17. Taylor Polynomials.

Objectives.

1. Use Taylor polynomials to approximate functions.

2. Consider the error term—how good is your approximation.

3. Calculate the error estimate (upper bound).

As we saw in the beginning of this section, a Taylor polynomial approximates a complicated function near a specified value of \( x \). As we move away from this value, the polynomial no longer performs this task. Today we will consider how good an approximation the Taylor polynomial is in a given range of values around \( x - c \).
Example

For the function \( f(x) = \sqrt{1+x} \), calculate \( p_3(x) \) about \( x=0 \), and use it to approximate the value of \( \sqrt{1.04} \).

Solution

\[
f(x) = (1+x)^{\frac{1}{2}} \quad \Rightarrow f(0) = 1
\]

\[
f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \quad \Rightarrow f'(0) = \frac{1}{2}
\]

\[
f''(x) = \frac{1}{4}(1+x)^{-\frac{3}{2}} \quad \Rightarrow f''(0) = \frac{1}{4}
\]

\[
f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \quad \Rightarrow f'''(0) = \frac{3}{8}
\]

So \( p_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} \)

\[
= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} .
\]

How do I use this to approximate \( \sqrt{1.04} \)?
Notice that \( \sqrt{1.04} = (1 + 0.04)^{\frac{1}{2}} \),
and that \( f(x) = (1 + x)^{\frac{1}{2}} \).

So the Taylor polynomial about \( x = 0 \) for \( f(x) \) should give us an approximation to \( \sqrt{1.04} \).

Here, we have \( x = 0.04 \)
so \( p_3(x) = 1 + \frac{x^2}{2} - \frac{x^3}{8} + \frac{x^4}{16} \)

\[ p_3(0.04) = 1 + \frac{0.04^2}{2} - \frac{(0.04)^3}{8} + \frac{(0.04)^4}{16} \]
\[ = 1 + 0.02 - 0.0002 + 0.000004 \]
\[ = 1.019804 \].

So using a 3\textsuperscript{rd} order Taylor polynomial, I have
\[ \sqrt{1.04} \approx 1.019804 \].

Calculator value \( \approx 1.019803903 \)

So the approximations agree up to 5 decimal places!!

- in fact calculators use Taylor polys. to calculate non-linear functions!

To get a better approximation, choose \( n > 3 \).
When we use $P_n(x)$ to approximate $f(x)$ near $x = 0$, there are many formulae for approximating $E_n$. The one that we will use is

$$E_n(x) = f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!}$$

where $c$ is some number: $x \leq c \leq 0$, $x < 0$

$0 \leq c \leq x$, $x > 0$

**Error term**

As we have seen, the value of the Taylor poly and the value of the function (for given $x$) is not the same. The actual error is

$$f(x) = P_n(x) + E_n(x)$$

This suggests that

$$f(x) = P_n(x) + E_n(x)$$

(don't forget this will change with $n$).
So in the previous example

\[ E_8(x) = f^{(8)}(c) \frac{x^8}{4!}, \quad 0 \leq c \leq 0.04. \]

Now \[ f^{(8)}(x) = -\frac{15}{16} (1 + x)^{-\frac{11}{2}}, \quad \text{so} \]

\[ E_8(x) = -\frac{15}{16(1+c)^{\frac{11}{2}}} \frac{x^8}{4!} \]

\[ = E_8(0.04) = -\frac{15 (0.04)^{\frac{11}{2}}}{16.4!(1+c)^{\frac{11}{2}}} \]

\[ = -\frac{10^{-7}}{(1+c)^{\frac{11}{2}}}, \quad 0 \leq c \leq 0.04. \]

This says that the error will be less than \( 10^{-7}. \)

\[
\begin{array}{c}
1.019804 000 \\
-1.019803 903 \\
0.000000 097 \\
10^{-7} \\
0.00000001 \\
0.00000097 < 10^{-7} \quad \checkmark \\
\end{array}
\]

So \( |E_8(x)| \) gives us a bound on the error.

Since \( 'c' \) is in the denominator

\[ |E_8(0.04)| \] will be largest when \( c = 0 \)

So \( |E_8(0.04)| \leq 10^{-7} \)
Example

For \( f(x) = \cos(3x) \n\)
Find the \( 4^{th} \) order Taylor polynomial about \( x=0 \) and use it to approximate \( \cos(0.3) \). Write down a formula for the error.

**Step 1** Calculate \( P_4(x) \).

\[
\begin{align*}
  f(x) &= \cos(3x) & \Rightarrow & \quad f(0) = \cos(0) = 1 \\
  f'(x) &= -3\sin(3x) & \Rightarrow & \quad f'(0) = -3\sin(0) = 0 \\
  f''(x) &= -9\cos(3x) & \Rightarrow & \quad f''(0) = -9\cos(0) = -9 \\
  f'''(x) &= +27\sin(3x) & \Rightarrow & \quad f'''(0) = 27\sin(0) = 0 \\
  f^{(4)}(x) &= 81\cos(3x) & \Rightarrow & \quad f^{(4)}(0) = 81\cos(0) = 81 \\
\end{align*}
\]

So \( P_4(x) = 1 - \frac{9x^2}{2} + \frac{81x^4}{8} \)

\[
= 1 - 9\frac{x^2}{2} + 27\frac{x^4}{8}
\]

**Step 2** approximate \( \cos(0.3) \)

\[ \cos(0.3) = \cos(3 \times 0.1) \], so using \( f(x) = \cos(3x) \) means that \( x = 0.1 \)

So \( P_4(0.1) = 1 - \frac{9(0.1)^2}{2} + \frac{27(0.1)^4}{8} \)

\[ = 1 - 0.45 + 3.375 \times 10^{-4} \]

\[ = 0.5503375 \]

**Step 3** Calculate \( E_4(x) \).

\[ E_4(x) = f^{(5)}(c) \frac{x^5}{5!} \quad \text{for some } c \in (0, 0.1) \]

\[ f^{(5)}(x) = -3 \times 81 \sin(3x) \]
\[ E_4(x) = \frac{-3 \times 81 \sin(3c)}{5!} \]
\[ = -\frac{27}{40} \sin(3c) x^5 \]

\[ \Rightarrow |E_4(\cdot 1)| \leq \frac{(0.1)^5 \times 81}{40} \sin(3c), \]

where \( 0 \leq c \leq 0.1 \).

Notice that sine is increasing from \( \sin(3 \times 0) \rightarrow \sin(3 \times 0.1) \)
\[ = \sin(0) \rightarrow \sin(0.3) \]
so \[ |E_4(\cdot 1)| \leq \frac{(0.1)^5 \times 81}{40} \sin(3 \times 0.1) \]
\[ = 2.025 \times 10^{-6} \sin(3) \]

New Taylor polynomials from old.

You have already calculated the Taylor polynomial for \( f(x) = e^x \)
and now you have been asked to calculate one for \( g(x) = e^{2x} \) ——
what do you do?
Noting that the T.P for \( f(x) = e^x \) is
\[
\frac{1}{n!} \quad \text{is } (y)^n
\]
\[
\text{for } g(x) = e^x \quad \text{we simply replace } "x" \text{ for } "x^2"
\]
\[
\frac{1}{n!} \quad \text{for } h(x) \text{ of order } 7 \text{ is }
\]
\[
\text{For } h(x) = x^2 e^x \quad \text{no problem. } \\
\text{for } g(x) = e^x \quad \text{we simply replace } "x" \text{ for } "x^2"
\]
\[
\text{So for example the T.P for } \\
\text{for } h(x) \text{ is }
\]
\[
\text{Too EASY!!}
\]
You may now finish Example Sheet 6.