In sections 5 and 6 of this course we were concerned with finding derivatives of functions (or slopes of curves) either explicitly:

\[ y = x^4 - 3x^3 + 2x + 1 \]

\[ \frac{dy}{dx} = 4x^3 - 9x^2 + 2 \]

\[ \left. \frac{dy}{dx} \right|_{x=2} = 32 - 36 + 2 = 2 \]

So the slope of the curve is 2 at (2,2).

Or implicitly: \( \frac{dy}{dx} \)

\[ x^3 + 5y^2 = 12 \]

\[ \Rightarrow 3x^2 + 10y \cdot \frac{dy}{dx} = 0 \]

\[ \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{10y} = \frac{dy}{dx} \bigg|_{x=2} = \frac{-12}{20} = \frac{-3}{5} \]

So the slope of the curve is \(-\frac{3}{5}\) at (-2,2).

Suppose now, that we are given an expression \( \frac{dy}{dx} \), in terms of \( x \) and \( y \).

That is, we know the slope of the curve(s) at any point \((x,y)\) in the plane.

How can we recover the curve? We want to find an expression for \( y \) in terms of \( x \).

A first order differential equation is an equation which involves some unknown function \( f \) (\( y = f(x) \)) and its derivative \( f' (\frac{dy}{dx} \cdot f(x)) \)
Examples.
1. If \( x = f(t) \), then
\[
\frac{dx}{dt} = x^2 + t^2, \\
(\frac{dx}{dt})^2 + \frac{dx}{dt} = t^2
\]
are both first order differential equations.

2. If \( y = f(x) \), then
\[
\frac{dy}{dx} + 3y = 2x^3 + 3x^2, \\
\frac{dy}{dx} = 2xy, \\
y \frac{dy}{dx} = x - y, \quad \text{are 1st order DE's.}
\]

DE's (first order and higher) are used as models for many situations in industry, science, engineering, and economics. The list goes on.

In a physical setting, we often start by knowing the DE (connecting two quantities \( x \) and \( y \)) and from this information we must recover the function \( y = f(x) \).

This function is called the solution to the DE—it is an explicit relation connecting \( x \) and \( y \), such that when substituted into the DE, it becomes an identity.

Example

The DE
\[
\frac{dy}{dx} + 3y = 3x^2 + 2x + 9
\]
has a solution \( y = x^2 + 3 \).

why?
\[
\frac{dy}{dx} = 2x, \quad \text{so}
\]
\[
\frac{dy}{dx} + 3y = 2x + 3\left(x^2+3\right) = 3x^2 + 2x + 9 \quad \text{RHS}
\]

There are two different types of tasks to consider:

1. Verify that a given function is a solution to the DE.
2. Find a solution to the given DE.

Task 1 is straightforward.

Task 2—may not always be possible!
Examples.

1. Show that $y = e^{2x}$ is a solution of $\frac{dy}{dx} = 2xy$.

Step 1.

$y = e^{2x}$

$\Rightarrow \frac{dy}{dx} = 2xe^{2x} \quad (\text{use chain rule})$

RHS$\Rightarrow 2xy = 2x(e^{2x})$

$\Rightarrow \text{LHS} = \text{RHS}$, so $y = e^{2x}$ is a solution of $\frac{dy}{dx} = 2xy$.

2. Show that $y = 5e^{2x}$ is a solution of $\frac{dy}{dx} = 2xy$.

Solution.

Since $y = 5e^{2x}$

1. $\frac{dy}{dx} = 10xe^{2x}$

$\Rightarrow$ RHS$\Rightarrow 2xy = 2x(5e^{2x})$

$\Rightarrow 10xe^{2x}$

So LHS$\Rightarrow$ RHS.

So $y = 5e^{2x}$ is also a solution of $\frac{dy}{dx} = 2xy$.

3. Is $y = e^{2x} + 2$ a solution of $\frac{dy}{dx} = 2xy$?

1. $y = e^{2x} + 2$

$\Rightarrow \frac{dy}{dx} = 2xe^{2x}$

So LHS$\Rightarrow \frac{dy}{dx} = 2xe^{2x}$

RHS$\Rightarrow 2xy = 2x(e^{2x} + 2)$

$\Rightarrow 2xe^{2x} + 4x$

So LHS $\neq$ RHS.

So $y = e^{2x} + 2$ is not a solution of the DE $\frac{dy}{dx} = 2xy$.

It turns out that DE’s have many solutions. In the example above ($\frac{dy}{dx} = 2xy$) any function $y = Ce^{2x}$ is a solution (C is some constant).

$y = Ce^{2x}$

$\Rightarrow \frac{dy}{dx} = 2Cxe^{2x}$

So LHS $\Rightarrow \frac{dy}{dx} = 2Cxe^{2x}$

RHS $\Rightarrow 2xy = 2x(Ce^{2x}) = 2Cxe^{2x}$

So LHS $\Rightarrow$ RHS.

This solution $y = Ce^{2x}$ is called the general solution of this DE.
When we choose a value for $C$ (say $5$) the solution
$y = 5e^{2x}$ is called a particular
solution of the DE.

**Example.**

$y = Ce^{-x^2/4}$ is the G.S of
the DE $\frac{dy}{dx} = \frac{-3x^2}{4} \cdot y$.

Verify: $y = Ce^{-x^2/4}$

(lhs) $\Rightarrow \frac{dy}{dx} = \frac{-3x^2}{4} \cdot Ce^{-x^2/4}$.

and $\frac{dy}{dx} \cdot \frac{du}{dx} \cdot \frac{du}{dx}$ $y = Ce^u$, $u = \frac{-x^2}{4}$

(LHS) $= Ce^u \cdot \frac{-3x^2}{4} = \frac{-3x^2}{4} \cdot Ce^{-x^2/4}$.

**Example.**

$\frac{dy}{dx} = y$

$y \cdot \frac{1}{2} e^{x-1}$

gives a family of solution curves.
(Editor to start: this is true since for each $y$ value we have the same slope at each point.)

The general solution to this DE is $y = Ce^x$ ($\Rightarrow \frac{dy}{dx} = Ce^x$ so $\frac{dy}{dx} = y$).

Each specific value of $C$ will correspond to a particular solution curve.

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**Slope Field.**

The slope field for a DE is a sketch showing the slope of the curve at various points selected in the plane.

It is usual to select these points in a grid formation and the slope is represented by a short straight line at each point.

We can find out a lot of information about a DE by looking at these slope fields.

A solution curve follows these line segments - any one of which is tangential to a solution curve.

To determine the value of $C$ which corresponds to a particular curve, we need an initial condition.

Suppose that we also know that when $x = 1$, $y = \frac{1}{2}$.

Then $y = Ce^x$

and $\frac{1}{2} = Ce^1$

$\Rightarrow C = \frac{1}{2} e^{-1}$.

So a particular solution is $y = (\frac{1}{2} e^{-1})e^x = \frac{1}{2} e^{x-1}$.

It is the curve which passes through the point $(1, \frac{1}{2})$. 
You may now attempt Sheet II

Q 1, 2

Q 3 (the general solution is given in Q 2.)