LECTURE 23:

Taylor Series.

1. Definition

2. Series Solutions
   - Polynomial solutions
   to DE's.

Important things to remember.

\[ p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n \]

So,

\[ p_n(c) \]

\[ p_n'(c) \]

\[ p_n''(c) \]

In general,

So this means that the value
of the function = the value
of the T.P. at \( x = c \) AND

So it is a very good
approximation.

Note:

useful for calculations.

Example

Calculate the 5th order TP
for \( \sin(x) \) about \( x = 0 \).

We already know that

\[ p_5(x) = x - \frac{x^3}{3!} \]
Taylor Series.

We have seen that, for a function \( f(x) \), the Taylor polynomial \( p_n(x) \) of order \( n \) can be used to approximate the function near some point \( x = c \).

Taylor's Theorem tells us that for some \( x \) in the region of \( x = c \),

The general term in the series is:

So

This is called a

- it is an infinite sum!

A Taylor polynomial is a finite sum!

Example.

Calculate the Taylor series for \( \log_e(x+5) \) about \( x = -2 \).

Solution

We know that

We need to determine the general term.

We need to try and spot a pattern for the derivatives:

* The denominator of each term

* The numerator of the \( (n^{th}) \)

So the general term is (coefficient)
Now the Taylor series is:

Observe that in the terms other than \( f(0) \), we have

in the coefficient.

Some Standard Taylor Series about \( x=0 \):

\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

\[ (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \]

\[ \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \]

Example.

Find the Taylor Series about \( x=0 \)

for \( y = x^2 - x - 2 \).
So the Taylor series for a quadratic terminates with the quadratic term - regardless of the value of \( C \).

In general, the Taylor series for a polynomial of order \( n \), will terminate at the term \((x-C)^n\).

Consider

Series solution and polynomial approximate solutions to DE's.

We start with an initial value problem:

\[
\frac{dy}{dx} = f(x,y) \\
y(C) = y_0
\]

This means that we can find \( y'(C) \) as \( f(C, y_0) \).

We can then differentiate the DE to find \( y'' \) in terms of \( x, y \) and \( y' \).

We can then substitute \( x = C, y = y_0 \) and \( y' = y'(C) \) to obtain \( y''(C) \).

We can repeat the process to find \( y'''(C), y^{(n)}(C) \), etc.

This means that we can calculate the coefficients for a Taylor series (or polynomial).

**Example.**

Find a series solution for the IVP

\[
\frac{dy}{dx} = x - y \quad , \quad y(0) = 0
\]

\[
y' = x - y \\
y'' = 1 - y' \\
y''' = -y'' \\
y^{(4)} = -y''' \\
\vdots \\
y^{(n)} = -y^{(n-1)}
\]

So \( y(x) = y(0) + y'(0)(x-0) + y''(0)(x-0)^2 \frac{y^{(3)}}{3!} + \cdots \)
Now we saw earlier that
\[ e^x = 1 + x + x^2 + \frac{x^3}{3!} + \ldots \]
which means that

\[ e^x \approx 1 + x \]

Hence

Find this solution is valid for all \( x \in \mathbb{R} \).

So \( y(x) \approx p_4(x) \)

**Example**

\[ \frac{dy}{dx} = 2xy, \quad y(0) = 1. \]

Find a quartic approximation for the solution \( y(x) \) near \( x=0 \).

\[ y' = 2xy \]
\[ y'' = 2y + 2xy' \]
\[ y''' = 2y' + 2y'' + 2xy'' \]
\[ = 4y' + 2xy'' \]
\[ y'' = 4y'' + 2y'' + 2xy'' \]
\[ = 6y'' + 2xy'' \]

We have \( c = 2 \) and \( y(2) = -1 \).

\[ y' = x^2 + y' \]
\[ y'' = 2x + \frac{2y}{a} \]
\[ y'' = 2x + y'' \]
\[ y''' = 2 + y'' + y'' \]
\[ = 2 + (y''^2 + y'y') \]
\[ y(4) = 2(y''^2 + y'y' + y''y') \]
\[ = 3y''^2 + y'y'' \]
\[ y(5) = 3y''^2 + 3y'y'' + y'y'' + y'y'' \]
\[ = 3(y''^2 + 4y'y'' + y'y'' \]
\( y_1(x, t) \geq p_2(x, t) \)