 &= \frac{15}{4} - \frac{1}{3} t \\
x_2 &= \frac{5}{8} + \frac{1}{2} t
\end{align*}
\]

And

\[ s_1 = \frac{17}{2} + 10t \geq 0 \]

This means that \( t \in [-\frac{17}{20}, \frac{45}{4}] \) So maximum is \( P = \frac{25}{2} \) at

\[ (x_1, x_2, x_3) = \left( \frac{15}{4} - \frac{t}{3}, \frac{5}{8} + \frac{t}{2}, t \right) \text{ with } t \in [-\frac{17}{20}, \frac{45}{4}] \]

Check

\[
P = 3 \left( \frac{15}{4} - \frac{t}{3} \right) + 2 \left( \frac{5}{8} + \frac{t}{2} \right) = 45/4 - t + 5/4 + t = 25/2 \ldots OK
\]

\[
6 \left( \frac{15}{4} - \frac{t}{3} \right) + 8 \left( \frac{5}{8} + \frac{t}{2} \right) - 12t = 60/4 - 2t + 5 + 4t - 12t = 20 - 10t \leq 36 \ldots OK
\]

\[
3 \left( \frac{15}{4} - \frac{t}{3} \right) - 2 \left( \frac{5}{8} + \frac{t}{2} \right) + 2t = 45/4 - t - 5/4 - t + 2t = 10 \ldots OK
\]

\[
3 \left( \frac{15}{4} - \frac{t}{3} \right) + 6 \left( \frac{5}{8} + \frac{t}{2} \right) - 2t = 45/4 - t + 15/4 + 3t - 2t = 15 \ldots OK
\]

6. (a) Solve graphically the following non-standard linear programming problem: Draw a graph with feasible region clearly marked and with all its corner points calculated. Write down all basic feasible solutions and the value of \( F \) at each of those solutions. Hence write down the values of \((x, y)\) for which \( F \) takes its minimum value.

Minimise \( F = y - 5x \)
subject to \[ \begin{align*} x + y & \geq 3 \\ -7x + 10y & \leq 30 \\ 4x - y & \leq 48 \\ 2x + y & \leq 30 \end{align*} \]

with \( x \geq 0 \) and \( y \geq 0 \).

(b) What is the solution to the linear programming problem with the same constraints as in part (a) but now one is required to find the maximum value of the same objective function \( F = y - 5x \) subject to those constraints?

(10 marks)

**SOLUTION:**

(a) Plot the feasible region:

- \(-7x + 10y = 30\) cuts the y-axis at \((0, 3)\)
- \(4x - y = 48\) cuts the x-axis at \((12, 0)\)
- \(x + y = 3\) cuts the x-axis at \((3, 0)\) and the y-axis at \((0, 3)\)
- \(2x + y = 30\) cuts the x-axis at \((15, 0)\)

To find the other corner points of the feasible region we need to calculate the intersection of the lines \(-7x + 10y = 30\) and \(2x + y = 30\) giving the point \((10, 10)\) and \(4x - y = 48\) and \(2x + y = 30\) giving the point \((13, 4)\).

So the plot is:

The corner points of the feasible region are

<table>
<thead>
<tr>
<th>corner</th>
<th>( F = y - 5x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0, 3) )</td>
<td>3</td>
</tr>
<tr>
<td>( (3, 0) )</td>
<td>-15</td>
</tr>
<tr>
<td>( (12, 0) )</td>
<td>-60</td>
</tr>
<tr>
<td>( (13, 4) )</td>
<td>-61</td>
</tr>
<tr>
<td>( (10, 10) )</td>
<td>-40</td>
</tr>
</tbody>
</table>

So the minimum is \( F = -61 \) at \((13, 4)\) and the maximum is \( F = 3 \) at \((0, 3)\).

(b) The maximum is \( F = 3 \) at \((0, 3)\).
7. (a) A cone is hanging from the ceiling so that solution is leaking from the tip. This causes the volume of solution in the cone to decrease at the rate of 15 cm³/min. How fast is the fluid level decreasing when the radius is 6 cm?

You may assume that the volume of the cone is $V = \frac{1}{3}\pi r^2 h$ where $h$ is the height of the fluid and $r$ is the radius.

You should leave your solution as a fraction.

(b) If the height of the cone is 24 cm and the radius is 10 cm, how long does it take for the volume of solution in the cone to decrease to 185π cm³? You may assume that the cone is full at time $t = 0$.

(15 marks)

SOLUTION: Two Versions of the answer

(a) Let $V$ be volume in cm³ and let $h$ be the height in cm.

We want $\frac{dh}{dt}$ when $r = 6$.

We know that $\frac{dV}{dt} = -15$.

Let $R$ be the base radius and $H$ be the height of the cone. Then by similar triangles we have:

$$\frac{H}{R} = \frac{h}{r}$$

where $h$ is the height of the fluid at time $t$ and $r$ is the radius at time $t$. Then $r = \frac{R}{H} h$ and $V = \frac{1}{3}\pi \left(\frac{R}{H} h\right)^2 h$. So

$$\frac{dV}{dh} = \pi \frac{R^2}{H^2} h^2 = \pi r^2.$$

At $r = 6$ we have $\frac{dV}{dh} = \pi r^2 = 36\pi$. So

$$\frac{dV}{dt}\big|_{r=6} = \frac{dV}{dh} \cdot \frac{dh}{dt}\big|_{r=6} \Rightarrow -15 = 36\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{15}{36\pi}.$$

ALTERNATIVELY

They could use $H = 24$ and $R = 10$ from part (b). So in the previous solution, $r = \frac{5}{12} h$ so $V = \frac{1}{3}\pi \left(\frac{55}{144}\right) h^3$ but $\frac{dV}{dt}$ still equals $36\pi$ and so the answer is the same.

(b) $V = \frac{1}{3}\pi 100 \times 24 = 800\pi$.

$V(t) = 800\pi - 15t$. When $V = 185\pi$ we have $15t = 800\pi - 185\pi = 615\pi$ so $t = 41\pi$.

8. (a) Solve the initial value problem

$$x \frac{dy}{dx} = x^3 \cos(x) + 2y \quad \text{and} \quad y\left(\frac{\pi}{2}\right) = \pi^2, \ x > 0.$$

You should write your solution for $y$ as a function of $x$.

Show all intermediate steps in your solution of the differential equation.
(b) Solve the initial value problem
\[ \frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}}, \quad \text{and} \quad y(0) = 0. \]
Show all intermediate steps in your solution of the differential equation. You should clearly state any restrictions on the values of \(x\).

(20 marks)

**SOLUTION:**

(a) This is an integrating factor problem.

\[ x \frac{dy}{dx} - 2y = x^3 \cos(x) \]
\[ \frac{dy}{dx} - \frac{2}{x}y = x^2 \cos(x) \]

So the integrating factor is:

\[ I(x) = \exp \left( -2 \int \frac{1}{x} \, dx \right) \]
\[ = \exp (-2 \log_e(x)) \]
\[ = \frac{1}{x^2} \]

Hence

\[ \frac{1}{x^2}y = \int \cos(x) \, dx \]
\[ \Rightarrow y = x^2 \sin(x) + c \]

Now using the initial value

\[ y \left( \frac{\pi}{2} \right) = \frac{\pi^2}{4} \sin \left( \frac{\pi}{2} \right) + c = \pi^2 \]
\[ \Rightarrow c = \pi^2 - \frac{\pi^2}{4} = \frac{3\pi^2}{4} \]

So \(y = x^2 \sin(x) + \frac{3\pi^2}{4} \).

(b)

\[ \frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} = \frac{e^x}{e^{2y}} \]

So

\[ \int e^{2y} \, dy = \int e^x \, dx \]
\[ \Rightarrow \frac{1}{2} e^{2y} = e^x + c \]
\[ \Rightarrow e^{2y} = 2e^x + 2c \]
\[ \Rightarrow 2y = \log_e(2e^x + 2c) \]
\[ \Rightarrow y = \log_e(2e^x + 2c)^{\frac{1}{2}} \]
Now

\[ y(0) = \log_e(2 + 2c)^{\frac{1}{2}} = 0 \]
\[ \Rightarrow (2 + 2c)^{\frac{1}{2}} = 1 \]
\[ \Rightarrow 2 + 2c = 1 \]
\[ \Rightarrow 2c = -1 \]
\[ \Rightarrow c = -\frac{1}{2} \]

Hence

\[ y = \log_e(2^{x} - 1)^{\frac{1}{2}}. \]

The restriction is \( x > \log_e \frac{1}{2} \) so that the log term is defined.

9. Consider the initial value problem

\[ \frac{dy}{dx} = 2y^2(x - 1), \quad \text{with} \quad y = -\frac{1}{2} \quad \text{when} \quad x = 2. \]

(a) (i) Find the Taylor polynomial of degree 3 for \( y \) about \( x = 2 \).

(ii) Use part (i) to find an approximate value for \( y \) when \( x = 2.2 \).

(b) Use Euler’s method to approximate \( y \) when \( x = 2.2 \) given that \( x_0 = 2, \ y_0 = -0.5 \) and \( \Delta x = 0.1 \).

(c) (i) Verify that

\[ y = \frac{1}{2x - x^2 - 2} \]

is the solution to the initial value problem.

(ii) Which of the two approximations that you have found above, gives the best approximation to the actual value of \( y \) when \( x = 2.2 \)?

(20 marks)

SOLUTION:

(a) (i) We have

At \( x = 2 \), \( y(2) = -\frac{1}{2} \).

\[ y'(x) = 2y^2(x - 1) \]

Sub \( x = 2, y = -\frac{1}{2} \) into the DE

\[ y'(2) = 2y(2)^2(2 - 1) = \frac{1}{2} \]
Differentiate the DE:
\[ y''(x) = 2y^2 + 4y(x-1)y'(x) \]
So when \( x = 2, y = -\frac{1}{2}, y'(2) = \frac{1}{2} \) we have
\[ y''(2) = 2\left(\frac{1}{4}\right) + 4\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2} \]

Differentiate again:
\[ y'''(x) = 4yy'(x) + 4yy'(x) + (x-1)(4y'(x)^2 + 4yy''(x)) \]
So at \( x = 2, y = -\frac{1}{2}, y'(2) = \frac{1}{2} \) and \( y''(2) = -\frac{1}{2} \) we have:
\[ y'''(2) = 4\left(-\frac{1}{2}\right)(\frac{1}{2}) + 4(-\frac{1}{2})(\frac{1}{2}) + 4\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 0 \]
Hence the Taylor polynomial around \( x = 2 \) is
\[ p_3(x) = y(2) + y'(2)(x-2) + \frac{1}{2}y''(2)(x-2)^2 + \frac{1}{6}y'''(2)(x-2)^3 \]
\[ = -\frac{1}{2} + \frac{1}{2}(x-2) - \frac{1}{4}(x-2)^2 \]
(ii) Sub \( x = 2.2 \) into the Taylor polynomial. This gives
\[ p_3(2.2) \approx -0.5 + 0.5 \times 0.2 - 0.5 \times 0.02 = -0.41 \]

(b) We are given the IVP
\[ \frac{dy}{dx} = 2y^2(x-1) \quad y(2) = -\frac{1}{2} \]
We use the Euler method to approximate the solution of this IVP at \( x = 2.2 \) with \( x_0 = 2, y_0 = -0.5 \) and \( \Delta x = 0.1 \).
Iterating this procedure gives the recurrence
\[ y_{n+1} = y_n + f(x_n, y_n)\Delta x \quad \text{and} \quad x_{n+1} = x_n + \Delta x \]
So
\[ n = 0, x_0 = 2, y_0 = -\frac{1}{2} \quad f(x_0, y_0) = 2\left(-\frac{1}{2}\right)^2(2-1) = 0.5 \quad y_1 = -\frac{1}{2} + 0.1 \times 0.5 = -0.45 \]
\[ n = 1, x_1 = 2.1, y_1 = -0.45 \quad f(x_1, y_1) = 2(-0.45)^2(2.1 - 1) = 0.4455 \quad y_2 = -0.45 + 0.1 \times 0.4455 = -0.40545 \]
\[ n = 2, x_2 = 2.2, y_2 = -0.40545 \]
(c) (i) $y(2) = \frac{1}{2.2 - 2^2} = -\frac{1}{2}$. So the initial condition is valid.

$$\frac{dy}{dx} = -\frac{1}{(2x - x^2 - 2)^2 \times (2 - 2x)}$$

$$= \frac{2(x - 1)}{(2x - x^2 - 2)^2}$$

$$= 2 \left( \frac{1}{2x - x^2 - 2} \right)^2 \times (x - 1)$$

$$= 2y^2(x - 1)$$

So it is a solution.

(ii) $y(2.2) = \frac{1}{4.4 - (2.2)^2 - 2} = \frac{1}{4.4 - 4.84} = \approx -0.40984$.

$p_3(2.2) = -0.41$

$y(2.2) = -0.40984$

$y_2 = -0.40545$

So

$$y(2.2) - y_2 = 0.00439$$

$$y(2.2) - p_3(2.2) = 0.00016$$

$y(2.2) - y_2 = .00439$, $y(2.2) - p_3(2.2) = .00016$.

So the Taylor approximation is best.

10. The natural change in the population of $p$ thousand lemmings in a colony after $t$ years is modelled by a logistic equation which is given by the differential equation

$$\frac{dp}{dt} = 6p - 5p^2.$$ 

(a) Find the population after $t$ years have elapsed if the initial population is 2 thousand. Show all intermediate steps in your solution of the differential equation.

(b) What happens to the population after a long time?

(c) Identify any points of equilibrium and decide if each is stable or unstable. You should draw a phase line to help you with your analysis.

(d) Suppose that we now add a harvesting term to the above equation so that the differential equation is now

$$\frac{dp}{dt} = 6p - 5p^2 - \gamma$$

where $\gamma > 0$. Find the minimum value for $\gamma$ such that if $\gamma$ is increased from this value, the population will die out regardless of the initial population.

SOLUTION:
(a) The DE is separable:

\[ \frac{dp}{dt} = 6p - 5p^2. \]

and \( p(0) = 2 \). Integrate both sides wrt \( t \)

\[ \int \frac{1}{p(6 - 5p)} \, dp = \int \frac{1}{dt} = \int 1 \, dt \]

\[ \int \frac{1}{p(6 - 5p)} \, dp = t + c \]

The LHS is a partial fractions integral.

\[ \frac{1}{p(6 - 5p)} = \frac{A}{p} + \frac{B}{6 - 5p} = \frac{6A - 5Ap + Bp}{p(6 - 5p)} \]

So \( 6A = 1 \) and \( B - 5A = 0 \) which gives \( A = \frac{1}{6} \) and \( B = \frac{5}{6} \).

Our integral can be rewritten:

\[ \int \frac{1}{p(6 - 5p)} \, dp = \frac{1}{6} \int \frac{1}{p} \, dp + \frac{1}{6} \int \frac{5}{6 - 5p} \, dp \]

\[ = \frac{1}{6} (\log_e |p| - \log_e |6 - 5p|) \]

So our DE is now

\[ \frac{1}{6} (\log_e |p| - \log_e |6 - 5p|) = t + c \]

\[ \log_e |p| - \log_e |6 - 5p| = 6t + 6c \]

\[ \left| \frac{p}{6 - 5p} \right| = e^{6c} e^{6t} \]

\[ \frac{p}{6 - 5p} = \pm e^{6c} e^{6t} \]

We can either impose the initial condition now, or after we isolate \( p \).

\[ \frac{2}{6 - 10} = \pm e^{6c} \]

So \( \pm e^{6c} = -\frac{1}{2} \), and we now isolate \( p \):

\[ \frac{p}{6 - 5p} = -\frac{1}{2} e^{6t} \]

\[ p = -3e^{6t} + 2.5pe^{6t} \]

\[ p(1 - 2.5e^{6t}) = -3e^{6t} \]

\[ p = \frac{3e^{6t}}{2.5e^{6t} - 1} = \frac{6}{5 - 2e^{-6t}} \]

Either of the last 2 lines will do.
(b) As \( t \to \infty \), \( e^{-6t} \to 0 \) and so \( p(t) \to 6/5 \).

(c) We consider \( 6p - 5p^2 = 0 \). So \( p = 0 \) or \( p = 6/5 \).

At \( p = 1 \), \( \frac{dp}{dt} = 1 \) and at \( p = 2 \), \( \frac{dp}{dt} = -8 \) so we have

\( p = 0 \) is unstable, \( p = 6/5 \) is stable.

(d) Add a harvesting term to the DE:

\[
\frac{dp}{dt} = 6p - 5p^2 - \gamma
\]

We find equilibrium points by setting \( 6p - 5p^2 - \gamma = 0 \). Now solving for \( p \) we have \( 3 \pm 2\sqrt{9 - 5\gamma} \). When \( \gamma \geq 9/5 \), \( \frac{dp}{dt} \leq 0 \), but when \( \gamma < 9/5 \), \( \frac{dp}{dt} \geq 0 \) from \( 3 - 2\sqrt{9 - 5\gamma} \) to \( 3 + 2\sqrt{9 - 5\gamma} \). So the minimum value of \( \gamma \) is \( 9/5 \).

11. A tank \( X \) contains 50 litres of saline solution (salt dissolved in water) which initially contains 25 kg of salt. A second tank \( Y \) contains 100 litres of saline solution which initially contains 25 kg of salt. The solutions in both tanks are kept uniform by constant mixing.

Pure water flows into tank \( X \) at a rate of 3 litres per minute and saline solution flows from tank \( X \) into tank \( Y \) at a rate of 4 litres per minute. One litre per minute of the mixture in tank \( Y \) flows from tank \( Y \) back into tank \( X \) whilst 4 litres per minute of the solution in tank \( Y \) is allowed to flow out of the tank.

Derive a pair of differential equations which give the number of kgs of salt in tank \( X \), \( (x(t)) \), and the number of kgs of salt in tank \( Y \), \( (y(t)) \), at time \( t \). Your solution should include a diagram, you should carefully explain any variables that you use. You should also write down the initial conditions for each tank.

YOU ARE NOT REQUIRED TO SOLVE THESE DIFFERENTIAL EQUATIONS.

Solution

Let \( x(t) \) be concentration in tank \( X \) and \( y(t) \) be concentration in tank \( Y \).

\[
\frac{dx}{dt} = \text{inflow} - \text{outflow}
\]

Inflow is \( \frac{y(t)}{100-t} \) Outflow is \( \frac{4x(t)}{50} \)

So \( \frac{dx}{dt} = \frac{y(t)}{100-t} - \frac{4x(t)}{50} \)

\[
\frac{dy}{dt} = \text{inflow} - \text{outflow}
\]

Inflow is \( \frac{4x(t)}{50} \) Outflow is \( \frac{4y(t)}{100-t} \)

So \( \frac{dy}{dt} = \frac{4x(t)}{50} - \frac{4y(t)}{100-t} \)