26. Complex Numbers

Operations:
- Subtraction
- Multiplication
- Complex conjugate
- Geometry
- Quotients
- Problems

Subtraction:
If \(a + ib\) is a complex number and we add
\((a + ib) + (-a) + (-ib)\)
we get zero. This means that
\((-a) + (-ib) = -(a + ib)\) is the additive inverse of \(a + ib\).
Subtraction of complex numbers is defined using additive inverses.
So \((a + ib) - (z + iy) = (a - x) + (b-y)i\).

Multiplication:
Two complex numbers are multiplied together according the usual algebraic rules for multiplying brackets; that is,
\((a + ib)(c + id)\)
\[= ac + iad + ibc + i^2bd\]
\[= (ac - bd) + i(ad + bc)\] (remember \(i^2 = -1\)).

\((3 + 2i)(2 + i)\)
\[= 3(2 + i) + 2i(2 + i)\]
\[= 6 + 3i + 4i + i^2 \cdot 2\]
\[= (6 - 2) + i(3 + 4) = 4 + 7i\].

Geometry

Suppose we wish to add \(A = 2 + i\) and \(P = 1 + 2i\) or subtract \(A\) from \(P\).
Complex conjugate:

Each \( z \in \mathbb{C} \) has an associate complex number \( \bar{z} \) which

\[ \text{is a real number:} \quad \bar{z} = x - iy \]

\[ \text{such that} \quad z \cdot \bar{z} = x^2 + y^2 \]

Complex conjugate:

Geometrically:

For any complex number \( z \), the complex conjugate \( \bar{z} \) is the real axis reflection through \( z \).

\[ z = 2 + i \]

\[ \bar{z} = 2 - i \]

\[ \text{with} \quad z \cdot \bar{z} = 4 + 1 = 5 \]

The complex conjugate through the real axis

Similarly, the reflection through \( z \) is simply the complex conjugate \( \bar{z} \) of \( z \).

\[ \bar{z} = 3 - 4i \]

\[ z \cdot \bar{z} = 17 \]

\[ \text{with} \quad z \cdot \bar{z} = x^2 + y^2 \]

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\[ \text{with} \quad z \cdot \bar{z} = x^2 + y^2 \]
**Quotients.**

Recall that the definition of a complex number is
\[ z = x + iy \] where \( x, y \in \mathbb{R} \).

This poses a difficulty when we divide one complex number by another. For example,

\[ z_1 = 3 + 2i, \quad z_2 = 1 - i \]

so
\[ \frac{z_1}{z_2} = \frac{3 + 2i}{1 - i} \]

It is not obvious how to write this in the form \( x + iy \) where \( x, y \in \mathbb{R} \).

Consider
\[ \frac{1}{z} \text{ where } z = a + ib. \]

Remember that, for any complex number \( z \), \( z \overline{z} \) is a real number. This is the key to our problem.

We write
\[ \frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z \overline{z}} \]

We have retained equality since \( \frac{\overline{z}}{z \overline{z}} = 1 \) and the denominator \( z \overline{z} \) is real.

**Examples.**

(i) \[ \frac{3 + 4i}{2 - i} = \frac{3 + 4i}{2 - i} \times \frac{2 + i}{2 + i} \]

\[ = \frac{(3 + 4i)(2 + i)}{4 + 1} \]

\[ = \frac{(6 - 4) + (6 + 3)i}{5} \]

\[ = \frac{2}{5} + \frac{9}{5}i. \]

(ii) \[ \frac{2 - 3i}{1 + 3i} = \frac{2 - 3i}{1 + 3i} \times \frac{1 - 3i}{1 - 3i} \]

\[ = \frac{(2 - 3i)(1 - 3i)}{1 + 9} \]

\[ = \frac{(2 - 9) + (-6 - 3)i}{10} \]

\[ = \frac{-7}{10} - \frac{9}{10}i. \]
Some problems:

1. Solve for $x$ and $y$ in the equation
   \[ i(2x-4y) = 4x + 2 + 3yi. \]
   Solution:
   \[
   \begin{align*}
   \text{LHS} &= i(2x-4y) \\
   &= 0 + i(2x-4y) \\
   \text{RHS} &= (4x+2) + 3yi
   \end{align*}
   \]

   Therefore
   \[
   \begin{align*}
   0 &= 4x + 2 \\
   \Rightarrow x &= -\frac{1}{2} \\
   3y &= 2x - 4y \\
   \Rightarrow 7y &= 2x \\
   \Rightarrow y &= -\frac{1}{7}. \\
   \end{align*}
   \]

2. Find all $z \in \mathbb{C}$ such that
   \[
   \frac{z - 2}{z} = 1 + i.
   \]
   Solution:
   \[
   \begin{align*}
   \frac{z - 2}{z} &= 1 + i \\
   \Rightarrow z - 2 &= z(1 + i) \\
   \Rightarrow z - 2 &= z + zi \\
   \Rightarrow 2i &= -2 \\
   \Rightarrow z &= 2i
   \end{align*}
   \]

3. Find all $z \in \mathbb{C}$ such that
   \[
   z^2 - 1 = (4 - i)^2.
   \]
   Solution:
   \[
   \begin{align*}
   z^2 - 1 &= (4 - i)^2 \\
   \Rightarrow z^2 - 1 &= 16 - 8i + i^2 \\
   \Rightarrow z^2 &= 15 - 8i \\
   \Rightarrow z &= 3 - 4i. \\
   \end{align*}
   \]

4. Verify that for any complex numbers $z, w \in \mathbb{C}$ that
   \[
   \overline{z \cdot w} = \overline{z} \cdot \overline{w}.
   \]

   Solution: Let $z = x + iy, \ w = a + ib$.
   \[
   \begin{align*}
   \text{LHS} &= \overline{z \cdot w} \\
   &= \overline{(x+iy)(a+ib)} \\
   &= \overline{(xa - by) + i(ay + bx)} \\
   &= (xa - by) - i(ay + bx). \\
   \end{align*}
   \]

   \[
   \begin{align*}
   \text{RHS} &= \overline{z} \cdot \overline{w} \\
   &= \overline{x+iy} \cdot \overline{a+ib} \\
   &= (x - iy)(a - ib) \\
   &= (xa - by) - i(ay + bx) \\
   &= \text{LHS as reqd.}
   \end{align*}
   \]