# Series Analysis of Kosterlitz-Thouless Transitions

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# Summary

- Potts models: standard and planar
- Planar Potts
- Kosterlitz-Thouless Transitions
- 6-state model: Monte Carlo and high-field
- 6-state model: Low-T series by finite lattice method
- Two-parameter 5-state model: series analysis and phase-space

# Potts Models

Potts model variables, $x_j$, at each lattice site take one of $q$ values: $1, 2 \dots q$, generalising $q = 2$ (Ising) case.
For $q > 3$ two natural generalisations (with scalar product interactions):

**Standard**: The $x_j$ are vectors in $(q-1)$-D space. Only two distinct pair-energies: $x_j$, $x_k$ are the same or different.
Widely-studied, generalised to non-integer $q$.
First-order transition for $q > 4$ on 2-D lattices (Baxter).

**Planar:** The $x_j$ are treated as vectors in 2-D space. Pair energy $\propto -\cos(2\pi(x_j - x_k))$.

# Planar Potts models

The $q$-state Planar Potts model has low-T ordered state.

Ordered state disappears as $q \to \infty$ (planar rotator).

Planar rotator has low-T spin-wave states (as does larger-$q$ planar Potts, above ordered phase).

Disordered phase at high-temperatures.

Expect 3 phases for $q$ *sufficiently large*, in practice for $q \geq 5$.

# Spin-wave states

Primary excitation of planar rotator (and planar Potts analogues) is spin-wave.

Spin-waves preclude long-range order in planar rotator (Mermin-Wagner theorem).

Spin-wave spectrum exhibits algebraic decay of correlations.

Bound vortex-pairs are additional excitations, ultimately leading to breakdown of spin-wave phase and a transition to a disordered state.

# K-T transitions

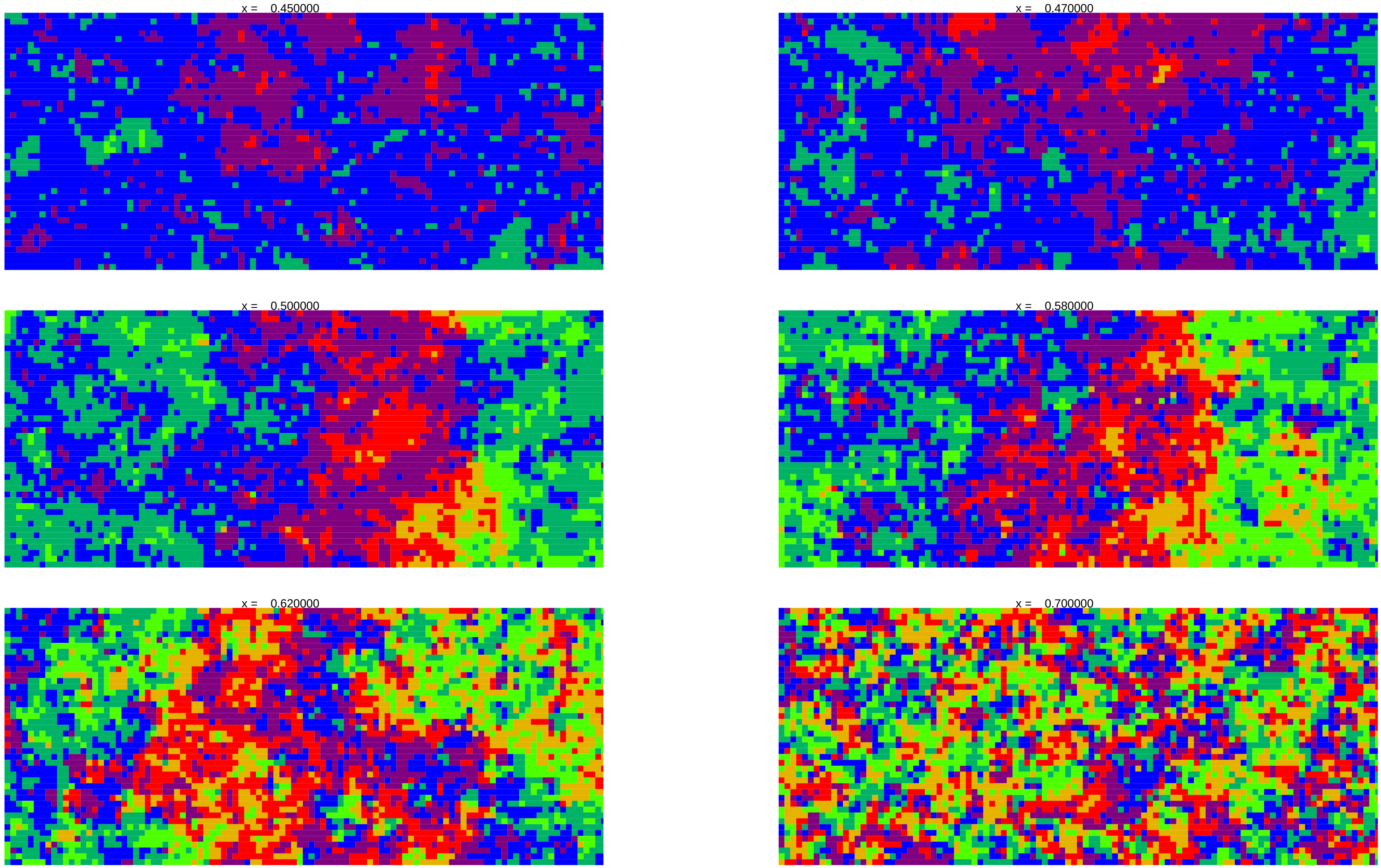
Phase between ordered and disordered phases has long-range algebraic decay of correlations (with varying exponent $\eta$). Termed 'massless' phase, based on particle physics identification of correlation length with inverse mass.
Onset of massless phase has expected critical behaviour with

$$M \sim \exp(-c/\sqrt{T_{\mathrm{C}} - T})$$

known as a Kosterlitz-Thouless transition.

In geology, K-T transition means Cretaceous-Tertiary, time of extinction of dinosaurs, with iridium-rich layer from asteroid impact.

# Monte Carlo: 6-state planar




Simulations for 0.45, 0.47, 0.50, 0.58, 0.62, 0.70.
c.f. $x_1 \approx 0.49$, $x_2 \approx 0.60$ series: Barber and Enting (1981).

# Series: 6-state planar model

Barber and Enting (1981) analysed massless phase $T_1 < T < T_2$ using high-field series for varying $\delta$ where $M \sim H^{1/\delta}$ as $H \to 0$, expecting $\delta = 1$ for $T > T_2$ and $\delta = \infty$ for $T < T_1$ (actually diagnosed by poor convergence of estimates). Hyperscaling $(\delta - 1)/(\delta + 1) = (2 - \eta)/d$.

Series to $\mu^9$ ($\mu = \exp(-H/k_\mathrm{B}T)$) from code method (partial generating functions: from sublattice summations).

Low temperature series for $M$ to $x^{16}$ too short to analyse (and wrong). With extension (2005) to $x^{31}$, analysis is still problematic.

# Finite Lattice Method

Expansions start (formally or actually) from partition function $Z$, or limit of $Z^{1/N}$, often as multi-variate series so derivatives can be taken.

Approximations from inclusion-exclusion relations extract $Z^{1/N}$ behaviour from finite rectangular lattices with $Z_{m,n}$ calculated by transfer matrix techniques: e.g.

$$Z^{1/N} \approx [Z_{m+1,m+1} Z_{m,m}]/[Z_{m+1,m} Z_{m,m+1}]$$

More efficient cutoff uses $Z_{m,n}$ with $m + n \leq k$.

# 6-state results (1/12/05)

Order parameter series extended to $x^{41}$.

Series inconsistent with power-law.

$M \sim \exp(-c/\sqrt{x_\mathrm{c} - x})$ implies

$X = \ln M \sim (x_\mathrm{c} - x)^{-0.5}$

Fit differential approximants to series $x^{-4}X$

No indication of confluent singularity.

| | $x_c$ | exponent |
|---|---|---|
| 1st order DA | $0.4881 \pm 0.0006$ | $0.55 \pm 0.05$ |
| 2nd order DA | $0.4888 \pm 0.0008$ | $0.50 \pm 0.05$ |

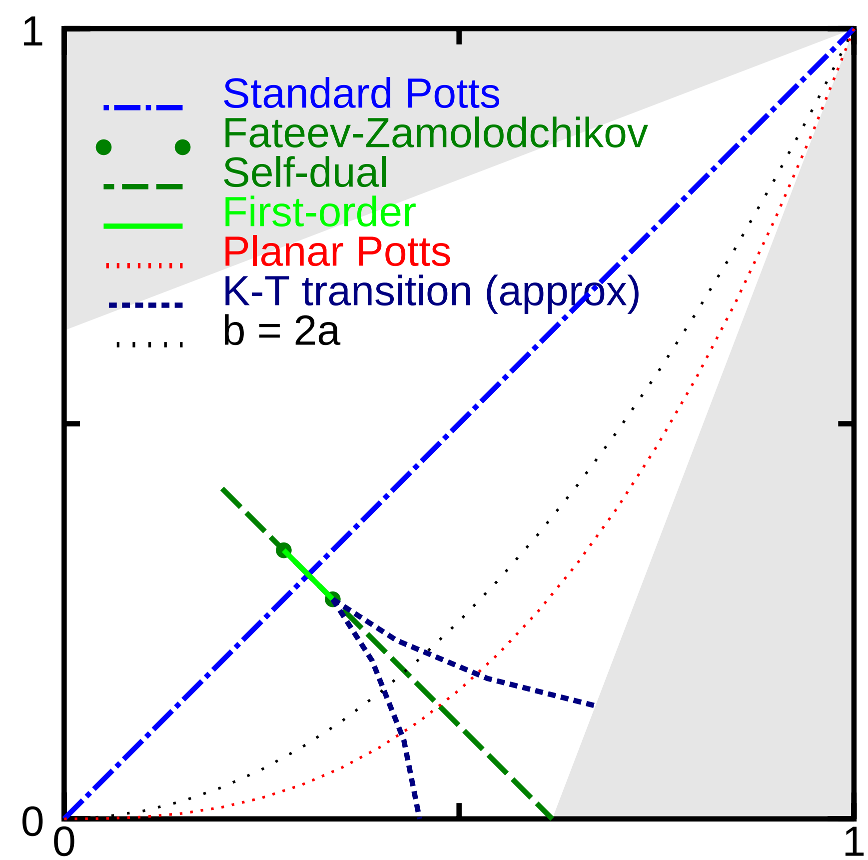
Padé approximants estimate $c = 0.02750$

# 6-state susceptibility: 4/12/05

- Low-T series for $\chi$ to order 35 — more field terms, so fewer temperature terms.
- Use known $x_c$ from $M$.
- Not a conventional singularity
- $\chi \sim \exp(A/(x_\mathrm{c} - x)^\alpha)$
- estimate from DAs to $\ln \chi$ suggest $\alpha \approx 0.9$
- probably $\alpha = 1$ ?????

# Phase space of 5-state models

Rotation/reflection symmetries imply 2 possible pair energies (relative to $x_j = x_k$). For $|x_j - x_k| = 1$ or 2, weights are $z_1$ and $z_2$. If energies sufficiently different, expect 2 phase transitions (Wu, Cardy). High-T vs low-T (and associated transition lines) related by duality (Wu).




Solution by Fateev and Zamolodchikov may mark bifurcation point.

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# Analysis: 5-state models

Low-$T$ series for order parameter using FLM expanded in $x$ with $z_1 = x^a$, $z_2 = x^b$.
$M$ to order $x^{41}$ for $a/b = 1/2, 1/3, 1/4$.

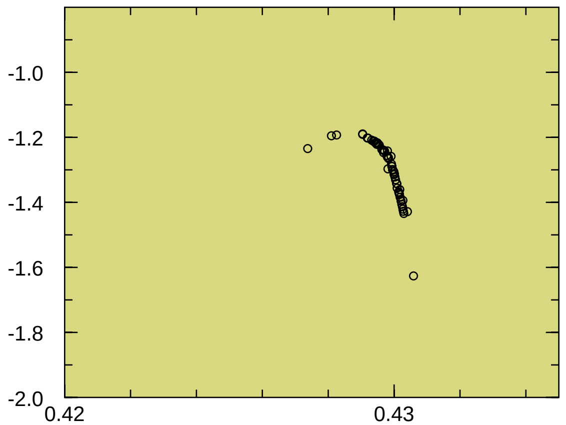
$M \sim \exp(-c/\sqrt{x_c - x})$ implies
$X = \frac{d}{dx} \ln M \sim (x_c - x)^{-1.5}$

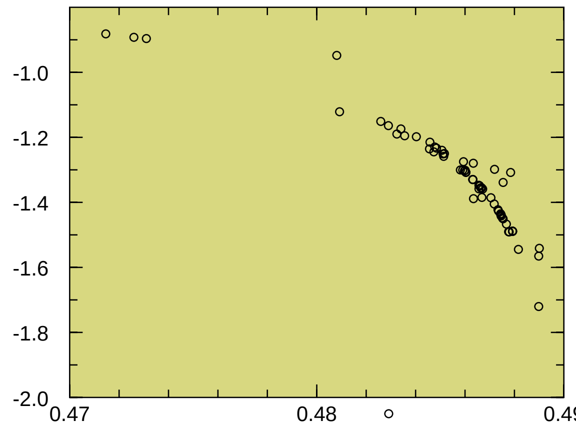
Fit differential approximants (DAs) to series $X$
$P_2(x)\frac{d^2}{dx^2}X + P_2(x)\frac{d}{dx}X + P_0(x)X = 0$

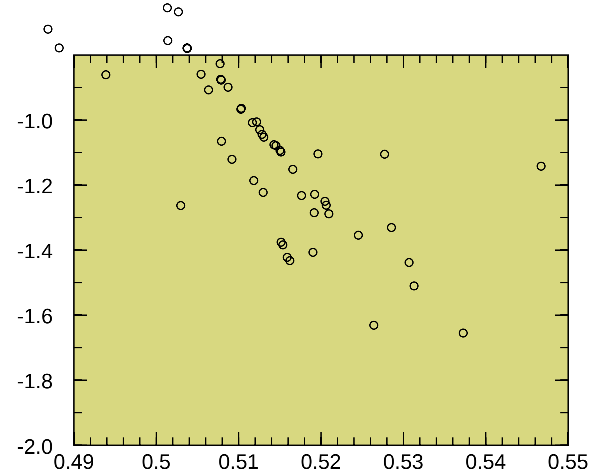
Roots of $P_2$ give $x_c$ and indicial equation gives exponent (-1.5 expected) (Guttmann 1989).
Poor convergence, especially if $a/b$ small.

# Results: 5-state models

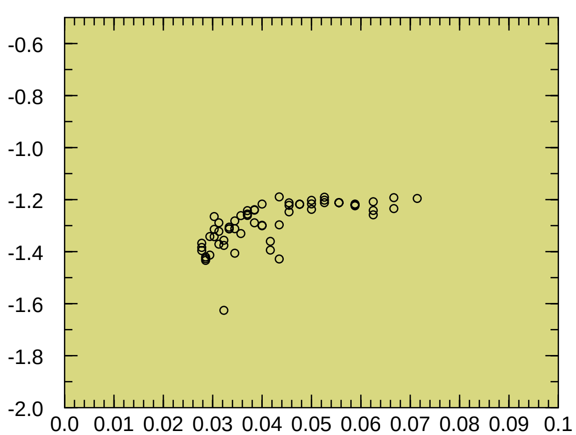
## Exponent vs critical point

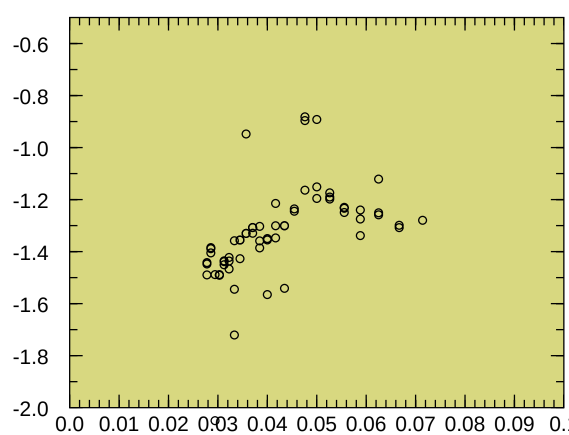


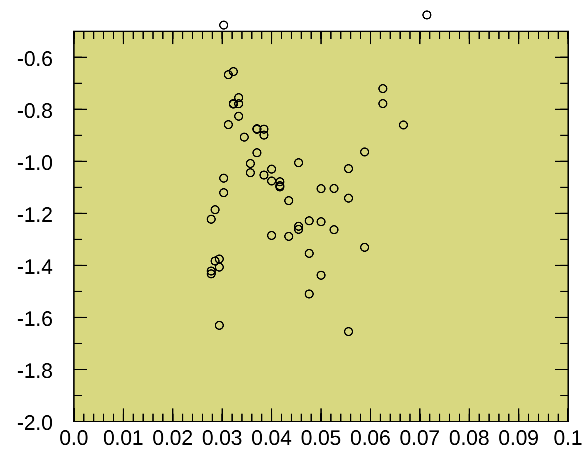




$a/b = 1/2$ $a/b = 1/3$ $a/b = 1/4$




## Exponent vs 1/number of terms fitted

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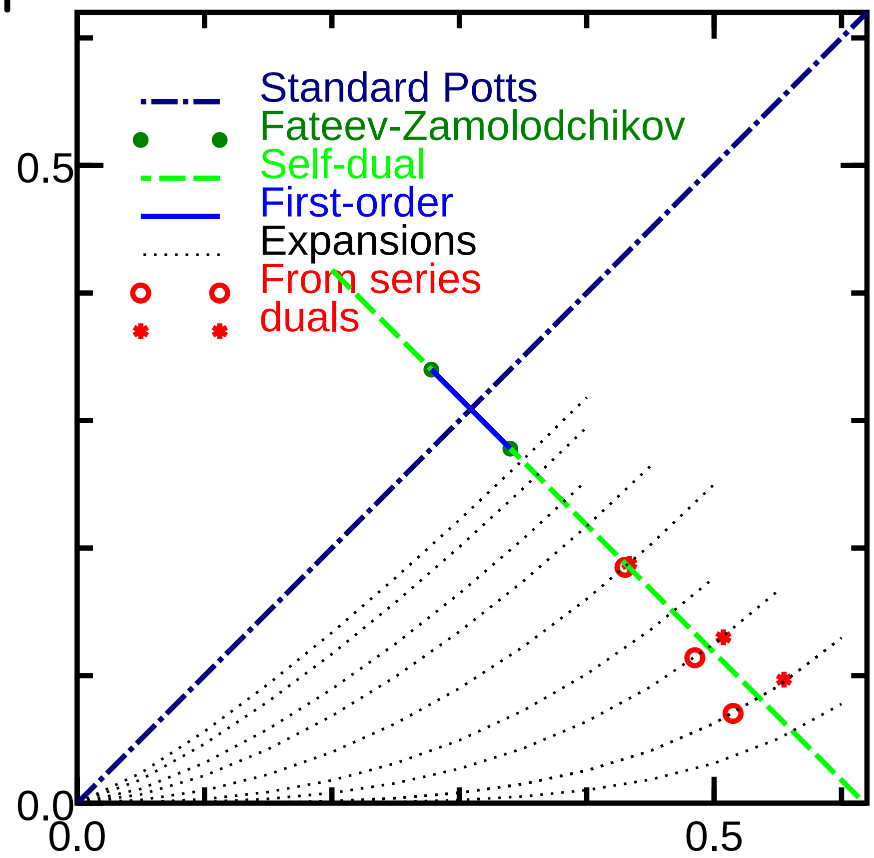
# Series: 5-state models

Low-$T$ expansions for order parameter using FLM. Expanded in $x$ with $z_1 = x^a$, $z_2 = x^b$ ($a$, $b$ small integers).

Estimated critical points for $a/b = 1/2,\ 1/3,\ 1/4$ suggest massless phase is narrow for $q = 5$.




Paths of fixed $a/b$ dotted.

# Implications

Kosterlitz-Thouless Transitions can be successfully studied by series analysis of favourable cases.
Prediction of $-1/2$ exponent in exponential confirmed.
It is likely to be hard to confirm the role of the Fateev-Zamolodchikov point.

# Possible future directions

- Longer low-$T$ series;
- Revisit high-field series analysis (as per Barber and Enting)
- More insight into structure of singularity: $\exp(-c/\sqrt{T_\mathrm{c} - T})$ vs $(T_\mathrm{c} - T)^\beta \exp(-c/\sqrt{T_\mathrm{c} - T})$
- Cases that are easier (for series): Z6 to Z5 (to Ashkin-Teller?).