Example (continued)....

We will rewrite the problem constraints as an underdetermined system of equations by introducing a new variable into each equation:

Thus

\[
\begin{align*}
4x_1 + 2x_2 & \leq 2000 \\
2x_1 + 3x_2 & \leq 1500
\end{align*}
\]

becomes

We call \( s_1 \) and \( s_2 \) slack variables, since they absorb the "slack" between the LHS & RHS of the inequalities.

Since LHS "\( \leq \)" RHS, the slack variables must be non-negative. Hence our new system of constraints becomes:

Now let's recalculate the intersections of our constraints but this time we incorporate \( s_1, s_2, s_3, s_4 \).

To do this, we set 2 variables at a time to zero, and calculate the other two.

\((x_1, x_2, s_1, s_2)\)
What do we notice?

1. We have calculated the intersections of our constraints as before. These solutions are called basic solutions.

2. Some of our values are negative – this violates the non-negativity conditions. They correspond to points outside of the feasible region.

Points with non-negative values are called basic feasible solutions – these are the ones that we are interested in.

Points with negative co-ordinates (at least one) are called basic infeasible solutions and we can disregard them.

So now to solve the LP problem, we have a way to list the intersection points and disregard those that are not feasible. We can then evaluate P at the basic feasible solutions to find $P^* \rightarrow$ no graph required!

**Problem**: Even though we now have a purely algebraic way of dealing with an LP problem – which will work in any dimension – the “size” quickly becomes impractical.

For example, a problem with 3 constraints and 2 decision variables has $4C_2 = 6$ basic solutions. For a problem with 5 constraints and 4 decision variables we have $9C_4 = 126$ basic solutions

(we take $n = \# $ variables (decision + slack) – $m = \# $ equations)

- many of these points will be infeasible.
The Simplex Method.

It is an efficient way of computing the optimal solution - it computes less than 2m basic solutions - all of them feasible.

Example continued -------
We rewrite the problem as

Here we treat the objective function as a constraint.
We move from one basic feasible solution to another, increasing the value of P as we go.
We stop when there are no more such basic feasible solutions.

This is, we have found the optimal solution.

To initiate the algorithm we need a starting basic feasible solution.

Flow Chart:

1. Initial basic feasible solution
2. Method to find another basic feasible solution
3. Method to find a better basic feasible solution
4. Test to see if optimal

IDEA: Interchange basic & non-basic variables to increase P.

The first step is to setup a Simplex Tableau.

So our initial basic feasible solution is

We want to interchange one basic variable with one non-basic variable in such a way that we increase P as much as possible.
For every unit increase in \( x_2 \), we increase \( P \) by 5.

For every unit increase in \( z_2 \), we increase \( P \) by 6.

This suggests that we should choose \( z_2 \) to become a basic variable. We say \( z_2 \) enters the basis.

Now we have to choose which basic variable (\( z_2 \) or \( x_2 \)) will leave the basis.

The second constraint is most restrictive, so we choose \( x_2 \) to leave the basis.

We have moved from one basic feasible solution to another in increasing \( P \) along the way.

How can we incorporate this information into our Simplex Tableau?

<table>
<thead>
<tr>
<th>BV</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( P )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2000</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1500</td>
</tr>
<tr>
<td>( P )</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

We now perform a pivot operation to move from one basic solution to the next.

The element circled is called the pivot element. Our first task is to make this a "1".
We can now read off the new basic solution:

So far so good, but what do we do now?
The bottom row says

We know that $x_1$ and $s_2$ are currently non-basic variables ($=0$). To increase $P$, we need to increase $x_1$ and leave $s_2 = 0$. This suggests that $s_1$ should enter the basis.

Let's examine the two problem constraints.

Keeping $s_2 = 0$ we have

1. \[\text{We want to increase } x_1 \text{ and keep } s_1 \text{ and } x_2 \text{ non-negative}\]

Equation 1. is most restrictive so we choose $s_1$ to leave the basis. So our new basic solution is:

The greedy rule chooses column 1 to be the pivot column, the ratio test chooses row 1 to be the pivot row -- so the pivot element is $3\frac{1}{2}$. We now update the tableau again:

<table>
<thead>
<tr>
<th>BV</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$P$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Pivot operation:

The bottom row now says: