The Least Spanning Area of a Knot
and the
Optimal Bounding Chain Problem

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Hyamfest, July 2011

Based on arXiv:1012.303

Slides available at http://dunfield.info
Knot: A smooth embedding of $S^1$ in a closed orientable 3-manifold $Y$.

Spanning surface: If $K = 0$ in $H_1(Y; \mathbb{Z})$, it is the boundary of an orientable embedded surface $S$.

Problem: Find the least genus $g(K)$ of such an $S$.

In the 1960s, Haken used normal surfaces to give an algorithm to compute $g(K)$. Here, $Y$ is given as a simplicial complex $\mathcal{T}$, and $K$ is a loop of edges in $\mathcal{T}^1$. 
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In the 1960s, Haken used normal surfaces to give an algorithm to compute $g(K)$. Here, $Y$ is given as a simplicial complex $\mathcal{T}$, and $K$ is a loop of edges in $\mathcal{T}^1$. 
**Knot Genus:** Given $K \subset T^1$ and $g_0 \in \mathbb{N}$, is $g(K) \leq g_0$?

**Agol-Hass-Thurston (2002)**

*Knot Genus is NP-complete.*

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**Decidable**

**Exp. time**  
Is $\dim (Kh_*(K)) \leq 10$?

**NP**

Is there a hamiltonian cycle?

Are two graphs isomorphic?

Traveling salesman

**P**  
Polynomial time

Is a list sorted?

Is $\Delta_K$ monic?

Word prob. in a hyp. group
When $Y$ is simple, e.g. $S^3$, then Knot Genus should be in $\textbf{NP} \cap \textbf{co-NP}$, and might even be in $\textbf{P}$.

Least area: $Y$ Riemannian, $K$ null-homologous. By geometric measure theory, there exists a spanning surface of least area.

Discrete version: Assign each 2-simplex in $\mathcal{T}$ an area (in $\mathbb{N}$), consider spanning surfaces “built out of” 2-simplices of $\mathcal{T}$.

Least Spanning Area: Given $K \subset \mathcal{T}^1$ and $A_0 \in \mathbb{N}$, is there a spanning surface with area $\leq A_0$?

*Least Spanning Area is NP-complete.*

**Thm (D-H)** *When $H_2(Y;\mathbb{Z}) = 0$, e.g. $Y = S^3$, Least Spanning Area can be solved in polynomial time.*
Algorithm uses linear programming.
Thm (D-H)  When \( H_2(Y; \mathbb{Z}) = 0 \), e.g. \( Y = S^3 \), Least Spanning Area can be solved in polynomial time.

Approach:

1. Consider the related Optimal Bounding Chain Problem, where \( S \) is a union of 2-simplices of \( \mathcal{T} \) but perhaps not a surface.

2. Reduce to an instance of the Optimal Homologous Chain Problem that can be solved in polynomial time. [Dey-H-Krishnamoorthy 2010]

3. Desingularize the result using two topological tools.
**Homology:** $X$ a finite simplicial complex, with $C_n(X;\mathbb{Z})$ the free abelian group with basis the $n$-simplices of $X$.

**Boundary map:** $\partial_n: C_n(X;\mathbb{Z}) \to C_{n-1}(X;\mathbb{Z})$

**Homology:**

$$H_n(X;\mathbb{Z}) = \frac{\ker(\partial_n)}{\text{image}(\partial_{n+1})}$$
Assign a “volume” to each $n$-simplex in $X$, which gives $C_n(X; \mathbb{Z})$ an $\ell^1$-norm.

$$\|c\|_1 = \sum |a_i| \text{Vol}(\sigma_i) \quad \text{where} \quad c = \sum a_i \sigma_i$$

**Optimal Homologous Chain Problem (OHCP)**
Given $a \in C_n(X; \mathbb{Z})$ find $c = a + \partial_{n+1}x$ with $\|c\|_1$ minimal.

**Optimal Bounding Chain Problem (OBCP)**
Given $b \in C_{n-1}(X; \mathbb{Z})$ which is 0 in $H_{n-1}(X; \mathbb{Z})$, find $c \in C_n(X; \mathbb{Z})$ with $b = \partial_n c$ and $\|c\|_1$ minimal.

**Thm (D-H)** *OHCP and OBCP are NP-hard.*

OHCP with mod 2 coefficients is NP-hard by [Chen-Freedman 2010].
Dey-H-Krishnamoorthy (2010) When $X$ is relatively torsion-free in dimension $n$, then the OHCP for $C_n(X; \mathbb{Z})$ can be solved in polynomial time.

Key: Orientable $(n+1)$-manifolds are relatively torsion-free.

**Thm (D-H)** When $X$ is relatively torsion free in dimension $n$ and $H_n(X; \mathbb{Z}) = 0$, then the OBCP for $C_{n-1}(X; \mathbb{Z})$ can be solved in polynomial time.

Compare

**Thm (D-H)** When $H_2(Y; \mathbb{Z}) = 0$, the Least Spanning Area problem for a knot $K$ can be solved in polynomial time.
Desingularization: a toy problem

In a triangulated rectangle $X$, find the shortest embedded path in the 1-skeleton joining vertices $p$ and $q$.

Consider $b = q - p \in C_0(X; \mathbb{Z})$, which is 0 in $H_0(X; \mathbb{Z})$. Let $c \in C_1(X; \mathbb{Z})$ be a solution to the OBCP for $b$.

Claim: $c$ corresponds to an embedded simplicial path.
Rest of desingularization

1. Pass to the exterior of the knot $K$.

2. Introduce a relative version of the Optimal Bounding Chain Problem.