Lecture 20:

Numerical Solutions of DE's.

→ Euler’s Method.

A differential equation
\[
\frac{dy}{dx} = x - y
\]
is valid for all \(x, y \in \mathbb{R}\) and has many solutions.
To get a "global picture" of the DE we use a slope field.

→ A picture of this function (perhaps an implicit function) is a curve in the plane.

Numerical Methods.

(a) Euler’s Method.

We start with an initial value problem (IVP)

and require a numerical procedure for constructing an approximate solution to it. The solution is a curve which goes through the point \((x_0, y_0)\).
Euler Method Idea:
Use tangent line as an approximate solution over a short distance \( \Delta x \).

Then calculate tangent line and repeat.

\[ x_1 = x_0 + \Delta x \]
\[ y_1 = y_0 + \Delta y \]

Euler Procedure:

1. Calculate slope of (unknown) solution curve at \((x_0, y_0)\) from the D.E. (eqn 1').

2. Hence find the equation to tangent line to this curve at \((x_0, y_0)\).

3. From the tangent line equation evaluate a new \( y \)-value \((y_1 = y_0 + \Delta y)\) at the \( x \)-value, \(x_1 = x_0 + \Delta x\).

4. This gives us a new point \((x_1, y_1)\).

5. Repeat steps 1 \(\rightarrow\) 5.

The point \( Q \) is on the tangent line.

We started at \( P = (x_0, y_0) \) as the initial point on the solution curve.
We have now found our next point.

Repeating the procedure gives us another point \((x_2, y_2)\) with
In general, the \((n+1)\)th point is given in terms of the \(n\)th point:

We call the initial point \((x_0, y_0)\) the 0th (zeroth) point!

Example.
Find an approximate solution to the following initial value problem:

\[
\begin{array}{c|c|c|c|c}
 n & \Delta x & (x_n, y_n) & y_{n+1} = y_n + f(x_n, y_n) \Delta x & y_{n+1} = f(x_n, y_n) \Delta x \\
0 & 1 & (x_0, y_0) & & \\
1 & 1 & (x_1, y_1) & & \\
2 & 1 & (x_2, y_2) & & \\
3 & 1 & (x_3, y_3) & & \\
4 & 1 & (x_4, y_4) & & \\
\end{array}
\]

The actual solution to this IVP is \(\ldots\). So the actual points on the curve are:
<table>
<thead>
<tr>
<th>n</th>
<th>x_n</th>
<th>y_n</th>
<th>f(x_n,y_n) = x_n + y_n</th>
<th>x_{n+1} = x_n + \Delta x</th>
<th>y_{n+1} = y_n + f(x_n,y_n) \Delta x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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</tr>
</tbody>
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