Families of $m$-convex polygons: $m = 1$.

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Abstract

Polygons are described as almost-convex if their perimeter differs from the perimeter of their minimum bounding rectangle by twice their ‘concavity index’, $m$. Such polygons are called $m$-convex polygons. We first use the inclusion-exclusion principle to rederive the known generating function for 1-convex self-avoiding polygons (SAPs). We then use our results to derive the exact anisotropic generating functions for osculating and neighbour-avoiding 1-convex SAPs, their isotropic form having recently been conjectured.

1 Introduction

In his seminal paper [1] Temperley introduced a number of combinatorial problems that endure to this day. These problems entail the enumeration of self-avoiding walks (SAWs) that are necessarily closed, forming self-avoiding polygons (SAPs). Despite a great deal of initial work, rigorous results remain elusive. What was more attainable was the proof [2] that there exists a certain exponential asymptotic growth in the number of SAWs, counted by their length, and SAPs, counted by their perimeter, which is known to be the same for a given lattice. Furthermore, for length $n$, it is believed that their asymptotic behaviour is described by $\mu^n n^{\gamma-1}$, where $\mu$ is the growth constant and $\gamma$ the critical exponent [3–5].

Exact results are more difficult and have so far required the restriction of the enumeration of SAPs to subclasses that are in some way convex. In two dimensions, convexity means that the perimeter is equal in length to the minimum bounding rectangle (MBR), such as in Figure 1. Progress in exact enumeration came via the now standard method for counting the sub-class of convex polygons called staircase polygons [6–8], which lead to the enumeration of convex polygons [9–12]. The convexity condition was then relaxed in one dimension, giving row- or column-convex polygons [13–17]. In 1997, Bousquet-Mélou and Guttmann (BMG) [18] gave exact results for convex SAPs of three dimensions and a method for their enumeration in an arbitrary dimension [18]. They used an inclusion-exclusion argument that is our primary tool in this paper.

In 1992, Enting et al. [19] classified self-avoiding polygons (SAPs) on the square lattice according to a ‘concavity index’, $m$. Almost-convex polygons are those SAPs whose perimeter differs from that of their minimum bounding rectangle (MBR) by twice their concavity index, if

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Figure 1: Examples of oriented convex polygons of perimeter 28.
this is non-zero. If their concavity index is \( m \), they are said to be \( m \)-convex polygons\(^1\). Enting et al. proceeded to derive the asymptotic behaviour of the number of \( m \)-convex polygons according to their perimeter, \( n \), for \( m = o(\sqrt{n}) \). The results were confirmed for the case \( m = 0 \) (i.e. convex polygons) by the known perimeter generating function. Soon after their paper was submitted, Lin [20] derived the exact generating function for 1-convex polygons, using a ‘divide and conquer’ technique introduced to the problem of convex animals (the interior of a convex SAP) by Klarner and Rivest [21]. His result provided support for a conjecture in [19], giving the next term in the asymptotic expansion.

Convex polygons, which we do not assume to be self-avoiding, can be uniquely determined by rooting them at an arbitrary vertex. The polygons in Figure 1, for example, are rooted at the bottom corner. If the polygons do not immediately reverse direction, forming a 1-dimensional loop, then they are either self-avoiding or have only 2-dimensional loops. Now, the inclusion-exclusion approach can be summarised by saying that one over-counts the class one is interested in (i.e. convex SAPs) by considering convex polygons that are not necessarily self-avoiding, and then one removes those that intersect. To do this, they factor intersecting convex polygons as unimodal polygons with unimodal loops at their root. A unimodal polygon can itself be factored as a staircase polygon with a unimodal loop. The approach therefore requires the enumeration of these subsets of the desired classes of polygons before obtaining the final results - a requirement that encapsulates the structure of this paper.

Section 3 extends the two-dimensional case of BMG’s argument to the case where \( m = 1 \). The motivation for this is to generalise this extension for cases where \( m > 1 \), which does not seem realistic using the ‘divide and conquer’ technique. These results are expected shortly. We begin, in Section 3.1, by defining ‘\( m \)-defective polygons’. These are characterised by the number of exceptions, \( m \) to their convexity constraint. Those which are not 1-staircase we label ‘corner-defective’ polygons. These motivate the definition of the series \( S_{g,b} \) which uncovers a useful change of variables that makes the generating function of many classes of convex polygons rational in these variables. We then enumerate the corner-defective polygons to leave us with 1-staircase SAPs. We then follow an analogous approach to enumerate 1-defective unimodal polygons in Section 3.2. Among these polygons there are also intersecting loops with no staircase factors, whose indent forms a ‘twist’ in the polygon. We enumerate these, as well as ‘corner-defective’ unimodal polygons, to obtain the generating function for 1-unimodal SAPs. The enumeration of 1-convex polygons is completed in Section 3.3, leading to the confirmation of Lin’s result for 1-convex SAPs in [20] as a corollary.

A fairly recent development of Jensen’s [22] was to obtain the isotropic generating function for two classes of polygons closely related to almost-convex SAPs, namely osculating polygons (OPs) and neighbour-avoiding polygons (NAPs). Osculating polygons are described by Jensen as the super-set of SAPs that can be described by a polygon which can touch at one or more vertices but does not overlap (share an arc) or cross. Neighbour-avoiding polygons must have any neighbouring vertices joined by an edge. He proved results for convex OPs and NAPs and gave conjectures for 1-convex OPs and NAPs. We examine, in Section 4, the structure of osculating 1-convex SAPs, as well as their staircase and unimodal subclasses. We are therefore able to easily express the generating functions of osculating 1-convex SAPs in terms of the subclasses of 1-convex SAPs that we enumerated in Section 3. In Section 5, we look at the known bijection between osculating and neighbour-avoiding convex polygons in order to extend it to the 1-convex case. The same bijection will work if a neighbour-avoiding constraint is imposed locally around the indent. We must therefore repeat the enumeration of almost-convex polygons done in Section 3 with this extra condition. We then construct OPs with a neighbour-avoiding condition on the indent and apply the bijection to obtain the anisotropic generating function of 1-convex NAPs, whose isotropic case verifies Jensen’s conjecture. Finally, we study the asymptotic behaviour of the osculating and neighbour-avoiding polygons in Section 6, giving the leading term for \( m = o(\sqrt{n}) \) and a conjecture for the first correction term.

\(^1\)‘Almost-convex’ has sometimes been used to describe the 1-convex case, which we consider to be one example of almost-convex polygons.
2 Notation and Definitions

2.1 Review of convex polygons

We assume that the reader is familiar with the basic definitions of polygons. Please refer to [18] for these definitions, and that of the operator $E_I$.

Notation (Word). A polygon or walk on the square lattice will often be represented as a word $u = u_1 u_2 \ldots u_n$ on the alphabet $\mathcal{A} = \{1, 2, \bar{1}, \bar{2}\}$. If $u_i = k$ (resp. $\bar{k}$), then $s_{i+1} - s_i = e_k$ (resp. $-e_k$), and we note that $\bar{k} = k$.

Notation (Negative alphabet, composite alphabet). If we take the alphabet $\mathcal{I} \subset \mathcal{A}$, we denote $\mathcal{I}^+ \subset \mathcal{I}$ to be the set of positive elements of $\mathcal{I}$, and $\mathcal{I}^-$ to be the set of negative elements. The negative alphabet, $\bar{\mathcal{I}}$ is $\{k | k \in \mathcal{I}\}$. Its composite alphabet, $\mathcal{I}^c$ is $\mathcal{A} \setminus \mathcal{I}$.

Notation (Direction). If a directed walk can be represented by a word that consists of, and only of, steps of the set $\mathcal{I}$ such that $\mathcal{I} \cap \bar{\mathcal{I}} = \emptyset$, then it is a walk in direction $I$. A 1-dimensional walk in direction $\{k\}$ is referred to simply as in direction $k$.

Notation (Spanning rectangle). The MBR of two vertices, $s_i$ and $s_j$ is called the $(s_i, s_j)$-spanning rectangle. SR is the acronym of the spanning rectangle, while the SR of a polygon $p$ is denoted SR($p$).

Notation (m.b.g.f., m.g.f., s.g.f.). The acronym m.b.g.f. is used for the (anisotropic) minimum bounding rectangle perimeter generating function. This is the multi-perimeter generating function, or m.g.f., for convex polygons. Also, s.g.f. refers to the spanning rectangle perimeter generating function.

Notation ($E$). For our two dimensional case, $E_{[1,2]}[f(x,y)]$ is denoted $E[f(x,y)]$.

2.2 Almost-convex polygons

A SAP on the square lattice is said to be $m$-convex if its perimeter differs from that of the MBR by twice its concavity index, $m$. These terms will be formally defined below.

Definition 1 (Almost-directed). Let us consider a walk, $w$, on the alphabet $\mathcal{A}$ and a direction $I$ such that $\bar{I} = I^c$. We say $|w|_k \in \mathcal{I}$, where $|w|_0 = 0$, is the concavity index of $w$ in direction $k$ w.r.t. $I$. The concavity index w.r.t. $I$ for the walk $w$ is then given by $\sum_{k \in I} |w|_k$. A walk is denoted $m$-directed, of the almost-directed class of walks, if its concavity index is $m > 0$. For example, the walk 22121211 is almost-directed w.r.t. $\{1, 2\}$, with concavity indices 2 and 1 in directions 1 and 2 respectively. The difference between the half-perimeter of the $w$-spanning rectangle and the length of the walk is twice the concavity index.

Notation (Indent). An $m$-convex polygon or $m$-directed walk is characterised by $m$ edges that are in the opposite direction to the other edges around it. Such edges we will call indents. If the edges around it are in direction $k$, the indent is also said to be in direction $k$, despite being a $\bar{k}$ step.

Definition 2 (Concavity index). Let $\mathcal{P}(w)$ be the set of all rooted cyclic permutations of a polygon $w$ and let $\mathcal{F}(w) = \{(u, v) \mid uv \in \mathcal{P}(w)\}$ be the set of all two-part cyclic factorisations of the polygon $w$. We define the concavity index in direction $k$ of $w$ to be

$$\min_{(u,v) \in \mathcal{F}(w)} (|u|_k + |v|_k),$$

and if $(u^*, v^*)$ is the factorisation which achieves this minimum, we denote $[w]_k = |u^*|_k$ and $[w]_k = |v^*|_k$. This is the number of indents that are neither among the first, nor the last elements of $u$ or $v$ respectively. The concavity index of the rooted polygon $w$, denoted $[w]$, is $\sum_{k \in \mathcal{A}} [w]_k$. For some $I$ such that $\bar{I} = I^c$, the concavity indices with respect to $I$ of the rooted polygon $w$ are $\{[w]_k, [w]_{\bar{k}}\}_{k \in I}$, and we write that $[w]_I = \sum_{k \in I} [w]_k$. Note that these definitions extend to dimensions greater than two.
Definition 3 (Almost-staircase). Let us consider those rooted polygons that factor as $uv$, where $u$ and $v$ are (almost-)directed walks in direction $I$ and $I^c$, respectively, between the root and the point $s_{|u|+1}$. If the concavity indices of $uv$ w.r.t. $I$ are $\{u[k], v[k]\}_{k \in I}$ such that $I = I^c$, then the rooted polygon $uv$ is called a rooted almost-staircase polygon, and we call $s_{|u|+1}$ the co-root. We note that this means that all the indents are inside the SR. If its concavity index, $[uv]$, is equal to $m > 0$, it is called a rooted $m$-staircase polygon. An almost-staircase polygon is one that can be represented by a rooted almost-staircase polygon. Such a polygon is also called an $m$-staircase polygon if its concavity index is $m$. An example of a 1-staircase polygon can be found in Figure 2(a).

Notation (co-root). The co-root, defined above, of an almost-staircase polygon will sometimes be marked by a dot. The staircase polygon, $uv$ whose co-root is $s_{|u|+1}$ is therefore denoted $u-v$.

Definition 4 (Almost-unimodal). A rooted polygon is $m$-unimodal in direction $k$ if it can be written as $uv$, where $u$ and $v$ are words on $A$ such that $[uv]_k = |u[k]|, [uv]_{k^c} = |v[k]|$ and $m = [uv]_k$. Now, let us take $I$ such that $I = I^c$. A rooted polygon is called $m$-unimodal in direction $I$ if it is $m_k$-unimodal in each direction $k \in I$ and $m = \sum_k m_k$. An $m$-unimodal polygon is one that can be represented by a rooted $m$-unimodal polygon. An almost-unimodal polygon is one that is $m$-unimodal for $m > 0$.

Definition 5 (Almost-convex). A polygon is $m$-convex in direction $k$ if it can be represented (after a cyclic permutation) by a rooted polygon that is $m$-unimodal in direction $k$. Let us take $I$ such that $I = I^c$. If there exists some cyclic permutation of the polygon that is $m_k$-unimodal for all directions $k \in I$, then the polygon is denoted $m$-convex, where $m = \sum_k m_k$. Such a polygon is also called an almost-convex polygon if its concavity index is greater than zero. An example of a convex polygon can be found in Figure 2(b).

3 The enumeration of polygons with concavity index, $m = 1$

3.1 Enumerating 1-staircase polygons

The function $Z_A$ is defined in [18], a special case of which is $Z_d$ that counts the number of staircase polygons in $d$ dimensions. Let us denote as $Z$ the multiperimeter generating function (m.g.f.) for 2-dimensional oriented staircase polygons, $Z_2$:

$$Z \equiv Z_2(x, y) = \sum_{n,m} \binom{n+m}{n} x^n y^m = \frac{1}{\sqrt{\Delta}}$$

where $\Delta = 1 - 2x - 2y - 2xy + x^2 + y^2$. This gives us the m.g.f. for oriented staircase SAPs, $S = 1 - \sqrt{\Delta}$, as $Z = 1 + SZ$ from the standard factorisation argument for counting staircase SAPs. In contrast, we denote the m.g.f. for non-oriented, 2-dimensional (i.e. not 11 or 22) staircase polygons to be $SP$ and note that $SP = (S - x - y)/2$.

One can easily see the bijection between staircase polygons and pairs of directed walks between opposite corners of the MBR, which gives us $Z$. For the almost-convex case, a directed walk and a 1-directed walk are taken between the root and co-root to generate all 1-staircase
polygons. However, if the last vertical step is the indent, then the resulting unrooted polygon is not almost-convex, it is unimodal (as in Figure 3). Similarly, it is not 1-staircase if the first vertical step is the indent. Let us begin, then, by enumerating this class, which we denote 1-defect staircase polygons, of the defective-convex class of polygons, and we will exclude the extra polygons later.

**Definition 6 (Defective-staircase).** If a rooted polygon factors as \( uv \), where \( u \) and \( v \) are (almost-) directed walks between a vertex taken as the root and a vertex as co-root in directions \( I \) and \( \overline{I} \) respectively such that \( \overline{I} = I^c \), then \( uv \) is said to be a rooted defective-staircase polygon in direction \( I \). As such, the polygon \( uv \) is defective-staircase in direction \( I \). Note that the concavity index of \( uv \) is not necessarily equal to the sum of the concavity indices w.r.t. \( I \) and \( \overline{I} \) of the walks \( u \) and \( v \) respectively, which is half the difference in length of the polygon and its SR. If this sum is equal to \( m \), then we denote the polygon \( uv \) as \( m \)-defect staircase and define \( [u-v]^m = m \). A defective-staircase polygon is, by definition, rooted. Defective-staircase polygons will usually be enumerated according to the SR of the root and co-root.

### 3.1.1 The inclusion-exclusion principle and 1-defective staircase polygons

We demonstrate the use of the inclusion-exclusion principle here by enumerating 1-defective staircase polygons, and we follow this demonstration mutatis mutandis to enumerate later classes of polygons that can be described simply by words on the alphabet \( A \).

**Lemma 3.1.** Let us consider the rooted 1-defect staircase polygons in direction \( A^+ \) such that the indent is not in a 1-dimensional loop. The (root, co-root)-spanning rectangle generating function (s.g.f.), denoted \( Z' \), is

\[
Z' = \frac{2}{y} \delta x^y Z.
\]

_Proof._ Let us take a rooted 1-defect staircase polygon, \( u\cdot v \), and assume, without loss of generality, that the indent is in direction 2. For an \( n \times m \) spanning rectangle, this puts \( u \) (resp. \( v \)) in one-to-one correspondence with the words \( u_1 u_2 \ldots u_{n+m+2} \) (resp. \( v_1 v_2 \ldots v_{n+m} \)) on the alphabet \( A^+ \) (resp. \( A^- \)) such that

- an occurrence of 2 in \( u \), corresponding to the indent, is distinguished, and
- \(|u_1| = |v_1| = n, |u_2| = m, |u_2| = m + 1, |u_2| = 1, |u_1| = |v_1| = |v_2| = 0.\]

Thus, the s.g.f. for 1-defect staircase polygons is

\[
\sum_{n,m} (m+2) \binom{n + m + 2}{n} \binom{n + m}{n} x^ny^m = \sum_{n,m} (n+1) \binom{n + 1 + m + 1}{n+1} \binom{n + m}{n} x^ny^m. \tag{1}\]

More generally, let us consider the set of directions in which there is a one dimensional loop containing the indent, denoted \( J \). Here the indent is in direction 2, so we have \( J \subseteq \{2, 3\} \). Thus, the m.g.f. for rooted 1-staircase polygons, \( uv \), having a factor \( k \) in \( u \) for all \( k \in J \) is

\[
S_J = \sum_{n,m} (n+1) \binom{n + 1 + m + 1 - |J|}{n+1} \binom{n + m}{n} x^ny^m.
\]
Figure 4: Examples of 1-staircase polygons that are formed by composing staircase polygons with a 1-defect SAP. If the indent is the first or last step of the 1-defect SAP, it can form a 1-dimensional loop with the adjacent self-avoiding loop.

Using the inclusion-exclusion principle, the s.g.f. for rooted 1-defect staircase polygons with no backtracks is
\[ \sum_{J \subseteq \{2,3\}} (-1)^{|J|} S_J = \sum_{n,m} \binom{n-1+m+1}{n-1} \binom{n+m}{n} x^n y^m = \frac{x}{\delta x} y \sum_{n,m} \binom{n-1+m}{m} \binom{n+m-1}{n} x^m y^n. \quad (2) \]

Note that the series is the generating function for the number of pairs of directed walks from the points (1, 0) and (0, 1), both going to the point (n, m). Consider the walk containing the point (1, 0) (resp. (0, 1)) as positively (resp. negatively) directed. Then there is a bijection between the objects counted by this series, and the number of polygons uv such that \( \ell \) is a (possibly empty) staircase polygon, and the polygon \( 1uv \bar{2} \) (resp. \( 2uv \bar{1} \)) is a 2-dimensional staircase loop (n.b. it is positively oriented and self-avoiding). The bijection is therefore with the composition of a non-oriented 2-dimensional staircase loop and an oriented staircase polygon, and the generating function is \( SP Z \).

3.1.2 Factorisation to yield self-avoiding 1-defect staircase polygons.

Definition 7 (Maximal staircase decomposition). For some rooted \( m \)-defective staircase polygon \( w \), in direction \( I \), let the set of staircase decompositions such that \( w \) contains all the indents be
\[ S_I(w) = \{(u, l_1, v, l_2) | pu_I v_q w, q_p = l_2, [u-v]^{(s)} = m \}. \]

The maximal staircase decomposition of a rooted defective-staircase polygon \( w \), which we denote \( w^{(s)} = (u^*, l_1^*, v^*, l_2^*) \), is defined as \( \max_{\text{MBR}(l_1)} \max_{\text{MBR}(l_2)} S_I(w) \), which maximises the size of the non-defective factors, while ordering symmetrical decompositions. For example, \( S_{A^+}(12 \hat{1} \hat{2} 122 \hat{1} \hat{2} 1 \hat{2} \hat{1}) = \{(12 \hat{1} \hat{2} 122, \emptyset, \hat{1} \hat{2} 1 \hat{2} \hat{1}), (1 \hat{2} 122, \emptyset, \hat{1} \hat{2} 1 \hat{2} \hat{1}), (1 \hat{2} 1 \hat{2} 1 \hat{2} 2, \emptyset, \hat{1} \hat{2} 1 \hat{2} \hat{1})\} \), and the maximal decomposition is \( 12 \hat{1} \hat{2} 1 \hat{2} 2 1 \hat{2} \hat{1} \hat{2} = (1 \hat{2} 122, \emptyset, \hat{1} \hat{2} 1 \hat{2} \hat{1}) \).

Proposition 3.2. If \( S' \) is the s.g.f. for 1-defect staircase SAPs in direction \( A^+ \) with a vertical indent, then
\[ S' = A Z' + 2 SP (3 SP + 2y)y. \]

Proof. Let \( w \) be an arbitrary 1-defective staircase polygon in direction \( A^+ \) such that the indent is not in a 1-dimensional loop. It has a unique maximal decomposition, \( w^{(s)} = (u^*, l_1^*, v^*, l_2^*) \). We assume without loss of generality that the indent occurs in direction 2, that is, it is in \( u^* \). The s.g.f. \( Z^2 S' \) generates \( w^{(s)} \) for all such \( w \). For example, the first decomposition in Figure 4(b) is maximal and is generated.

However, \( Z^2 S' \) also generates decompositions, \( (u, l_1, v, l_2) \) where the first (resp. last) step of \( u \) is the indent, that is, a \( \hat{2} \), and where the last (resp. first) step of \( l_2 \) (resp. \( l_1 \)) is a 2. In this case, a 1-dimensional loop is formed in the composition \( ul_1vl_2 \). The second and third polygons in Figure 4(b) and both polygons in Figure 4(c) are examples of such polygons, which are enumerated by the s.g.f. \( 2Z SP (SP + y)Z \).
Now consider those maximal decompositions of 1-staircase polygons whose last step of $u$ is the indent. If the last step of $l_1$ is also a 2, while its first step is a 1, then this decomposition is not maximal. Take the last polygon of Figure 4(b) as an example. This polygon is the same as the first polygon in the figure, but the way of factorising it is not maximal. These non-maximal compositions are generated by the s.g.f. $Z^2SP^2$. We are therefore left with the identity $Z' + 2ZSP(3SP + 2y)Z = ZS'Z$. 

3.1.3 Corner self-avoiding polygons

Definition 8 (Corner-staircase, corner-unimodal). Any 2-dimensional rooted polygon that has a cyclic permutation of its vertices, $uu$, such that

- $v = i\ldots ju_1\ldots j$ forms a unimodal polygon in direction $I = \{i, j\}$ such that $|v|_{1,1} = a$ and $|u|_{1,2} = b$, and

- $w$ is an almost directed walk in direction $\{i, j\}$ that contains the root,

is denoted a (rooted) $(a, b)$-corner unimodal polygon in direction $I$. If the polygon $v$ is staircase, it is denoted $(a, b)$-corner staircase in direction $I$. Note that if this walk is $m$-directed, then the corner unimodal (resp. staircase) polygon will be $m$-convex (resp. $m$-unimodal).

We also want to classify all such polygons for a given root. We therefore denote all corner unimodal (resp. staircase) polygons in direction $I$ whose root is a distance of $(\pm a, \pm b)$ from the root of $v$ (which is the corner of the MBR that is minimal with respect to $I$) as $(a, b)$-corner unimodal (resp. staircase) polygons. For fixed $m$, polygons which are $(a, b)$-corner unimodal (resp. staircase) such that $a + b = m$ are called $m$-corner unimodal (resp. staircase) polygons.

3.1.4 The series $S_{a,b}$

We denote the generating function for pairs of directed walks going from $(a, 0)$ and $(0, b)$ to $(n, m)$, and therefore of $(a, b)$-corner staircase polygons, by $S_{a,b}$. The m.b.g.f. is therefore defined by the formal power series

$$S_{a,b} = \sum_{n,m} \binom{n - a + m}{m} \binom{n + m - b}{n} x^n y^m$$

where $a \geq 0$ and $b \geq 0$. Expanding the first binomial term, rearranging, and repeating for the second term, one arrives at

$$S_{a,b} = S_{a+1,b} + yS_{a,b-1} = S_{a-1,b} - yS_{a-1,b-1} = S_{a,b-1} - xS_{a-1,b-1},$$

which gives us the following recurrence relation that is independent of $b$:

$$S_{a+1,b} = (1 + x - y) S_{a,b} - x S_{a-1,b}. \quad (3)$$

Together with the symmetrical case for $S_{a,b+1}$ and the boundary cases $S_{0,0}$, $S_{1,0}$ and $S_{0,1}$, this will fully determine $S_{a,b}$. Clearly, the boundary case $S_{0,0}$ is simply the staircase polygon m.g.f., $Z$, and $S_{0,1}$ is symmetrical to $S_{1,0}$.

Now, $S_{1,0}$ enumerates the polygons $\ell_e v$ such that $\ell$ is a (possibly empty) staircase polygon, and the polygon $1w$ is a positively oriented self-avoiding staircase loop, or the one dimensional loop $\Pi$. These are enumerated by $(x + SP)Z$. Defining $u = x + SP$ and $v = y + SP$, we have $S_{1,0} = uS_{0,0}$, and, by symmetry, $S_{0,1} = vS_{0,0}$. We are therefore left with

$$S_{a,b} = u^a v^b Z. \quad (4)$$

We then factorise to obtain the m.g.f. for $(a, b)$-staircase SAPs, $u^a v^b$. We can also rearrange our equations for $u$ and $v$ to obtain $x = u(1 - v)$ and $y = v(1 - u)$. As $\sqrt{\Delta}$ is simply $1 - u - v$, we can then write the generating functions for various m.g.f.s as rational functions in $u$ and $v$, including $SP = uv$, $S = u + v$ and $Z = 1/(1 - u - v)$.
3.1.5 Enumerating 1-staircase SAPs

**Lemma 3.3.** The m.b.g.f. for 1-staircase SAPs is

\[
\frac{2}{y^2} \left( -\Delta + x ((1 - x)^2 - 4y) + \frac{A(x, y)}{\sqrt{\Delta}} \right)
\]

where \( A(x, y) = 1 - 4x + 6x^2 - 4x^3 + x^4 - 3y + 5xy - x^2y - x^3y + 3y^2 - xy^2 - y^3 \).

**Proof.** We note that if a 1-defect SAP \( uv \) has its indent in \( u \) such that it is neither the first or last step then it must be 1-staircase. Thus, the required m.b.g.f. must be \( S' = 4S_{[0,1]} \), where \( S_{[0,b]} \) is the s.g.f. for \([a, b]\)-corner staircase polygons.

All \([0, b]\)-staircase loops look like Figure (5a) near their root, \((0, b)\), except for the possibility of a 1-dimensional loop. And so, if we allow 1-dimensional loops at the root and then exclude such polygons by multiplying by \((1 - x)\), we can write

\[
S_{[0,b]} = \frac{(1 - x)}{y} \sum_a \left( \frac{b - 1 + a}{a} \right) S_{a+1,b} = \frac{(1 - x)uv^b}{y(1 - u)^b},
\]

for \( b > 0 \). Letting \( b = 1 \) gives the case we require, which then yields the given result. \( \Box \)

3.2 Enumerating unimodal polygons

**Definition 9 (Defective-unimodal).** For some rooted polygon \( w \), let \( W(w) = \{ (u,v) \mid uv = w \} \) be the set of all two-part (non-cyclic) factorisations of \( w \). The first vertex of \( w \) is denoted its root. Now, let us take \( I \) such that \( \bar{I} = I^c \). We define the number of defects of \( w \) in direction \( I \) to be \( [w]_I = \sum_{k \in I} [w]_{k}^{(u)} \) where \( [w]_{k}^{(u)} = \min_{(u,v) \in W(w)} (|u|_k + |v|_k) \). A rooted polygon, \( w \) is called \( m \)-defect unimodal in direction \( I \) if \( m = [w]_I^{(u)} > 0 \). We define

\[
[w]_I^{(u)} = \min_{I=I^c} [w]_I^{(u)}
\]

and call \( w \) \( m \)-defect unimodal. An \( m \)-defect unimodal polygon is one that can be represented by a rooted \( m \)-defect unimodal polygon. A defective-unimodal polygon is one that is \( m \)-defect unimodal for \( m > 0 \). The spanning rectangle of a defective-staircase polygon is defined to be the SR of the root and the vertex of the MBR farthest from the root. We note that the difference between the lengths of the polygon and its spanning rectangle is twice its number of defects, \([w]_I^{(u)}\).

3.2.1 Corner-defective unimodal SAPs

**Proposition 3.4.** Let us consider those rooted, non-oriented, 1-defect unimodal SAPs that are not 1-unimodal. These are \([0, 1]\)-corner unimodal SAPs, whose s.g.f. we denote \( U_{[0,1]} \). We have

\[
U_{[0,1]} = \frac{1 - x}{y} \left( E \left[ \frac{xy(1 - y)}{1 - x - y} \right] - \frac{uv}{1 - u} E \left[ \frac{xy}{1 - x - y} \right] \right) - \frac{2xP(x, y)}{y} + \frac{x^2}{1 - x},
\]
where \( P(x, y) = \frac{x(1-x)y}{(1-x-y)^2} \), the m.g.f. for pyramid polygons.

Proof. We begin by noting that the form of \([0,1]\)-unimodal polygons around the root is the same as that of \([0,1]\)-staircase polygons as depicted in Figure 5. Reproducing our inclusion-exclusion argument, mutatis mutandis, we can immediately enumerate all our required polygons by enumerating the paths with no 1-dimensional loops from \((\alpha, 0)\) to \((0, 1)\) for some \(\alpha \geq 0\) with the s.g.f.

\[
\frac{1}{y} E \left[ \frac{x^{\alpha+1}(1-x)y(1-y)}{1-x-y} \right].
\]

We must now exclude those polygons that intersect. If they have a corner-staircase polygon as a factor, then they must intersect after the indent. (See Figure 6(a).) Such staircase factors are enumerated by \( S_{\alpha + 1, 1} \), and so adding a unimodal loop gives \( x^{\alpha+1}E \left[ \frac{xy}{1-x-y} \right] \). The only other way these polygons can intersect is before the indent, forming a height one loop in the bottom corner. These either have non-empty pyramid factors, as in Figure 6(b), or the SR is of height zero. If the indent occurs on the edge of the MBR (forming a negatively oriented height one loop), these polygons are enumerated by the s.g.f. of fixed width \(\alpha\)

\[
[x^\alpha] \left( \frac{xP(x, y)}{(1-x)^3} + \frac{x^2}{(1-x)^2} \right) = [x^\alpha] \frac{xP(x, y)}{(1-x)^2}.
\]

If the orientation of the loop is positive, the exclusion case is enumerated by the s.g.f.

\[
\frac{x^{\alpha+1}}{1-x} [x^\alpha] \frac{P(x, y)}{(1-x)^2}.
\]

Summing the above cases over all possible values of \(\alpha\) generates polygons that can have a horizontal 1-dimensional factor at the root, which we can exclude by multiplying by \((1-x)\).

3.2.2 Turning-defective unimodal SAPs

Unimodal polygons that are 1-defect can usually be factored into a 1-defect staircase polygon and a unimodal SAP, or a staircase polygon and a 1-defect unimodal SAP. Now, the polygons that do not factor as either of these forms must factor as a 1-unimodal polygon with a height one loop. (See Figure 7.) We would like to classify these three classes according to some decomposition.

Definition 10 (Maximal unimodal decomposition). For some \( w \) that is a rooted \( m \)-defective unimodal polygon in direction \( I \), let

\[
\mathbb{U}(w) = \left\{ (u, \ell, v) \mid u\ell v = w, [u \cdot v]^{(s)} + [\ell]^{(u)} = m, \ell \text{ is a non-empty polygon} \right\}
\]

be its set of unimodal decompositions, such that if \((s_i, s_{i+1})\) is an indent of \(u\ell v\), then \(|\text{SR}(s_i, s_{i+1})| < |\text{SR}(w)|\) \(\forall j\). The maximal unimodal decomposition of \( w \), denoted \( w^{(u)} = (u^*, \ell^*, v^*) \), is defined as

\[
w^{(u)} = \max_{\mathbb{U}(w)} \max_{[u \cdot v]^{(s)}} \mathbb{U}(w).
\]

We define the maximal unimodal decomposition of an unrooted \( m \)-unimodal polygon to be that of its \( m \)-unimodal cyclic permutation that is rooted in the corner of its SR.
Definition 11 (Turning-defective unimodal). If $w$ is almost-unimodal and its maximal unimodal decomposition, $(u^*, \ell^*, v^*)$ is such that $\ell^*$ is self-intersecting, then we call $w$ a turning-defective unimodal polygon. Note that this means that $\ell^*$ is almost-convex and can be factored as $u^+lu^-$ such that $u^+u^-$ is a unimodal SAP and $l$ is a height one loop containing the indent.

Proposition 3.5. Let $U'_{\text{torn}}$ be the s.g.f. for rooted 1-defect unimodal polygons with no 1-dimensional loops whose maximal decompositions are of the form $(\emptyset, \ell^*, \emptyset)$, such that $\ell^*$ is self-intersecting. Thus, $\ell^*$ contains a height one loop that includes the indent, and

$$U'_{\text{torn}} = \frac{2x}{(1-x)^2} \left( \left(1 + 3x\right)v \mathcal{SP} Z + 2U - yS_{[0,1]} - (1-x)\mathcal{SP} \right).$$

Proof. Given a turning-defective, 1-defect unimodal polygon with no staircase factor, if the indent touches the top of the SR, then the polygon must be of the form depicted in Figure 8(a). Their m.b.g.f. we denote $U'_{\text{corren}}$, and we have

$$U'_{\text{corren}} = \frac{2x}{(1-x)^2} (yS_{[0,1]} - (1-x)\mathcal{SP}).$$

Now, let us assume that the indent does not touch the top of the SR and, without loss of generality, that the orientation is positive. As such, the indent is one of the first $m$ occurrences of 2 and is of the form depicted in Figure 7(b). And so, we can factor $\ell^*$ as $u^+lu^-$ with $l$ being of height one. It is either adjacent to a right-edge arc of the SAP $u^+u^-$, or is attached to one corner of the right-edge. Considering the latter case, when $l$ touches the top-most vertex of the right-edge of $u^+u^-$ but is not adjacent to a right-edge arc, the m.b.g.f. is $\frac{x}{(1-x)^2}(U - 2yS_{[0,1]})$. When $l$ touches the bottom corner, the m.b.g.f. is simply $\frac{x}{(1-x)^2}U$. Now let us consider the former case, where the loop is next to a right-edge arc. Given such an arc, the loop is either immediately adjacent, or has two possible orientations while connecting to either the upper or lower vertex of the adjacent edge. These cases are therefore enumerated by the s.g.f.

$$\frac{x}{(1-x)^2} \left( \left(1 + 3x\right) \left( \sum c \mathcal{U}_c - \mathcal{SP} \right) + 2(U - yS_{[0,1]} \right),$$

Figure 8: The turning-defective polygons when the indent touches the top of the MBR.
where $U_c$ is the m.b.g.f. of rooted unimodal polygons with right perimeter $c$. We transform these unimodal polygons to be in corner-staircase form, as is illustrated in Figure 9. Hence, $U_c = x^c + \sum_{a,b} x^{(a+1)v+b+c+1} = xv^c(1 + \mathcal{S}_P Z)$, which leaves us with

$$
\sum_c cU_c - \mathcal{S}_P = \frac{xv}{(1-v)^2}(1 + \mathcal{S}_P Z) - \mathcal{S}_P = \mathcal{S}_P ((1-u)Z - 1) = v \mathcal{S}_P Z.
$$

3.2.3 Enumerating 1-unimodal SAPs

**Lemma 3.6.** The m.b.g.f. for 1-unimodal SAPs is

$$
\frac{x(2(1-x)^2-x)}{(1-x)y} + \frac{2xP(x,y)}{y} + \frac{xU_A(x,y)}{(1-x)y} \Delta^{3/2}
$$

where $U_A = 2(1-x)^5 - y(1-x)(7-10x-x^2+4x^3) + y^2(1-x)(9-x) - y^3(5+3x+2x^2) + y^4$

and $P(x,y) = \frac{x(1-x)y}{(1-x)^2-y}$, which is the m.b.g.f. for pyramid polygons.

**Proof.** Let $Q'$ denote the s.g.f. for rooted 1-defect unimodal polygons in direction $A^+$ with no 1-dimensional loops. Then,

$$
Q' = E \left[ \delta \frac{x(1-y)^2}{y(1-x)(1-x-y)} + \frac{x(1-y^2)}{(1-x)(1-x-y)} \right]
$$

from Lemma 3.1, *mutatis mutandis*. Let us then denote the s.g.f. for rooted 1-defect unimodal SAPs that have a vertical indent as $U'$ and consider those polygons with maximal decomposition $(u^*, l^*, v^*)$. Following Proposition 3.2 *mutatis mutandis*, the s.g.f. of such polygons with their indent in $u^*v^*$ is $2Z'U - Z \mathcal{S}_P U/y - ZU_{\text{turn}}$, and that of those with their indent in $l^*$ is $Z(U' + U'_{\text{turn}}) - Z(\mathcal{S}_P + v)U/y$. Therefore,

$$
U' = \frac{Q' - 2Z'U}{Z} + \frac{U}{3\mathcal{S}_P + y} + U'_{\text{turn}} - U'_{\text{turn}}
$$

If the indent of a rooted 1-defect unimodal SAP in neither its first, nor its last vertical step, then it must be 1-unimodal. Otherwise, it must be corner-defective. Thus, the m.b.g.f. for 1-unimodal SAPs is the difference between $U'$ and the s.g.f.s of corner-defective unimodal SAPs given in Proposition 3.4. \qed

3.3 Enumerating 1-convex polygons

**Definition 12 (Maximal convex decomposition).** Let $w$ be some $m$-convex polygon that is rooted at its first vertical step. We define

$$
C(w) = \{(u_{\ell_1}v, \ell_2) | w = p\ell_1q, (u_{\ell_1}v, \ell_2, u) \in \mathcal{U}(u_{\ell_1}v), (v, \ell_2, u) \in \mathcal{U}(pq)\}
$$

to be the set of convex decompositions of $w$, such that if the indent of $u_{\ell}v$ is $(s_i, s_{i+1})$, then $s_{i+1} \notin \text{MBR}(u_{\ell}v)$. 

11
The maximal convex decomposition of a rooted convex polygon, $w$, which we denote $w^{(c)} = (u^c, \ell^*_1, v^*, \ell^*_2)$, is defined as

$$w^{(c)} = \max_{[\ell_1, [u, v]_{x^0}]} \max_{[\ell_2, [u, v]_{x^0}]} C(w).$$

We arbitrarily take the maximal decomposition of an unrooted $m$-convex polygon to be that of the cyclic permutation rooted at its first positive vertical step.

**Definition 13 (Turning-defective).** Let $w$ be an almost-convex polygon whose maximal convex decomposition, $(u^*, \ell^*_1, v^*, \ell^*_2)$ is such that $\ell^*_1$ or $\ell^*_2$ is self-intersecting. Note that this means that $\ell^*_1$ or $\ell^*_2$ is almost-convex and can be factored as $u^+u^-$ such that $u^+u^-$ is a unimodal SAP and $l$ is a height one loop containing the indent. We say that $w$ is an (intersecting) turning-defective convex polygon.

Furthermore, if $w$ is almost-convex, intersecting and admits no convex maximal decomposition in any direction, then we call it a turning-defective (else self-avoiding) convex polygon. For example, a 1-unimodal polygon with maximal unimodal decomposition of the form $(0, \ell, 0)$ such that $\ell$ is intersecting has no maximal convex decomposition. Such polygons can be factored as $e^+le^-$ such that $e^+e^-$ is a convex SAP and $l$ is a height one loop containing the indent, and are referred to as a 1-turning-defect convex polygons.

**Proposition 3.7.** Let $C^{i}_{i\text{ren}}$ be the m.b.g.f. for rooted, 1-turning-defect convex polygons with no 1-dimensional loops. Then,

$$C^{i}_{i\text{ren}} = \frac{4x}{(1-x)^2} \left( 1 + 3x \left( \sum_m mR_m - \left( U - \frac{xy}{1-x} \right) \right) + 2(C-yU[0,1]) - U + x \left( U - \frac{2xy}{1-x} \right) \right),$$

where

$$\sum_m mR_m = \frac{xy(1-x)(1-x-y)^2}{\Delta^2} + \frac{4xy^2(U/2-SP-x)}{\Delta^{3/2}}.$$

**Proof.** Considering the polygons enumerated by $C^{i}_{i\text{ren}}$, let us assume, without loss of generality, that their orientation is positive, and the indent is one of the first $m + 1$ occurrences of 2. The cardinality of these polygons is therefore four.

If the indent touches the top of the MBR, then the polygon must be of the form depicted in Figure 8(b). We denote their m.b.g.f. $C^{i}_{i\text{con}}$ which, similarly to Equation 6, is equal to

$$\frac{x}{(1-x)^2} \left( 2yU[0,1] - U + x \left( U - \frac{2xy}{1-x} \right) \right).$$

Now, if we assume that the indent does not touch the top or bottom of the MBR, then, as in Proposition 3.5, we can factor (a cyclic permutation of) the polygon as $u^+u^-$ such that $l$ is a height one loop and $u^+u^-$ is self-avoiding. In fact, these polygons are the same as their unimodal analogues in the previous proposition, except for the fact that the SAPs, $u^+u^-$ are now convex and the indent can touch neither the top, nor the bottom of the MBR. Thus, we can write the m.b.g.f., directly from Equation 7, as

$$\frac{x}{(1-x)^2} \left( 1 + 3x \left( \sum_m mR_m - \left( U - \frac{xy}{1-x} \right) \right) + 2(C-2yU[0,1]),$$

where $C(x, y)$ is the m.b.g.f. for oriented convex SAPs, and $R_m(x, y)$ is the m.b.g.f. for non-oriented convex SAPs with right-perimeter $m$, which is well known. Referring to [10], we have

$$R_m = \frac{x}{\Delta^2} E(x) \left[ (1-x)^2(1-x-y)^2((1+x)^2-y)^2 \left( \frac{y}{1-x} \right)^m \right] - \frac{4x^2y(1-x-y-SP)y^m}{\Delta^{3/2}},$$

and, consequently, the given m.b.g.f. for 1-convex SAPs. $\blacksquare$
3.3.1 Enumerating 1-convex SAPs

**Corollary.** The m.b.g.f. for non-directed 1-convex SAPs with a vertical indent is

\[
\frac{8x^2yA}{(1-x)^{\Delta^5/2}} + \frac{2x^2yB}{(1-x)((1-x)^2 - y)^{\Delta^3}},
\]

where

\[
A(x, y) = (1 - x)^\delta - y(1 - x)^2(4 + 3x) + 6y^2(1 - x) - 2y^3(2 - x + x^2) + y^4
\]

and

\[
B(x, y) = -4(1 - x)^2 + 8y(1 - x)^3(3 + 2x) - y^2(1 - x)^4(60 + 35x + 10x^2 - x^3) + y^3(1 - x)^2(80 - 3x + 28x^2 + 9x^3 - 2x^4) + 2y^4(-30 + 31x + 13x^2 + 37x^3 + 7x^4) + 2y^5(12 - 5x + 28x^2 + 13x^3) - y^6(4 - 5x - 24x^2 + 13x^3 - 3y^7x).
\]

**Proof.** Let \( R' \) denote the m.b.g.f. for 1-convex polygons with no 1-dimensional loops and a vertical indent. Then,

\[
R' = E \left[ x^2 y^3 \delta y \left( \frac{(1 - y)^2}{\delta x (1 - x)(1 - x - y)} + \frac{2(1 - y)^2}{(1 - x)^2(1 - x - y)} + \frac{4x^4 y^2}{(1 - x)^3} \right) \right],
\]

from Lemma 3.1, mutatis mutandis.

Now, let \( u \) be an arbitrary 1-convex polygon with no 1-dimensional loops that is rooted at the vertex preceding the first positive vertical step. Its maximal convex decomposition, \( u^{(c)} = (u^*, \ell_1^*, v^*, \ell_2^*) \), is unique if it exists and must have its root in \( \ell_2^* \). Hence, \( \ell_1^* \) is (almost-)unimodal and \( u^*v^* \) (almost-)staircase, both either in direction \( \{1, 2\} \) or \( \{\bar{1}, \bar{2}\} \).

We begin by enumerating those 1-convex polygons which have a maximal decomposition. We assume, without loss of generality, that \( \ell_1^* \) is in direction \( A^+ \), and follow Proposition 3.2 mutatis mutandis. We note, however, that those decompositions which have the indent as the first vertical step of \( \ell_2^* \) (but not the very first) and have \( |u^*\ell_2^*v^*| = 1 \) are invalid decompositions. If \( \ell_2^* \) is self-avoiding, as is illustrated in Figure 8(b), the polygons are enumerated by \( C'_{\text{pyramid}} \), given in Equation 9. If \( \ell_2^* \) is intersecting, the maximal decomposition cannot admit an almost-staircase factor, as \( u^*v^* = \emptyset \), and there must therefore be another height one loop in the horizontally adjacent corner. These are enumerated by the m.b.g.f.

\[
C'_{\text{pyramid}} = 4 \left( \frac{x}{1 - x} \right)^2 \left( P - \frac{xy}{1 - x} \right),
\]

where \( P = \frac{x(1-x)y}{(1-x)(1-x-y)} \), the m.b.g.f. for pyramid polygons. Thus, we have the m.b.g.f. for intersecting 1-convex polygons in direction \( A^+ \) with no 1-dimensional loops and with their indent in \( u^* \) or \( \ell_2^* \) of their maximal decomposition as \( U(Q' - Z'U - U/(2y)) = C'_{\text{pyramid}} - C'_{\text{pyramid}} \), and via symmetry arguments, we obtain the m.b.g.f. of the 1-convex polygons with maximal decompositions. Those without are either self-avoiding or turning-defective. The latter case is enumerated by \( C'_{\text{turn}} \), given in Equation 8. Thus, 1-convex SAPs are enumerated by the m.b.g.f.

\[
R' - 4U(Q' - Z'U) + 3U^2/y + 4(C'_{\text{pyramid}} + C'_{\text{pyramid}}) - C'_{\text{turn}}.
\]

\( \square \)

4 The enumeration of 1-convex osculating polygons

Another interesting class of polygons similar to SAPs is that of osculating polygons (OPs), which are polygons that touch at one or more vertices but do not overlap (share an arc), or cross. A convex OP must either be a SAP, or be composed of two unimodal loops, joined by an arbitrary number of 2-dimensional staircase SAPs such that each loop has the same orientation. This is illustrated in the example in Figure 10(a). Their m.g.f., denoted \( C^{(0)} \), can therefore be written as \( C + U^2/(1 - SP)^2 \), which is of the form

\[
\frac{A(x, y)}{(x + y + xy)^{\Delta^3/2}} + \frac{B(x, y)}{(x + y + xy)^{\Delta^2}}.
\]
Figure 10: Osculating polygons. The asterisks in (b) indicate that there are zero or more factors.

We note that the possibility of an osculating staircase factor, $\frac{1}{1-SP} = \frac{x+y+SP}{x+y+xy}$, explains the factor $(x + y + xy)$ in the denominator, this being the only difference in the algebraic form of its m.g.f. and that of convex polygons. Recently, Jensen [22] obtained the isotropic generating function for 1-convex OPs. The results obtained in Section 3 can now be used to determine the anisotropic generating function and provide a combinatorial proof of Jensen’s result.

4.1 1-staircase osculating polygons

Any 1-staircase OP is composed of a 2-dimensional, 1-defective staircase loop with a possibly empty, 1-staircase OP at the root and co-root. Note that if the 1-defective loop has the indent in the corner, then there must be an adjacent non-empty, 2-dimensional loop. The two possible cases are depicted in Figure 10(b). Hence, the m.g.f. for 1-staircase OPs with a vertical indent, which we denote $S_{[0,1]}^{(1)}$, is

$$S_{[0,1]}^{(1)} = \frac{S + 4SP (S_{[0,1]} - SP / y)}{(1 - SP)^2} = \frac{2(A_{os} \sqrt{\Delta} + B_{os})}{y^2(x + y + xy)^2 \sqrt{\Delta}}$$

where $A_{os} = -x(1 - x)^3 - (1 - x)(1 - 2x - 3x^2 + 2x^3)y + (2 + x - 2x^2 - 3x^3 + x^4)y^2 - (1 + 2x + x^2 + x^3)y^3$,

and $B_{os} = (x(1 - x)^4 + (1 - x)^2(1 - 3x - 4x^2 + 2x^3)y - (3 - 2x - 6x^2 - 2x^3 + 6x^4 - x^5)y^2$

$$+ (3 + 4x + x^2 - x^2 - 2x^4)y^3 - (1 + 2x + x^2 - x^3)y^4.$$

4.2 1-unimodal osculating polygons

A 1-unimodal polygon that osculates must factor as either: an osculating staircase factor and a 1-defective unimodal loop; or a 1-staircase OP and a unimodal loop; or an osculating staircase factor, a unimodal loop and a height 1 loop. This last case is the osculating equivalent of a turning-defective unimodal polygon. All three cases are illustrated in Figure 11. Thus, the m.g.f. for 1-unimodal OPs with a vertical indent we denote $U_{[0,1]}^{(1)}$ and is given by

$$U_{[0,1]}^{(1)} = S_{[0,1]}^{(1)} \frac{U}{2} + \frac{U(S_{[0,1]} - SP / y) + SP (2U_{[0,1]} - U / y)}{(1 - SP)^2} + \frac{U_1 + \frac{2x}{y^2}(U - (1 - SP)P - SP - S_{[0,1]})}{(1 - x)(1 - x^2 - y)(x + y + xy)^2}$$

where $A_{ou} = 2(1 - x)^4x(1 + x) + (1 - x)(2 - 9x^2 + x^3 + 4x^4)y + (-2 - x + 5x^2 + 6x^3 - 8x^4 + 2x^5)y^2$

$$+ (1 + 3x + x^2 + 3x^3 + x^4)y^3 + x^2(1 + x)(3 + 2x)y^4$$

and $B_{ou} = 2(1 - x)^3x(1 + x) + (1 - x)^2(2 - 4x - 19x^2 - 5x^3 + 4x^4)y + (1 - x)(-6 - 5x + 19x^2$

$$+ 31x^3 - 13x^4 - 12x^5 + 2x^6)y^2 + (-6 - 9x - 9x^2 + 17x^3 + 4x^4 + 5x^5 - 5x^6)y^3 + (2 + 3x - 9x^2$

$$- 18x^3 - x^4 + 5x^5)y^4 - x(1 - 3x + 2x^2 + 3x^3)y^5 + x^2(1 + x)y^6.$$
Figure 11: The form of 1-unimodal OPs, and those that are only 1-convex when adjacent to a unimodal SAP. The asterisk in a staircase polygon again indicates that there are zero or more staircase loops at that root.

4.3 1-convex osculating polygons

An osculating polygon which is 1-convex must be: a 1-convex SAP; or an osculating 1-unimodal factor and a unimodal loop; or an osculating unimodal factor and a 1-defective unimodal loop; or a self-avoiding convex polygon with a height 1 loop forming the indent.

We generate these last two classes of polygon by composing a non-directed unimodal SAP with a 1-unimodal OP (a class of compositions with cardinality 4). However, we note that, for a given diagonal direction, this generates polygons with the indent in the staircase factor twice. Also, the required staircase factor in Figure 11(b) is no longer necessary, and we need an adjusting term. Their m.b.g.f. will therefore be

\[ U(2U_{[0,1]}^{(c)}) + 4U_{[0,1]} - 2U - S_{[0,1]}(U/2). \]

When a height one loop at one side forms the indent and it is not self-avoiding, its form is similar to the case depicted in Figure 11(c), only with a unimodal loop at the root. These are generated by the above composition, except for the case depicted in Figure 11(d), where the osculating 1-defective factor is not 1-convex. Here, if the unimodal loop is of height one itself, then there is a vertical symmetry. This case is generated by the m.b.g.f.

\[ \frac{4xP}{1-x} \left( U - \frac{xy}{1-x} \right). \]

Now consider those that are composed of self-avoiding convex polygons with a height 1 loop at one corner. We note that such a convex SAP cannot be [0,1]-corner unimodal, else it would factor as a unimodal loop and a 1-unimodal osculating polygon. Such polygons are therefore generated by the m.b.g.f.

\[ 4x/(1-x)(C - U - 2U_{[0,1]}), \]
Figure 12: The bijection, denoted \( \nu \), between osculating and neighbour-avoiding convex polygons. If the vertices around the indent are not neighbour-avoiding, the mapping applied by the bijection, \( \nu \) does not change this.

leaving us with the m.b.g.f. for 1-convex OPs with a vertical indent, denoted \( C_{[0,1]}^{(O)} \), as

\[
C_{[0,1]}^{(O)} = C' + U \left( 2U_{[0,1]} + 4U_{[0,1]} - S_{[0,1]}^{(O)}U/2 \right) + \frac{4x}{1-x} \left( C - 2U_{[0,1]} + P \left( U - \frac{xy}{1-x} \right) \right)
\]

\[
= \frac{4x^2(1+x+y)A_{oc}}{(1-x)(x+y+xy^2)\Delta^3/2} + \frac{4x^2B_{oc}}{(1-x)^2((1-x)^2-y)(x+y+xy)^2\Delta^3},
\]

where \( A_{oc} = (1-x)^2x(1+x)(1-x)(1-4x-6x^2+2x^3)y \)

\[
+ (1-x)(-4 + x + 8x^2 + 16x^3 - x^4 - 9x^5 + x^6)y^2 + 2(3 - x^2 + 8x^3 - 3x^4 - 5x^5 + 2x^6)y^3
\]

\[
+ (-4 - 2x + 12x^2 + 15x^3 + 3x^4 - 6x^5)y^4 + (1 + 2x - 4x^2 + 4x^4)y^5 - x^2(1 + x)y^6
\]

and \( B_{oc} = -(1-x)^2(1+2x+y)(1-4x-15x^2-4x^3+2x^4)y \)

\[
-(1-x)^2(-5 - 6x + 36x^2 + 76x^3 + 31x^4 + 21x^5 - 8x^6 + x^7)y^2
\]

\[
+ (1-x)^2(-9 - 26x + 31x^2 + 51x^3 + 116x^4 - 52x^5 - 3x^7 + 4x^8)y^3
\]

\[
+ (1-x)(-5 - 41x + 69x^2 - 83x^3 + 34x^4 + 66x^5 - 43x^6 - 75x^7 + 9x^8 + 5x^9)y^4
\]

\[
+ (5 - 47x + 89x^2 - 37x^3 + 15x^4 - 47x^5 - 97x^6 + 21x^7 - 28x^8 - 2x^9)y^5
\]

\[
+ (1 + x)(-9 + 27x - 34x^2 - 21x^3 - 83x^4 - 98x^5 + 8x^6 + 2x^7)y^6
\]

\[
+ (5 + 7x - 54x^2 - 79x^3 - 21x^4 - x^5 - 3x^6 - 6x^7)y^7
\]

\[
+ (-1 + 3x + 51x^2 + 117x^3 + 87x^4 + 32x^5 + 7x^6)y^8
\]

\[
- x(1 + x)(3 + x)(1 + 6x + 4x^2)y^9 + x(1 + x)^3y^{10}.
\]

5 The enumeration of neighbour-avoiding polygons

Neighbour-avoiding polygons (NAPs) are polygons that have a step between all neighbouring vertices. Hence,

\[
|i - j| > 1 \iff |s_i - s_j| > 1 \forall 1 \leq i < j \leq n + 1.
\]

A bijection, which we denote \( \nu \), between OPs and NAPs, is illustrated in Figure 12. This bijection simply adds a step along the MBR whenever the polygon touches it (once for each side). Thus, the m.g.f. for convex NAPs, denoted \( C^{(NA)} \), is given by

\[
C^{(NA)} = 1 + xyC^{(O)},
\]

where the 1 corresponds to the empty polygon, which is considered neighbour-avoiding.

If the vertices around the indent of a 1-convex OP, \( w \), are not neighbour-avoiding, then \( \nu(w) \) is not neighbour-avoiding. We must therefore enumerate 1-convex SAPs that are locally
neighbour-avoiding around the indent — i.e. with only horizontal steps within two steps of the indent — in order to enumerate the OPs we require. Now, 1-unimodal and 1-convex polygons can have different words representing the same polygon. As such, when the indent touches the top of the MBR, there are two top edges where the bijection \( \nu \) could place the extra step. As the indent is already locally neighbour-avoiding, we can add the extra step without ambiguity to the edge not adjacent to the indent.

5.1 1-staircase neighbour-avoiding polygons

**Lemma 5.1.** If \( S_{1}^{(na)} \) is the m.b.g.f. for 1-staircase neighbour-avoiding polygons, then

\[
S_{1}^{(na)} = \frac{x(A_{NAS} \sqrt{\Delta} + B_{NAS})}{y(x + y + xy)^2 \sqrt{\Delta}},
\]

where \( A_{NAS} = -(x + 1)^2 y^3 + (x + 1)(x^2 + 3x + 4)y^4 + (x^2 - x - 3)y^3 \\
+ (x - 2)(x - 1)(x^2 + x - 1)y^2 - (1 - x)(2x^2 - x^2 - 4x + 1)y - (1 - x)^3 \)

and \( B_{NAS} = (1 - x)^4 x + (1 - x)^3 (1 - 5x - 2x^2 + 2x^3)y + (-5 + 11x + 4x^2 - 7x^3 - 2x^4 + x^5)y^2 \\
+ (10 + 7x - 3x^2)y^3 - (10 + 14x + 6x^2 + 2x^3 + x^4)y^4 \\
+ (1 + x)(5 + 5x + 2x^2)y^5 - (1 + x)^2 y^6. \)

**Proof.** Let us first consider 1-defective staircase polygons with a vertical, neighbour-avoiding indent. Denoting their s.g.f. \( Z'_{1}^{(na)} \), from Lemma 3.1 mutatis mutandis, we have

\[
Z'_{1}^{(na)} = 2y - \frac{2((1 - y)^{-3}(1 + y) - 3x(1 - y^2)^{2} - x^{3}(1 + y^2) + x^{2}(3 + 7y + 3y^2 + 3y^3))}{y^{3/2}}.
\]

We recall that there are only two possible forms of staircase OPs, which are illustrated in Figure 10(b). And so, we need to find the m.b.g.f. of (non-corner) 1-staircase polygons with a neighbour-avoiding indent, and the s.g.f. of [0,1]-corner polygons with at least two horizontal steps between the indent and the root. We note that their sum, which we denote \( S_1' \), is the s.g.f. for 1-defect staircase SAPs with a neighbour-avoiding indent, and comes from Proposition 3.2, mutatis mutandis.

\[
S_1' = \Delta Z'_{1}^{(na)} + 2u^2 v^2 (2 + 4u - u^2)/y.
\]

Let us now consider just the corner polygons. If the first (resp. second) step was the indent, then the s.g.f. would be \( u^2 v/y \) (resp. \( u^3 v^2/y \), and there are at least two horizontal steps on each side of the indent, then the s.g.f. is \( u^4 v^2/(1 - u)/y \). Now, we sum these cases to get the s.g.f. for the corner polygons, and subtract this (with a factor of 2 for the orientation of the polygon) from \( S_1' \) to find the m.b.g.f. of 1-staircase SAPs with neighbour-avoiding indents:

\[
((1 - x)^4 - (5 - x)(1 - x)^2 y + (10 - 9x + x^2)y^2 - (10 - x)y^3 + (5 + x)y^4 - y^5) xZ/y \\
- ((1 - x)^3 + 2(2 - x)(1 - x)y - 3(2 - x)y^2 + 4y^3 - y^4) x/y.
\]

If the OP has a 1-corner staircase factor, then it must have a staircase self-avoiding factor adjacent, or the polygon will be convex. We therefore have the m.b.g.f. for the composition of this factor and its adjacent factor as \( u^5 v^3/(1 - u)/y \).

Adding an arbitrary number of self-avoiding loops to each end of either of the above type of polygons gives all 1-staircase OPs with a neighbour-avoiding indent. And so, one obtains the stated result by multiplying the sum by \( 1/(1 - \mathcal{SP})^2 \) and then \( xy \) to apply the bijection \( \nu \).  \( \square \)
5.2 1-unimodal neighbour-avoiding polygons

5.2.1 Corner-defective unimodal SAPs with a neighbour-avoiding indent.

**Proposition 5.2.** Let us take a characteristic word \( w = w_1w_2 \) with an \( \alpha \times \beta \) SR and consider those rooted corner-defective unimodal polygons with no 1-dimensional loops\(^2\) of the form \( u = w_2w_1 \) whose root are at \((a, b)\), whose indents are neighbour-avoiding, and whose maximal decompositions are of the form \((0, u, \emptyset)\). Now, let us assume that their characteristic words are minimal, such that they do not share an arc with the MBR. The s.g.f. of such polygons, which we denote \( U^{[na]}_{[a,b]^0} \), is

\[
U^{[na]}_{[a,b]^0} = E \left[ \frac{x^{\alpha+2a_0}(1-x)y^{\beta+2b_0}(1-y)}{1-x-y} \right] - u^{\alpha+2a_0, \beta+2b_0} E \left[ \frac{xy}{1-x-y} \right]
- a_0 x^{\alpha+1} \sum_{\beta \geq \beta + 2b_0} P_{\beta}(x,y) - b_0 y^{\beta+1} \sum_{\alpha' \geq \alpha + 2a_0} P_{\alpha'}(y,x) - a_0 b_0 \left( \frac{\alpha - 1 + \beta - 1}{\alpha - 1} \right) x^{\alpha+1} y^{\beta+1},
\]

where \( a_0 = \min(1, b) \), \( b_0 = \min(1, a) \) and \( P_i(x,y) \) is the pyramid m.g.f. with left perimeter \( l \).

**Notation.** We denote the s.g.f. of those polygons that are corner-staircase and satisfy a similar neighbour-avoiding condition on the indent \( S^{[na]}_{[a,b]^0} \).

**Proof.** Reproducing an inclusion-exclusion argument similar to that of Lemma 3.1, we can immediately enumerate all our required polygons by enumerating the paths with no 1-dimensional loops from \((\alpha + 2a_0, 0)\) to \((0, \beta + 2b_0)\) with the generating function

\[
E \left[ \frac{x^{\alpha+2a_0}(1-x)y^{\beta+2b_0}(1-y)}{1-x-y} \right]. \tag{11}
\]

If they have a corner-staircase polygon as a factor, then the maximal decomposition cannot be of the form \((0, v, \emptyset)\), for some \( v \). Such staircase factors are enumerated by \( S_{\alpha+2a_0, \beta+2b_0} \), and so adding a unimodal loop gives

\[
u^{\alpha+2a_0, \beta+2b_0} E \left[ \frac{xy}{1-x-y} \right].
\]

Finally, as \( w \) is minimal, if \( a \) (resp. \( b \)) is non-zero, its first (resp. last) step must be an indent of \( u \). This means that there must be two steps perpendicular to this indent at the end (resp. start) of \( u \). These will form a 1-dimensional loop if the polygon is of height \( \beta + 1 \) (resp. width \( \alpha + 1 \)). After its 1-dimensional loop, the polygon is unimodal in the other direction, and is therefore enumerated using the pyramid generating function by width and left perimeter. If there is both a horizontal and vertical 1-dimensional loop, then the polygon \( u \) must be a directed walk. These cases give the last three terms in the proposed result. \( \square \)

We note that the polygons enumerated above are either self-avoiding or are ‘turning-defective’, due to the condition on the maximal decomposition. We note that these polygons can have a horizontal 1-dimensional loop at the root, which we can remove by multiplying the generating function by \((1-x)\).

**Corollary.** If we define s.g.f. \( U^{[na]}_{[a,b]} = \sum_w U^{[na]}_{[a,b]^0} \), we can take \( a = 0 \) and \( b = 1 \) to obtain the m.b.g.f.

\[
yU^{[na]}_{[0,1]} = E \left[ \frac{x^2 y (1-y)}{1-x-y} \right] - \frac{u^2 v}{1-u} E \left[ \frac{xy}{1-x-y} \right] - \frac{v}{1-x} P(x,y),
\]

and

\[
U^{[na]}_{[0,1]} = (1-x)U^{[na]}_{[0,1]} - x^2/(1-x)^2 P(x,y),
\]

where \( P(x,y) \) is the pyramid m.g.f.

\(^2\)Note that there can be a 1-dimensional factor at the root.

\(^3\)For clarity, the asterisk indicates the possibility of self-intersection.
Figure 13: 1-defective unimodal polygons with a neighbour-avoiding condition around the indent.

5.2.2 Enumerating 1-unimodal NAPs

Lemma 5.3. If \( U_1^{(na)} \) is the m.b.g.f. for 1-unimodal neighbour-avoiding polygons with a vertical indent, then

\[
U_1^{(na)} = \frac{x A_{NAU}}{y(1-x)(1-y)((1-x)^2-y)(x+y+xy)^2} + \frac{x B_{NAU}}{y(1-x)(1-y)(x+y+xy)^2}\Delta^3/2,
\]

where

\[
A_{NAU} = 2(1-x)^4 x(1+x) + y(1-x)^2(2 - 6x - 7x^2 + 3x^4) - y^2(3 - 13x + 19x^2 + 30x^3 - 10x^2 + 2x^6) + y^2(12 + x - 28x^2 + 8x^3 + 18x^4 - 6x^5 - 2x^6)
\]

and

\[
B_{NAU} = 2(1-x)^5 x(1+x) + y(1-x)^3(2 - 10x - 17x^2 + x^3 + 2x^4) - y^2(12 - 21x - 47x^2 + 23x^3 + 19x^4 - 4x^5 + 2x^6)
\]

\[
+ y^3(30 - 7x - 85x^2 + 5x^3 + 39x^4 - 2x^5 - 6x^6 + 2x^7)
\]

\[
- y^4(40 + 52x - 43x^2 - 47x^3 + 12x^4 + 7x^5 + x^6)
\]

\[
+ y^5(30 + 66x + 20x^3 - 40x^3 - 28x^4 - 7x^5 - 2x^6)
\]

\[
- y^6(1 + x)^2 (12 - 2x + 4x^2) + y^7(1 + x^2) (2 + x - 2x^2).
\]

Proof. Let \( Q_{(na)}' \) denote the s.g.f. for rooted 1-defect unimodal polygons with a vertical indent that is neighbour-avoiding. From Lemma 3.1 mutatis mutandis, we can enumerate all 1-defect polygons with a neighbour-avoiding indent. We then note the possibility of a 1-dimensional loop being formed by the two horizontal steps around the indent, if they are the \( n^{th} \) and \( n + 1^{st} \). This cannot be excluded via our inclusion-exclusion argument above, as fixing \( a \) after the \( n^{th} \) occurrence of 1 to prevent this would produce a forbidden factor. These three cases, illustrated in Figure 13, are enumerated respectively by the s.g.f.s: \( 2uv^3vz^2; \frac{xuv^2z}{1-u} \); and \( x^2 (1 + uz)(1 + vz) \). We therefore have

\[
Q_{(na)}' = E\left[ y^{\delta} \frac{x^2(1-y)(1-y-xy)}{(1-x)(1-x-y)} + \frac{x^2(1-y)(1+y+xy)}{(1-x)(1-x-y)} \right]
\]

\[- x - x^2 (1 + uz)(1 + vz) - xu^2 SP Z \left( 2Z - \frac{1}{1-u} \right).\]

Now, let \( U_{(na)}' \) be the s.g.f. for 1-defect unimodal SAPs with a vertical neighbour-avoiding indent. Following the factorisation argument of Proposition 3.2 mutatis mutandis, we obtain the generating function for all 1-defect unimodal loops with neighbour-avoiding indents, which are either self-avoiding or have a maximal decomposition (0, w, \emptyset), for some \( w \). If we enumerate the latter class of polygons with \( U_{(na)}' \), we have

\[
U_{(na)}' = \sqrt{A} \left( Q_{(na)}' - 2Z_{(na)}' U \right) - U_{(na)}' + \frac{2xu^2v}{(1-x)(1-u)} + (2 + u) \frac{u^2v}{y} \left( U - \frac{xy}{1-x} \right) + 2vU_{(na)|0}|_1 + 2(v + uy)U_{(na)|0}|_2 - (xv + yu) \frac{x^2}{1-x},
\]
where \( U_{\text{turn}}^{[n_a]} = \frac{x^2}{1-x} \left( 2Z \left( \frac{4 - (2 + x)(1 - u) - (1 + x)v}{1 - u} - \frac{u(1 + u)}{1 - u} - 2u \left( \frac{U}{x} + \frac{vy}{1-y} \right) \right) \),

from Proposition 3.5, \textit{mutatis mutandis}.

A 1-unimodal OP has certain possible factorisations that are illustrated in Figure 11. Thus, we can just apply the neighbour-avoiding condition to all instances of the indent in the expression for \( U^{[O]}_{[0,1]} \) given in Section 4.2. Thus, after applying the bijection, we have the identity

\[
U_1^{[n_a]} = S_1^{[n_a]} U + \frac{xy}{1-SP} \left( \frac{U - \frac{xy}{1-x}}{y - \frac{1}{1-x}} \right) \left( U^{[n_a]}_{[0,1]} - U^{[n_a]}_{[0,1]/2} - U^{[n_a]}_{[0,1]/2} \right) + \frac{xy}{1-SP} \left( 2SP U^{[n_a]}_{[0,1]} - U^{[n_a]}_{[0,1]/2} - U^{[n_a]}_{[0,1]/2} + U_{[n_a]} - U^{[n_a]}_{[0,1]} \right),
\]

which is the stated result.

\( \square \)

5.3 1-convex neighbour-avoiding polygons

Proposition 5.4. Let \( C_{tw}^{[n_a]} \) be the m.b.g.f. for rooted 1-turning-defect convex polygons with a neighbour-avoiding indent and no 1-dimensional loops. Then,

\[
C_{tw}^{[n_a]} = \frac{4x^2}{(1-x)^2} \left( 1 + x + 2x^2 \right) \left( \sum mR_m - \left( U - \frac{xy}{1-x} \right) \right) + x(1+x) \left( U - \frac{xy}{1-x} \right)
\]

\[
+ \frac{4x^2}{(1-x)^2} \left( 2(C-yU_{[0,1]} - U_{[0,1]} - (1-x)(C_{2T2} - U^{2T2}_{[0,1]}) \right),
\]

where \( C_{2T2} = C - 2R_1 - yU(1 + x/(1-y)) \) is the m.g.f. of convex SAPs with a factor \( \pm 2\overline{T2} \) touching the minimal vertical edge of the MBR, \( U^{2T2}_{[0,1]} = \frac{SP U - \frac{xy}{1-x}}{1-y} \) is the m.g.f. of the non-oriented \([0,1] \)-unimodal subset of these polygons, and \( R_m \) is given by Equation 10.

\( \text{Proof.} \) We construct these turning-defective polygons as in Section 3.3. To enforce the neighbour-avoiding condition on the indent, we require that there are two horizontal steps on either side. We therefore add a horizontal step to either side of the maximal vertical step by multiplying the m.b.g.f., \( C_{tw}^{[n_a]} \) (from Equation 8) by \( x \). We then exclude those polygons, \( c^+c^- \), whose indent neighbours the self-avoiding factor \( c^+c^- \). Those with the indent neighbouring a right-most edge of \( c^+c^- \) are enumerated by

\[
\frac{2x^3}{(1-x)} \left( \sum mR_m - \left( U - \frac{xy}{1-x} \right) \right).
\]

In the remaining cases, \( \ell \) must be diagonally adjacent to \( c^+c^- \). These are enumerated by

\[
\frac{x^3}{(1-x)^2} \left( \frac{U - \frac{xy}{1-x}}{1-x} \right)
\]

when the indent touches the top of the MBR. Otherwise, \( \ell \) must be self-avoiding, enumerated by

\[
\frac{x^2}{(1-x)} \left( C_{2T2}^{2T2} - U^{2T2}_{[0,1]} \right),
\]

where \( C_{2T2} \) is the m.g.f. of convex SAPs with a factor \( \pm 2\overline{T2} \) touching the maximal (and by symmetry, minimal) vertical edge of the MBR, and \( U^{2T2}_{[0,1]} \) is the m.g.f. of the non-oriented subset of these polygons whose \( 2\overline{T2} \) factor touches the top of the MBR. These are straightforward to enumerate via basic inclusion-exclusion arguments. \( \square \)

Lemma 5.5. If \( C_1^{[n_a]} \) is the m.b.g.f. for 1-convex neighbour-avoiding polygons, then

\[
C_1^{[n_a]} = \frac{2xy}{(1-x)(1-y)} \left( x + y + xy \right)^2 \Delta^3 \left( A \Delta + B \right),
\]

where

\[
A \Delta = (1-x)^5 x^2 (1+x)^3 (1-x)^3 x^2 (1+x)(1+3x-2x^2 + x^3)y
\]

\[
+ (1+9x+45x^2-42x^3-46x^4+31x^5-16x^6+17x^7-6x^8 + x^9)y^3
\]

\[
+ (-4+12x-31x^2-46x^3+46x^4+18x^5-10x^6-5x^7-2x^8)y^4
\]

\[
+ (5+7x+9x^2+31x^3+18x^4+36x^5-11x^7)\Delta + x(-17-29x-16x^2-10x^3+24x^5)y^6
\]

\[
+ (-5+3x+20x^2+17x^3-5x^4-11x^5)y^7 - 2(-2+2x^2+3x^3+4x^4)y^8 + (-1+x)(1+x)^2 y^9
\]

\( \Delta \) is given by Equation 10.

\( \square \)
and \[ B_{SAC} = -(1 - x)^9 x^3 (1 + x)^2 - (1 - x)^7 x^2 (1 + x) (-9 x + 14 x^2 + 3 x^3 + x^4) y \]
+ (1 - x)^6 x (-1 + 15 x - 34 x^3 - 103 x^5 - 45 x^7 + 35 x^9 + 29 x^{11} - x^9 y^2)
+ (1 - x)^3 (-1 + 12 x - 80 x^2 + 63 x^3 + 374 x^4 - 94 x^5 - 120 x^6 - 100 x^7 + 60 x^8 - 4 x^9 + x^{10} + x^{11} y^3)
+ (1 - x) (-7 + 47 x - 180 x^2 - 113 x^3 + 247 x^4 - 91 x^5 - 361 x^6 + 50 x^7 + 17 x^8 + 22 x^9 - 100 x^{10} + 37 x^{11} + x^{12} y^4) + (-19 + 55 x - 252 x^2 + 947 x^3 + 2162 x^4 - 424 x^5 - 1611 x^6 + 1023 x^7 - 179 x^8 - 96 x^9
- 34 x^{10} - 52 x^{11} + 18 x^{12} + x^{13} y^5 + (21 + 75 x - 354 x^2 + 1221 x^3 - 554 x^4 - 1611 x^5 + 1272 x^6 - 427 x^7
- 81 x^8 - 143 x^9 + 177 x^{10} - 43 x^{11} - x^{12} y^6 + (6 - 218 x + 249 x^2 - 396 x^3 - 61 x^4 + 1023 x^5 - 427 x^6
+ 4 x^7 + 655 x^8 - 140 x^9 + 210 x^{10} - 36 x^{11} + x^{12} y^7 + (-42 + 140 x + 173 x^2 + 94 x^3 + 233 x^4 - 179 x^5
- 81 x^6 + 655 x^7 - 136 x^8 - 209 x^9 + 61 x^{10} - 3 x^{11} y^8 + (42 + 78 x - 176 x^2 - 364 x^3 - 5 x^4 - 96 x^5
- 143 x^6 - 140 x^7 - 200 x^8 + 50 x^9 + 2 x^{10} y^9 + (-6 - 155 x - 85 x^2 + 293 x^3 + 131 x^4 - 34 x^5 + 177 x^6
+ 210 x^7 + 61 x^8 + 2 x^9 y^{10} + (-21 + 75 x + 130 x^2 - 74 x^3 - 146 x^4 - 52 x^5 - 43 x^6 - 36 x^7 - 3 x^8 y^{11}
+ (19 - 6 x - 44 x^2 + 4 x^3 + 36 x^4 + 18 x^5 - x^6 + x^7) y^{12} + (-7 - 5 x + 6 x^2 + 2 x^3 + x^4 + x^5) y^{13}
+ (1 - x) (1 + x)^2 y^{14}.

Proof. Let \( R'_{[na]} \) denote the m.b.g.f. for 1-convex polygons with a vertical neighbour-avoiding indent. Then,
\[
R'_{[na]} = x^3 E \left[ x y \frac{2 \delta}{\delta x} \left( \frac{1}{y} \left( \frac{1}{1 - x} \right) \left( \frac{1}{1 - x - y} \right) + \frac{1}{(1 - x)^2} \right) + \frac{y}{(1 - x - y)^2} \right],
\]
from Lemma 3.1, mutatis mutandis. If \( C'_{[na]} \) is the m.b.g.f. for directed 1-convex SAPs with a vertical neighbour-avoiding indent, its derivation follows from Section 3.3.1, mutatis mutandis.

A 1-convex OP with a neighbour-avoiding indent is constructed in exactly the same way as in Section 4.3, only applying the neighbour-avoiding condition to the indent. Thus, after applying the bijection, we have
\[
S_1^{[na]} U^2 + x y C'_{[na]} + \frac{2 x y U}{1 - SP} \left( U'_{[na]} + (S'_{[0,1]}^{[na]} - S'_{[0,1]2}^{[na]} - S'_{[0,1]12}^{[na]}) \left( \frac{U}{y} - \frac{x}{1 - x} \right) \right)
+ 4 x y (U'_{[0,1]}^{[na]} - U'_{[0,1]2}^{[na]} - U'_{[0,1]12}^{[na]}) \left( \frac{U}{1 - SP} - \frac{x y}{1 - x} \right).
\]
This is the generating function with a vertical indent. Adding the symmetric function which generate the polygons with a horizontal indent, we obtain the given result. \(\square\)

6 Asymptotics

In this section we calculate the asymptotics for convex and 1-convex osculating and neighbour-avoiding polygons, and give the corresponding general result for \(m\)-convex osculating polygons with \(m = O(\sqrt{n})\). The calculation is simplified by observing the following:
\[
[x^n](1 - 4x)^{-k} = \begin{pmatrix} n + k - 1 \\ k - 1 \end{pmatrix} 4^n,
\]
\[
[x^n](1 - 4x)^{-3/2} = \sqrt{n/\pi} (2 + \frac{3}{4n} - \frac{7}{64n^2} + O(\frac{1}{n^3})) 4^n, \tag{12}
\]
\[
[x^n](1 - 4x)^{-5/2} = \sqrt{n/\pi} (\frac{4n}{3} + \frac{5}{2} + \frac{65}{96n} + O(\frac{1}{n^2})) 4^n.
\]

The asymptotic expression for the number of polygons then follows by expanding the factors multiplying the above terms about \( x = \frac{3}{4}, \) using the above asymptotic expressions and collecting terms. In this way we find the number of convex osculating polygons of perimeter \(2n\) to be given by
\[
p_{2n}^{(c)} = \frac{n 4^n}{128} \left[ 1 - \frac{4}{\sqrt{n/\pi}} + \frac{17}{6n} + O(n^{-3/2}) \right], \tag{13}
\]
Interestingly, the first two terms are precisely equal to the corresponding terms in the case of ordinary convex SAPs, though the third term is different, having coefficient $\frac{2}{7}$ in the case of convex polygons [19]. Thus we see that, asymptotically, there are the same number of osculating convex polygons as ordinary convex polygons. This is not the case [22] for SAP’s compared to osculating SAPs, where osculations allow for exponentially more SAPs.

The corresponding result for convex, neighbour avoiding polygons follows similarly as

$$p_{2n}^{(\text{NA})} = \frac{n^{4n}}{2048} \left[ 1 - \frac{4}{\sqrt{n\pi}} + \frac{5}{6n} + O(n^{-3/2}) \right].$$

(14)

For neighbour-avoiding convex polygons we see that the exponential growth is the same as for convex polygons, but that the amplitude is smaller by a factor of 16.

For 1-convex polygons the calculations proceed similarly, but are just a little more tedious because of the high degree polynomials that occur in the numerators. The results are surprisingly simple. The number of 1-convex, osculating polygons of perimeter $2n$ is given by

$$p_{2n,1}^{(\text{O})} = \frac{n^{2n}}{256} \left[ 1 - \frac{4}{\sqrt{n\pi}} - \frac{9677}{4608n} + O(n^{-3/2}) \right].$$

(15)

Similarly, the number of 1-convex, neighbour-avoiding polygons of perimeter $2n$ is given by

$$p_{2n,1}^{(\text{NA})} = \frac{n^{2n}}{16384} \left[ 1 - \frac{4}{\sqrt{n\pi}} - \frac{46541}{4608n} + O(n^{-3/2}) \right].$$

(16)

Again, we observe that 1-convex osculating polygons have the same first two terms in their asymptotic expansion as have their convex counterpart, while the amplitude of 1-convex neighbour-avoiding polygons is down by a factor of 64 compared to 1-convex polygons.

In [19] it was proved that the number of $m$-convex polygons of perimeter $2n$ is given by $p_{2n,m} = \frac{n^{m+1}}{2m+1}$, for $m = o(\sqrt{n})$, and that the radius of convergence of the series counting all such polygons with $m = o(n)$ is $\frac{1}{4}$. Based on numerical data it was conjectured further [19] that

$$p_{2n,m} = \frac{n^{m+1}}{2m+1} \left[ 1 - \frac{4}{\sqrt{n\pi}} + O\left(\frac{m^2}{n}\right) \right],$$

for $m = o(\sqrt{n})$. That is to say, the first correction term is independent of $m$. For both osculating and neighbour-avoiding polygons, the result that the radius of convergence of the series counting all such polygons with $m = o(n)$ is $\frac{1}{4}$ follows virtually without change. The result for osculating polygons that $p_{2n,m}^{(\text{O})} = \frac{n^{m+1}}{2m+1}$, for $m = o(\sqrt{n})$, follows mutatis mutandis, as does the conjecture that

$$p_{2n,m}^{(\text{O})} = \frac{n^{m+1}}{2m+1} \left[ 1 - \frac{4}{\sqrt{n\pi}} + O\left(\frac{m^2}{n}\right) \right],$$

for $m = o(\sqrt{n})$.

In the case of neighbour avoiding polygons the topmost, bottommost, leftmost and rightmost straight segments of the polygon (these are the sections that coincide with the minimum bounding rectangle) must be at least of length two, otherwise the neighbour avoiding constraint is violated. This reduces the choices at precisely four vertices. At these vertices no bend is permitted. This gives rise to an extra factor $2^{-4}$ compared to the convex and osculating convex case. In the convex and osculating convex cases, when inserting non-convex pieces, there are four step sequences that are forbidden. These are 11, 11, 22 and 22. These give rise to the factor $2^{-m}$, as explained in [19]. For neighbour-avoiding convex polygons, there are eight further forbidden sequences, such as 121, and seven other similar such sequences that violate the neighbour avoiding condition. These then give rise to a factor $2^{-2m}$. So for neighbour-avoiding convex polygons of convexity $m$, we arrive at the result that the number of such polygons is $p_{2n,m}^{(\text{NA})} = \frac{n^{m+1}}{2m+1}$, for $m = o(\sqrt{n})$. Our exact results for the cases $m = 0$ and $m = 1$ also support the conjecture that

$$p_{2n,m}^{(\text{NA})} = \frac{n^{m+1}}{2m+1} \left[ 1 - \frac{4}{\sqrt{n\pi}} + O\left(\frac{m^2}{n}\right) \right],$$

for $m = o(\sqrt{n})$. 

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References


