Out of 8 - Parts (a) and (b) each out of 4.

(a) We have $f(x, y) = x^3 + y^3 - 3xy$. Stationary points occur where $f_x(x, y) = f_y(x, y) = 0$. So we want

\[ f_x(x, y) = 3x^2 - 3y = 0 \quad \Rightarrow \quad x^2 = y \]  
\[ \text{and} \quad f_y(x, y) = 3y^2 - 3x = 0 \quad \Rightarrow \quad y^2 = x \]  

Substituting (1) into (2) we get

\[ x^4 = x , \]

which has solutions $x = 0$ and $x = 1$, and thus $y = 0$ and $y = 1$ accordingly. Hence there are two stationary points: $(0, 0)$ and $(1, 1)$.

To classify these stationary points we need the second partial derivatives:

\[ A = f_{xx}(x, y) = 6x \]
\[ B = f_{xy} = -3 \]
\[ C = f_{yy} = 6y \]

and so

\[ AC - B^2 = (6x)(6y) - (-3)^2 = 36xy - 9 . \]

At $(0, 0)$, we have

\[ AC - B^2 = 0 - 9 = -9 < 0 \]

so $(0, 0)$ is a saddle point.

At $(1, 1)$, we have

\[ AC - B^2 = 36 - 9 = 27 > 0 \quad \text{and} \quad A = 6 > 0 \]

so $(1, 1)$ is a local minimum point.

(b) (i) We want to minimise $f(x, y) = -6x^2 + 2y^2$ subject to the constraint $2x + y = 4$, or $2x + y - 4 = 0$. Using the method of Lagrange multipliers, define $F(x, y, \lambda) = (-6x^2 + 2y^2) + \lambda(2x + y - 4)$. Then we want

\[ F_x(x, y, \lambda) = -12x + 2\lambda = 0 \]
\[ F_y(x, y, \lambda) = 4y + \lambda = 0 \]
\[ F_\lambda(x, y, \lambda) = 2x + y - 4 = 0 \]

The first equation gives $\lambda = 6x$ and the second gives $\lambda = -4y$, so equating these gives $y = -\frac{3}{2}x$. Substituting this into the third equation gives:

\[ 2x + \left( -\frac{3}{2} \right)x - 4 = 0 \quad \Rightarrow \quad \frac{1}{2}x = 4 \quad \Rightarrow \quad x = 8 . \]

Then $y = -\frac{3}{2}(8) = -12$, and so $(x, y) = (8, -12)$ is the required minimum point (as it is the only stationary point of $F$). The minimum value of $f$ is $f(8, -12) = -6(8)^2 + 2(-12)^2 = -96$. 

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(ii) We want to minimise \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to \( 2x + y - z = 3 \), or \( 2x + y - z - 3 = 0 \). Define \( F(x, y, z, \lambda) = (x^2 + y^2 + z^2) + \lambda(2x + y - z - 3) \). Then we want:

\[
\begin{align*}
F_x(x, y, z, \lambda) &= 2x + 2\lambda = 0 \\
F_y(x, y, z, \lambda) &= 2y + \lambda = 0 \\
F_z(x, y, z, \lambda) &= 2z - \lambda = 0 \\
F_\lambda(x, y, z, \lambda) &= 2x + y - z - 3 = 0 .
\end{align*}
\]

The first equation gives \( \lambda = -x \), the second gives \( \lambda = -2y \) and the third \( \lambda = 2z \). Equating these yields, for example, \( x = 2y \) and \( z = -y \). Substituting into the final equation, we get:

\[
2(2y) + y - (-y) - 3 = 0 \quad \Rightarrow \quad 6y = 3 \quad \Rightarrow \quad y = \frac{1}{2} .
\]

Then \( x = 2(\frac{1}{2}) = 1 \) and \( z = -\frac{1}{2} \), so that \( (x, y, z) = (1, \frac{1}{2}, -\frac{1}{2}) \) is the required minimum point (as it is the only stationary point of \( F \)). The minimum value of \( f \) is therefore \( f(1, \frac{1}{2}, -\frac{1}{2}) = 1^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2 = \frac{3}{2} . \)