21. Functions of several variables.

Objectives

1. What are they?
   - Some introductory examples.
2. Visualization
3. Partial derivatives

Introduction.

Many problems we encounter in real life depend on more than one variable.

For example,

(i) the area of a rectangle depends on its height and width

\[ A(w,h) = w h. \]

(ii) the volume of a cylinder depends on the radius and the height.

\[ V(r,h) = \pi r^2 h. \]

(iii) the volume of a box depends on height, length and width.

\[ V(w,l,d) = w l d. \]

(iv) A linear programming problem may depend on any number of variables.

\[ z = 3x_1 + 2x_2 + 4x_3 + x_4. \]

(Here \( z(x_1, x_2, x_3, x_4) \) is a function of four variables.

In some cases these variables may be related. For example, in the last lecture we considered the surface area of a cylinder.

Although this problem depends on two variables \([r, h]\) in our previous
Examples.

i) \( f(x, y) = 5x^2y + 2x \).
Evaluate \( f \) at the point \((2, 1)\).
\[
\begin{align*}
  f(2, 1) &= 5(2)^2(1) + 2(2) \\
          &= 60 + 4 = 64.
\end{align*}
\]

ii) \( f(x, y) = 5x + \sin(xy) + y^2 \).
Evaluate \( f \) at the point \((3, 0)\).
\[
\begin{align*}
  f(3, 0) &= 5\cdot 3 + \sin(3\cdot 0) + 0^2 \\
          &= 15 + \sin(0) + 0 \\
          &= 9.
\end{align*}
\]

Sketching functions of two variables.

Recall that to sketch a function with one variable, we need 1 axis; we evaluate the function at some point \([x]\), and denote its value by \([y]\).

So the points on the graph of \( f \) are ordered pairs \((x, y)\).

For a function of two variables we need 3 axes; we evaluate the function at the point \([x, y]\) and denote its value as \([z]\).

This gives us a picture in 3-dimensions.

If the function has more than 2 variables, we cannot sketch it.
The graph of a function $f(x,y)$ is called a surface and in general, will be difficult to sketch. However, we can get some idea of the shape of $f(x,y)$ by drawing 2D pictures called level curves or contours of $f$.

To do this, we let $f(x,y) = $ constant and plot those points $(x,y)$ which are mapped to that constant. In this way we get a 2D cross section of the surface.

**Level curves**

![Level curves diagram]

**Partial derivatives**

Recall that for a function of one variable, $y = f(x)$, we defined the derivative of $y$ with respect to $x$ as

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This was the ratio of the change in $y$ values to the change in $x$ values.

![Partial derivatives diagram]
Functions in 2 variables

For a function \( z = f(x, y) \)
we define
\[
\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}
\]
- this is the partial derivative
of \( z \) with respect to \( x \).

Here, we are calculating
the ratio of the change in \( z \)
with the change in \( x \). For a
fixed value of \( y \). (\( y \) is constant)

Similarly, we define
\[
\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}
\]
as the partial derivative of
\( z \) with respect to \( y \).

This measures the ratio of the
change in \( z \) with the change in \( y \).

Of course if \( f \) is a function
with \( n \) variables, \( f(x_1, x_2, \ldots, x_n) \)
we can define the \( i \)th
partial derivative of \( f \) with
respect to \( x_i \) as
\[
\lim_{h \to 0} \frac{f(x_1, \ldots, x_i+h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}
\]

Examples

1. \( z = x^3 + 3xy^2 + y^3 \)
   \( \frac{\partial z}{\partial x} = 3x^2 + 3y^2 \) \( \Rightarrow \) differential wrt \( x \), keeping \( y \) constant.
   \( \frac{\partial z}{\partial y} = 6xy + 3y^2 \) \( \Rightarrow \) differential wrt \( y \), keeping \( x \) constant.

2. \( z = xe^{xy} \)
   \( \frac{\partial z}{\partial x} = e^{xy} + yxe^{xy} \) \( \Rightarrow \) differential wrt \( x \), keeping \( y \) constant.
   \( \frac{\partial z}{\partial y} = xe^{xy} \) \( \Rightarrow \) differential wrt \( y \), keeping \( x \) constant.

3. \( z = \sin(xy) \)
   \( \frac{\partial z}{\partial x} = \cos(xy) \cdot y \) \( \Rightarrow \)
   where \( u = xy \),
   \( \frac{du}{dx} \),
   \( \frac{\partial z}{\partial y} = \cos(xy) \cdot x \)
4. \( z = 5x^2y + 2x \).

Calculate \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) from first principles.

\[
\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}
\]

\[
= \lim_{h \to 0} \frac{5(x+h)^2y + 2(x+h) - 5xy^2 - 2x}{h}
\]

\[
= \lim_{h \to 0} \frac{5x^2y + 5x^2h + 2x + 2h - 5xy^2 - 2x}{h}
\]

\[
= \lim_{h \to 0} \frac{5x^2h}{h} = 5x^2.
\]

\[
\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{5x^2y + 5x^2h + 2x - 5x^2y - 2x}{h}
\]

\[
= \lim_{h \to 0} \frac{5x^2h}{h} = 5x^2.
\]

5. \( w = 3x^2y + z^2 + xy + \varepsilon \) (\( w(x,y,z) \))

\[
\frac{\partial w}{\partial x} = 6xy + 0 + yz = 6xy + yz.
\]

\[
\frac{\partial w}{\partial y} = 3x^2 + 0 + xz = 3x^2 + xz.
\]

\[
\frac{\partial w}{\partial z} = 0 + 2z + xy = 2z + xy.
\]

Alternative notation:
When \( y = f(x) \), we say \( \frac{dy}{dx} \) or \( f'(x) \) for the derivative with respect to \( x \).

Here \( z = f(x,y) \), we write \( \frac{\partial z}{\partial x} = f_x(x,y), f_x \)
for the partial derivative with respect to \( x \) and \( \frac{\partial z}{\partial y} = f_y(x,y), f_y \) for the partial derivative with respect to \( y \).

You may now attempt Sheet 8
Q's 2, 4 (i)

1. \(- \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\)

For (i), (ii), (iii), (iv)