LECTURE 24:

1. A last example of series solutions
2. Integration techniques
   - some standard antiderivatives
   - partial fractions
   - examples.

\[ B, T, B \] chapters 11 and 12

Thomas & Finney: Calculus §4.3
   and Chpt 7.

Integration Techniques:

If we have a function

then we know that

So a solution to a differential equation of the form

That is:

In other words: \( f(x) \) is an antiderivative of \( g(x) \).

More precisely:

Example:

Let \( \frac{dy}{dx} = 3x^2 \).

In this example \( c \in \mathbb{R} \) is an undetermined constant.

To determine an exact value of \( c \), we need an initial condition.

For example,

then

and is the solution to the initial value problem.
3 cardinal rules of integration:

1. \( \int k \, f(x) \, dx \)
2. \( \int (f(x) + g(x)) \, dx \)
3. \( \int f(x) \cdot g(x) \, dx \)

Some standard indefinite integrals:

1. \( \int k \, dx = \)
2. \( \int x^a \, dx = \)
3. \( \int e^{kx} \, dx = \)
4. \( \int \cos (kx) \, dx = \)
5. \( \int \sin (kx) \, dx = \)

What about \( \int \frac{1}{x} \, dx \)?

What about \( x < 0 \)?

\( \frac{1}{x} \) is defined but \( \log_e(x) \) isn't.

So

Therefore

So
Techniques for finding antiderivatives

(a) Partial Fractions   \(\text{rational functions}\)
(b) Integration by Parts \(\text{"aux" product rule}\)
(c) Substitution \(\text{"aux" chain rule}\)

A. Partial Fractions pp 569-577
Thomas A. Finney.

We wish to calculate
\[
\int f(x) \, dx
\]
where
\[
f(x) = \frac{P(x)}{Q(x)}
\]
and \(P(x)\) and \(Q(x)\) are both polynomials.

Procedure:

1. Is degree \(P < \text{degree } Q\)?
   - No? Then divide \(P(x)\) by \(Q(x)\) to give
   \[
   \frac{P(x)}{Q(x)} = \quad =
   \]

2. Now write \(Q(x)\) as a product of linear and quadratic factors (sometimes this is tough).
   \[
   \]


In general
\[
Q(x) = A (x - r_1)^{m_1} (x - r_2)^{m_2} \ldots \times (x^2 + bx + c_1)^{n_1} (x^2 + bx + c_2)^{n_2} \ldots
\]

(i) Do partial fractions.

Let \(b(x) = \text{sum of terms of the form } \frac{a}{(x^2 + bx + c)^n}\)

(ii) For each quadratic factor in \(Q(x)\) we include terms on the RHS.

4. Find all constants:

- Multiply both sides of the equation by \(Q(x)\) and equate coefficients of powers of \(x\) to give equations for the constants.
Example 1

\[ \int \frac{4x-5}{(x-1)(x-2)} \, dx \]

Solution.
\text{deg} (P) < \text{deg} (Q) \text{ so no need to divide.}

This equation must hold for all \( x \).

So...

Example 2.

\[ \int \frac{2x+1}{(x-3)^2} \, dx \]

Solution.
\text{again deg} (P) < \text{deg} (Q)

The denominator \( Q(x) \) is a repeated linear factor:
We write:

Again, this must hold for all \( x \).

Hence

Consider \[ \int \frac{2x+1}{x^2-12x+4} \, dx \].

If \( \text{then} \)
\text{we can factorise} \( Q(x) \) \text{ into 2 distinct linear factors}

(A)

If \( \text{then} \)
The \( Q(x) \) can be factorised into a repeated linear factor

(B)

If \( \text{then} \)
cannot be factorised into real linear factors.

(C)
Example 3.
\[ \int \frac{1^3}{x^2-2x-3} \, dx. \]

Solution
\[ \text{deg}(f) = \text{deg}(g). \]

Example 4.
\[ \frac{x^2 + x + 1}{(x-1)(x+2)^3} = \]
\[ \Rightarrow x^2 + x + 1 = \]

Example 5.
\[ \frac{x^4 + 2x^3}{(x-1)^3(x^2+2x+5)} \]
\[ \Delta = 2^2 - 4(5) < 0 \text{ so cannot factorise.} \]
\[ \text{So} \]
\[ \frac{x^4 + 2x^3}{(x-1)^3(x^2+2x+5)} \]

and we proceed as in previous examples.

Now the question arises ---
what is the antiderivative of
\[ \frac{\text{Dart} E}{x^2+2x+5} ? \]