22. Functions of several variables

Objectives:

1. Higher order partial derivatives

2. Examples

3. What does it mean geometrically?

Recall that, for functions of one variable \( y = f(x) \), we can calculate the second derivative as

\[
\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2},
\]

that is, we differentiate the first derivative.

For a function in two variables \( z = f(x, y) \) we have two first (partial) derivatives \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

In this case, we have four second order partial derivatives.

\[
\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2},
\]

\[
\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y},
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x},
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2}.
\]

Example

\[
z = x^3 + 3xy^2 + y^4.
\]

(i) \( \frac{\partial z}{\partial x} = 3x^2 + 3y^2 \)

(ii) \( \frac{\partial z}{\partial y} = 6xy + 4y^3 \)

Now

(i) \( \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) \)

\[= \frac{\partial}{\partial x} (3x^2 + 3y^2) = 6x \]

(ii) \( \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \)

\[= \frac{\partial}{\partial y} (6xy + 4y^3) = 6x + 12y^2 \]
Alternative notation:

Recall that if \( z = f(x,y) \)
then \( \frac{\partial z}{\partial x}, f_x, f_x(x,y) \)
all mean 1st partial derivative
with respect to \( x \) AND
\( \frac{\partial z}{\partial y}, f_y, f_y(x,y) \) all mean
1st partial derivative with
respect to \( y \).

Also
\[
\frac{\partial^2 z}{\partial x^2}, f_{xx}, f_{xx}(x,y) \text{ 2nd p.d. wrt } x
\]
\[
\frac{\partial^2 z}{\partial y^2}, f_{yy}, f_{yy}(x,y) \text{ 2nd p.d. wrt } y
\]

AND
\[
(f_x)_y, (f_y)_x, \frac{\partial^2 z}{\partial y \partial x} \text{ diff. 1st wrt } x \text{ then wrt } y.
\]
\[
(f_y)_x, (f_x)_y, \frac{\partial^2 z}{\partial x \partial y} \text{ diff. 1st wrt } y \text{ then wrt } x
\]

Note that \( f_{xy} = f_{yx} \) here anywazy.

Example

Find the partial derivatives
of \( f(x,y) = \sqrt{16-x^2-y^2} \)
at \((x,y) = (1,3)\).

Solution

Write \( f(x,y) = 16-x^2-y^2 \) as \( f(u) = u^{1/2}, \ u(x,y) = 16-x^2-y^2 \).

Then \( \frac{\partial f}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} \)
\[= \frac{1}{2} u^{-1/2} \cdot (-2x) = \frac{-2x}{2(16-x^2-y^2)} \]
\[= -\frac{x}{16-x^2-y^2} \]
The surface \( z = \sqrt{16 - x^2 - y^2} \) is a hemisphere of radius 4.

This calculation is straightforward, but what does it mean?

Notice here that we have two tangents to the surface at the point \( P \)!

(In fact, there is a tangent plane at the point \( P \) — both of these tangent lines are in the same plane.)

Example

Find all 2nd order partial derivatives of \( f(x, y) = x^2 e^{xy^2} \).

Solution

\[
\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \left( x^2 e^{xy^2} \right) = 2x e^{xy^2} + x^2 \frac{\partial e^{xy^2}}{\partial x}
\]

Product Rule

\[
= 2x e^{xy^2} + x^2 y^2 e^{xy^2}
\]

\[
\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \left( x^2 e^{xy^2} \right) = x^2 \frac{\partial e^{xy^2}}{\partial y}
\]

Product Rule

\[
= 0 + x^2 \cdot 2xy e^{xy^2}
\]

Chain Rule

\[
= 2x^2 y e^{xy^2}
\]
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( 2xe^{xy^2} + x^2y^2e^{xy^2} \right)
= \frac{\partial}{\partial x} \left( 2xe^{xy^2} + 2x \frac{\partial e^{xy^2}}{\partial x} + \frac{\partial}{\partial x} x^2y^2e^{xy^2} \right)
= 2e^{xy^2} + 2xe^{xy^2} + x^2y^2 \frac{\partial e^{xy^2}}{\partial x}
= 2e^{xy^2} + 2xe^{xy^2} + x^2y^2 \cdot y^2e^{xy^2}
= e^{xy^2} (2 + 4xy^2 + x^2y^4).
\]

\[
\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( 2xe^{xy^2} + x^2y^2e^{xy^2} \right)
= 2x \frac{\partial e^{xy^2}}{\partial y} + x \left( \frac{\partial}{\partial y} e^{xy^2} + y^2 \frac{\partial e^{xy^2}}{\partial y} \right)
= 2xe^{xy^2} + xe^{xy^2} \cdot 2y^2e^{xy^2} + y^2 \cdot 2xe^{xy^2}
= e^{xy^2} (4x^2y + 2xy^2 + 2x^3y^2)
= e^{xy^2} (6x^2y + 2x^3y^2).
\]

\[
\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( 2x^3y^2e^{xy^2} \right)
= 2x^3y^2 \frac{\partial e^{xy^2}}{\partial y} + 2x^3y \cdot y^2 \frac{\partial e^{xy^2}}{\partial y}
= 2x^3y^2 \cdot y^2e^{xy^2} + 2x^3y \cdot y^2 \cdot y^2e^{xy^2}
= e^{xy^2} (2x^3 + 6x^2y).
\]

You may now attempt.

Sheet 8
Q1. (second p.d's)
Q4.

Find some more examples in Calculus Books!!