Veering triangulations admit strict angle structures

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Joint work with
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Geometric Triangulations

We want to understand the relationship between the geometry and combinatorics of triangulations of 3-manifolds. Today we look at the following:

**Question.** Given an ideal triangulation $\mathcal{T}$ of a orientable cusped hyperbolic 3-manifold $M$, is the triangulation *geometric*, i.e. realized by positively oriented ideal hyperbolic tetrahedra in the complete hyperbolic structure on $M$?
Ideal tetrahedra in $\mathbb{H}^3$

have shapes parametrized by 3 dihedral angles $\alpha, \beta, \gamma \in (0, \pi)$ as shown below.

The sum of angles at each vertex is

$$\alpha + \beta + \gamma = \pi,$$

and opposite edges have the same dihedral angle.
To show that an ideal triangulation $\mathcal{T}$ is geometric we need to find dihedral angles for the ideal tetrahedra such that:

1. the sum of dihedral angles around each edge is $2\pi$,
2. there is no translational holonomy ("shearing") along the edges,
3. each "cusp" is complete (i.e. the holonomy of each peripheral curve is parabolic).

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Angle structures

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- angles attached to opposite edges of \( \sigma \) are equal,
- angles add up to \( \pi \) at each vertex of \( \sigma \), and \((*)\)
- the sum of all angles around each edge of \( \mathcal{T} \) is \( 2\pi \).
Remarks

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2. An angle structure gives a hyperbolic structure on $M \backslash \{\text{the edges of the triangulation}\}$, but with possible shearing type singularities around these edges.

3. If an angle structure exists, then Casson showed that $M$ is irreducible, atoroidal and not Seifert fibred, so $M$ admits a hyperbolic structure by the uniformization theorem of Thurston.
**Step 2.** Solve the *non-linear* equations (2) and (3) by a volume maximization procedure.

We define a volume function $V: A \to \mathbb{R}$ by adding up the hyperbolic volumes of the ideal tetrahedra in $H^3$ given by a point in $A$. This volume function is smooth and strictly concave on $A$ and extends to a continuous function on the closure $\overline{A}$ of $A$ (a compact convex polytope), so attains a maximum on $\overline{A}$.

Theorem [Rivin, Casson, Ken Chan-H]. If $V: A \to \mathbb{R}$ attains its maximum at a point of $A$, then this gives the complete hyperbolic structure on $M$; in particular the triangulation $T$ is geometric.
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**Theorem** [Rivin, Casson, Ken Chan-H]. If \( V : \overline{\mathcal{A}} \to \mathbb{R} \) attains its maximum at a point of \( \mathcal{A} \), then this gives the complete hyperbolic structure on \( M \); in particular the triangulation \( \mathcal{T} \) is geometric.
This program has been carried out very successfully by François Guéritaud for the standard layered triangulations of once-punctured torus bundles, and by Dave Futer for 2-bridge knot and link complements.

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We would like to extend this to other classes of 3-manifolds!

A year or so ago, Ian Agol introduced a class of “veering taut triangulations” with nice combinatorial properties and showed that every hyperbolic bundle over the circle admits such an ideal triangulation, possibly after removing a suitable knot or link.
Veering triangulations

Last year, Rubinstein, Segerman and Tillmann and I introduced a new class of “veering triangulations”, which includes the class of triangulations considered by Agol, and proved the following:

**Main Theorem** [HRST]

*Veering triangulations admit angle structures.*
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Open Question. Is every veering triangulation geometric?
Some definitions

Definition: A *taut angle structure* on an ideal triangulation $\mathcal{T}$ is a solution to the angle equations $(\ast)$ where each angle is 0 or $\pi$. Then each tetrahedron has 2 opposite edges with angle $\pi$ and the other 4 edges with angle 0. For an oriented tetrahedron we have the following standard picture with the 0 angles at the sides of the square, and the $\pi$ angles at the diagonals.

Definition: We colour the zero angle edges red and blue as shown. Then the red edges are called right-veering and the blue edges are called left veering.
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![Diagram of a taut angle structure](image)
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![Diagram of tetrahedron angles](image)

**Definition:** We colour the zero angle edges *red* and *blue* as shown. Then the red edges are called *right-veering* and the blue edges are called *left veering*.
Note:
As viewed from a red edge: the triangles at the edge move to the right going from bottom to top. As viewed from a blue edge: the triangles at the edge move to the left going from bottom to top.
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Definition: A veering structure on an oriented ideal triangulation $\mathcal{T}$ is a choice of taut angle structure together with an assignment of a colour to every edge of the triangulation so that the zero angle edges of each tetrahedron are coloured as above. We also say that $\mathcal{T}$ is a veering triangulation for short.
Normal surfaces and angle structures

A *normal surface* in a triangulation $\mathcal{T}$ of $M$ is a surface that intersects each tetrahedron in a collection of *normal triangles* and *normal quadrilaterals (quads)* as shown below. There are 4 types of normal triangles and 3 types of normal quads in each tetrahedron.
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Each quad type $q$ meeting an edge $e$ has a sign $\varepsilon(q) = +1$ or $-1$ as shown below, where $e$ is the dotted vertical edge.
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Then: the sum of the numbers $\varepsilon(q)x_q$ around each edge $e$ is zero.
Note:

The set of all solutions to the $Q$-matching equations is a vector subspace $Q(T) \subset \mathbb{R}^{3n}$ where $n$ is the number of tetrahedra in $T$. Any admissible solution to these equations in non-negative integers with only one quad type per tetrahedron gives a spun normal surface made up of finitely many quads and (possibly infinitely many) triangles.
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Any *admissible* solution to these equations in non-negative integers with only one quad type per tetrahedron gives a *spun normal surface* made up of finitely many quads and (possibly infinitely many) triangles.
Convex duality (from linear programming) gives the following:

**Proposition.** [Rubinstein-Kang, Luo-Tillmann] Assume that $\mathcal{T}$ has a taut angle structure. Then $\mathcal{T}$ admits an angle structure if there is no non-negative solution $x$ to the Q-matching equations with $\chi^*(x) = 0$ and at least one quad coordinate positive, where

$$2\pi \chi^*(x) = \sum_{\text{quads } q} -2\alpha(q)x_q$$

and $\alpha(q)$ is the angle on the two edges opposite $q$ in the tetrahedron containing the quad $q$. 

Note: The generalised Euler characteristic $\chi^*(\mathcal{T}): \mathcal{Q}(\mathcal{T}) \to \mathbb{R}$ is a linear function which agrees with the Euler characteristic for embedded and immersed normal surfaces in $\mathcal{T}$. 
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**Note:** The generalised Euler characteristic $\chi^* : Q(\mathcal{T}) \rightarrow \mathbb{R}$ is a linear function which agrees with the Euler characteristic for embedded and immersed normal surfaces in $\mathcal{T}$. 
For a triangulation with *taut angle structure* the quads are of three types:

**Horizontal**

\[
\begin{align*}
\alpha(q) &= \pi \\
+1 &-1 +1 -1
\end{align*}
\]

**Vertical type 1**

\[
\begin{align*}
\alpha(q) &= 0 \\
-1 &+1 -1 +1
\end{align*}
\]

**Vertical type 2**

\[
\begin{align*}
\alpha(q) &= 0 \\
1 &-1 -1 -1
\end{align*}
\]

The quad types in an angle-taut tetrahedron, with angles \(\alpha(q)\) of opposite edges and associated signs in the \(Q\)-matching equations.
Since any horizontal quad gives a negative contribution to $\chi^*$, the previous result can be reformulated as follows:

**Proposition.** Assume that $\mathcal{T}$ has a taut angle structure. Then $\mathcal{T}$ admits an angle structure if there is no non-negative solution $x$ to the Q-matching equations *consisting of only vertical quads* and with at least one quad coordinate positive.
Sketch of the proof of Main Theorem

Let $\mathcal{T}$ be a veering triangulation and assume (for a contradiction) we have a solution to the Q-matching equations where all $x_q \geq 0$, at least one $x_q > 0$ and $q$ is a vertical quad type whenever $x_q > 0$. 
Sketch of the proof of Main Theorem

Let \( \mathcal{T} \) be a veering triangulation and assume (for a contradiction) we have a solution to the Q-matching equations where all \( x_q \geq 0 \), at least one \( x_q > 0 \) and \( q \) is a vertical quad type whenever \( x_q > 0 \).

From above, there are two types of vertical quads:

- type 1: zero angle edges are red and contribute \(-1\) to the Q-matching equations
- type 2: zero angle edges are blue and contribute \(+1\) to the Q-matching equations
Now consider the Q-matching equations:

At red edges, we get a negative contribution from angle 0 edges of vertical type 1 quads. This cannot be compensated for by vertical type 2 quads so must cancel with contributions from vertical type 1 quads.
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Thus

\[(\text{the sum of negative contributions from type 1 vertical quads along all red edges}) + (\text{the sum of positive contributions from type 1 vertical quads along all red edges}) = 0.\]
This means that if \( q \) is a type 1 vertical quad with \( x_q > 0 \) then all corners of the quad are on red edges, and the quad lies in a tetrahedron with 4 red edges and 2 blue edges as shown below.
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Similarly for type 2 vertical quads with \( x_q > 0 \), all 4 corners are on blue edges.
It follows that the Q-matching equations split into two subsets:

- the equations for red edges and type 1 vertical quads
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Thus, without loss of generality, we can assume we have a solution with only type 1 vertical quads.
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Thus, without loss of generality, we can assume we have a solution with only type 1 vertical quads.

Then there is only one quad type in each tetrahedron, so we have an “admissible solution” \( x \) to the Q-matching equations. By standard results about these equations we can assume \( x \) gives

an embedded spun normal surface \( S \), connected and oriented, made up of type 1 vertical quads and triangles.
Now each quad occurring in $S$ lies in a tetrahedron with 4 red edges and has 4 red corners:

- 2 corners with angle $\pi$ and sign $+1$ in the Q-matching equations, and
- 2 corners with angle 0 and sign $-1$ in the Q-matching equations.

This implies that in the cell decomposition of $S$ into normal quads and triangles:

- at each vertex we have at most 2 quads with angle $\pi$ (since angle sum is $2\pi$) and each of these contributes sign $+1$ to Q-matching equations so needs another quad corner with sign $-1$ to cancel.

So we have at most 4 quads at each vertex (and perhaps some triangles).
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Now the veering condition gives:

Lemma. Let $e$ be an edge with angle 0 in a tetrahedron with 4 red edges. Then the two neighbouring tetrahedra around $e$ have angle $\pi$ at $e$. 
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This implies if $q$ is a quad in $S$ with angle 0 and sign $-1$ at a vertex $v$, then both adjacent cells around $v$ have angle $\pi$ at $v$ and at least one of these is a quad with sign $+1$ at $v$. 
This leads to the following:

**Lemma.** For some starting quad $q_0$, we can find an arbitrarily long strip of adjacent quads in $S$ continuing in the NE direction or in the SE direction.
Orientability of $S$ implies that this strip must consist of quads facing a single edge $e$ of the triangulation:

The dual edge remains the same as we walk across quads.
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But then the angle sum around $e$ would be zero, contradicting the definition of a taut angle structure. This completes the proof of the main theorem. □
Some open questions

1. If $M$ has a triangulation $\mathcal{T}$ with an angle structure, is $\text{vol}(\text{angle structure}) \leq \text{vol}(\text{complete hyperbolic structure on } M)$? This would give useful lower bounds on hyperbolic volume of $M$. (True if the triangulation $\mathcal{T}$ is geometric.)

2. Is every veering triangulation geometric? The case of Agol’s triangulations for hyperbolic bundles is particularly interesting. Here there is a lot of additional structure that might be useful.
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