Diffeomorphisms and Heegaard splittings of 3-manifolds

Hyamfest

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Some philosophy

*Adding geometric structure tends to restrict automorphisms.*

\[
\begin{array}{ccc}
\text{topological manifold } M & \longleftrightarrow & \text{Homeo}(M) \\
\downarrow & & \downarrow \\
\text{smooth manifold } M & \longleftrightarrow & \text{Diff}(M) \\
\downarrow & & \downarrow \\
\text{Riemannian manifold } M & \longleftrightarrow & \text{Isom}(M)
\end{array}
\]
But adding symmetry tends to create automorphisms.

*Notation:* $\text{isom}(S^2) = \text{connected component of } 1_{S^2} \text{ in } \text{Isom}(S^2)$, similarly for $\text{diff}(M) \subseteq \text{Diff}(M)$.

<table>
<thead>
<tr>
<th>metric</th>
<th>random</th>
<th>( \text{isom}(S^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ellipsoid</td>
<td>$S^1 = \text{SO}(2)$</td>
<td></td>
</tr>
<tr>
<td>round</td>
<td></td>
<td>$\text{SO}(3)$</td>
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An example

By Perelman’s Geometrization Theorem, a closed 3-manifold with finite fundamental group is of the form $S^3/G$, with $G \subset SO(4)$ acting freely. Consequently, such a manifold has Riemannian metrics of constant positive curvature.

We call these manifolds elliptic 3-manifolds.

M (2002): Calculated $\text{Isom}(M)$ for all elliptics.

— This is “folklore”. Hyam and others understood the $\text{Isom}(S^3/G)$ decades ago.

— $\text{Isom}(S^3/G) = \text{Norm}(G)/G$, where $G$ is the normalizer of $G$ in $\text{Isom}(S^3) = O(4)$.

— Compute $\text{Norm}(G)/G$ using the quaternionic description of $SO(4)$:

$$
S^3 = \text{unit quaternions},
$$

$$
SO(4) = (S^3 \times S^3)/\langle(-1, -1)\rangle
$$
\[
\begin{array}{|l|l|l|}
\hline
L(m, q) & \text{Isom}(L(m, q)) & \dim(\text{Isom}(L(m, q))) \\
\hline
L(1, 0) = S^3 & O(4) & 6 \\
L(2, 1) = \mathbb{RP}(3) & (SO(3) \times SO(3)) \circ C_2 & 6 \\
L(m, 1), m \text{ odd}, m > 2 & O(2)^* \tilde{\times} S^3 & 4 \\
L(m, 1), m \text{ even}, m > 2 & O(2) \times SO(3) & 4 \\
L(m, q), 1 < q < m/2, q^2 \not\equiv \pm 1 \mod m & \text{Dih}(S^1 \times S^1) & 2 \\
L(m, q), 1 < q < m/2, q^2 \equiv -1 \mod m & (S^1 \tilde{\times} S^1) \circ C_4 & 2 \\
L(m, q), 1 < q < m/2, q^2 \equiv 1 \mod m, \\
gcd(m, q+1) \gcd(m, q-1) = m & O(2) \tilde{\times} O(2) & 2 \\
L(m, q), 1 < q < m/2, q^2 \equiv 1 \mod m, \\
gcd(m, q+1) \gcd(m, q-1) = 2m & O(2) \times O(2) & 2 \\
\hline
\end{array}
\]

Table 1: Isometry groups of \(L(m, q)\)

\[
\begin{array}{|l|l|l|l|}
\hline
G & M & \text{Isom}(M) & \dim(\text{Isom}(M)) \\
\hline
Q_8 & \text{quaternionic} & SO(3) \times S_3 & 3 \\
Q_8 \times C_n & \text{quaternionic} & O(2) \times S_3 & 1 \\
D_{4m}^* & \text{prism} & SO(3) \times C_2 & 3 \\
D_{4m}^* \times C_n & \text{prism} & O(2) \times C_2 & 1 \\
\text{index 2 diagonal} & \text{prism} & O(2) \times C_2 & 1 \\
T_{24}^* & \text{tetrahedral} & SO(3) \times C_2 & 3 \\
T_{24}^* \times C_n & \text{tetrahedral} & O(2) \times C_2 & 1 \\
\text{index 3 diagonal} & \text{tetrahedral} & O(2) & 1 \\
o_{48}^* & \text{octahedral} & SO(3) & 3 \\
o_{48}^* \times C_n & \text{octahedral} & O(2) & 1 \\
I_{120}^* & \text{icosahedral} & SO(3) & 3 \\
I_{120}^* \times C_n & \text{icosahedral} & O(2) & 1 \\
\hline
\end{array}
\]

Table 2: Isometry groups of elliptic 3-manifolds other than \(L(m, q)\)
For reducible 3-manifolds, the gap between isom\((M)\) and \(\text{diff}(M)\) tends to be large: For most reducible \(M\), \(\text{isom}(M) = \{1\}\) for any metric, while \(\pi_1(\text{diff}(M))\) is not finitely generated (Kalliongis-M 1996)

But for an irreducible 3-manifold with a metric of “maximal” symmetry, we often see a close connection between isom\((M)\) and \(\text{diff}(M)\), and sometimes even Isom\((M)\) and Diff\((M)\).

Let’s start with dimension 1:
\(\text{Isom}(S^1) = O(2) \hookrightarrow \text{Diff}(S^1)\) is a homotopy equivalence.

— The subspace of orientation-preserving diffeomorphisms that take the basepoint 1 to a given point \(p\) canonically deformation retracts to the unique rotation that rotates 1 to \(p\) (a straight-line homotopy between lifts to the universal cover \(\mathbb{R}\) is an equivariant isotopy, so defines a canonical isotopy on \(S^1\)).

— Similarly the orientation-reversing diffeomorphisms taking 1 to \(p\) canonically deformation retract to the reflection taking 1 to \(p\).

— These deformation retractions all fit together continuously to give a deformation retraction of all of \(\text{Diff}(S^1)\) to \(O(2)\).
This tells us the *homeomorphism* type of $\text{Diff}(S^1)$ with the $C^\infty$-topology:

— With the $C^\infty$-topology, $\text{Diff}(M)$ is a separable Fréchet manifold (locally $\mathbb{R}^\infty$) for any closed $M$.

— $\text{Diff}(S^1) \simeq \text{O}(2) \simeq \text{O}(2) \times \mathbb{R}^\infty$.

— Homotopy equivalent (infinite-dimensional) separable Fréchet manifolds are homeomorphic, so $\text{Diff}(S^1) \approx \text{O}(2) \times \mathbb{R}^\infty$.

What about *isomorphism*? If $\text{Diff}(M)$ and $\text{Diff}(N)$ are abstractly isomorphic, then $M$ is diffeomorphic to $N$ (Filipkiewicz, 1982).

— The hard part of the argument is to show that an isomorphism from $\text{Diff}(M)$ to $\text{Diff}(N)$ takes the point stabilizer subgroups $\text{Diff}(M, x)$ to point stabilizer subgroups of $\text{Diff}(N)$.

— In this way an isomorphism from $\text{Diff}(M)$ to $\text{Diff}(N)$ gives a bijective correspondence between the points of $M$ and those of $N$.

— This correspondence turns out to be a diffeomorphism.
The Smale Conjecture

S. Smale (1959): Isom($S^2$) = O(3) $\hookrightarrow$ Diff($S^2$) is a homotopy equivalence (so Diff($S^2$) $\approx$ O(3) $\times$ $\mathbb{R}^\infty$).

Smale conjectured that Isom($S^3$) = O(4) $\hookrightarrow$ Diff($S^3$) is a homotopy equivalence.

This was proven by J. Cerf and A. Hatcher:

— Cerf (1968): $\pi_0$(Isom($S^3$)) $\rightarrow$ $\pi_0$(Diff($S^3$)) is an isomorphism (the “$\pi_0$-part” of the conjecture).

— Hatcher (1983): $\pi_q$(Isom($S^3$)) $\rightarrow$ $\pi_q$(Diff($S^3$)) is an isomorphism for all $q \geq 1$.

Terminology: A (Riemannian) manifold $M$ satisfies the Smale Conjecture (SC) if Isom($M$) $\hookrightarrow$ Diff($M$) is a homotopy equivalence.

$M$ satisfies the weak Smale Conjecture (WSC) if isom($M$) $\hookrightarrow$ diff($M$) is a homotopy equivalence.
The case of infinite fundamental group

1. Hatcher, N. Ivanov (independently, late 1970’s): Haken manifolds satisfy the WSC.

Key ideas in the proofs:

— Let \( F^2 \hookrightarrow M \) be incompressible. Use the *Cerf-Palais fibration*:

\[
\text{Diff}(M \text{ rel } F) \subset \text{Diff}(M) \quad f
\]

\[
\downarrow \quad \downarrow
\]

\[
\text{Emb}(F, M) \quad f|_F
\]

to relate \( \text{Diff}(M) \) to embeddings of \( F \) into \( M \).

— Analyze parameterized families of embeddings of \( F \) into \( M \). Show that the components of \( \text{Emb}(F, M) \) are contractible, deduce that

\[
\text{diff}(M \text{ rel } F) \hookrightarrow \text{diff}(M \text{ rel } \partial M)
\]

is a homotopy equivalence.

— This eventually reduces the result to knowing that \( \text{Diff}(B^3 \text{ rel } \partial B^3) \) is contractible, which is equivalent to the SC for \( S^3 \).

In general, Haken manifolds do not satisfy the SC: \( \pi_0(\text{Isom}(M)) \) is finite, but \( \pi_0(\text{Diff}(M)) \) can be infinite.

   
   — The proof utilizes Gabai’s methodology.
   
   — Hyam had the idea of how to do this years earlier.

The case of finite fundamental group

1. Ivanov (around 1980): Adapted the Hatcher-Ivanov method to some of the elliptic $M$ that contain a one-sided geometrically incompressible Klein bottle, to prove SC for many of the prism manifolds (Seifert-fibered over $S^2$ with 2, 2, $n$ cone points) and announced the result for the lens spaces $L(4n, 2n - 1), n \geq 2$.

2. M-Rubinstein (starting in 1980’s): Extended Ivanov’s method to all elliptic $M$ containing one-sided Klein bottles, except for $L(4, 1)$. This includes all prism manifolds and all $L(4n, 2n - 1), n \geq 2$.

A key ingredient is a Cerf-Palais fibration $\text{Diff}_f(M) \to \text{Emb}_f(K, M)$, where the “$f$” subscript indicates the fiber-preserving diffeomorphisms for a Seifert fibering of $M$. This “folklore” theorem took a lot of effort to prove (Kalliongis-M).

3. M (2002): For elliptic $M$, $\text{Isom}(M) \to \text{Diff}(M)$ is a bijection on path components.
   - The proof uses the calculation of $\text{Isom}(M)$ and applies many people’s results on $\pi_0(\text{Diff}(M))$ to establish that $\pi_0(\text{Isom}(M)) \to \pi_0(\text{Diff}(M))$ is an isomorphism.
   - This is the “$\pi_0$-part” of the SC for all elliptic 3-manifolds. It reduces the SC to the WSC.
4. Hong-M-Rubinstein (2000’s): SC for all lens spaces (except $L(2, 1) = \mathbb{RP}^3$).

The proof is unfortunately very long and technical. The key ideas:

— By M (2002), it suffices to prove the WSC for $L$. For this it suffices to prove that

$$\pi_q(\text{isom}(L)) \rightarrow \pi_q(\text{diff}(L))$$

is an isomorphism for all $q \geq 1$.

— For a certain Seifert fibering of $L$, every isometry is fiber-preserving (this fails for $L = L(2, 1)$), so

$$\text{isom}(L) \subset \text{diff}_f(L) \subset \text{diff}(L).$$

It’s not too hard to prove that

$$\pi_q(\text{isom}(L)) \rightarrow \pi_q(\text{diff}_f(L))$$

is an isomorphism, so it remains to prove that $\pi_q(\text{diff}_f(L)) \rightarrow \pi_q(\text{diff}(L))$ is an isomorphism.

— This reduces the problem to proving that all $\pi_q(\text{diff}(L), \text{diff}_f(L))$ are zero. An element of

$$\pi_q(\text{diff}(L), \text{diff}_f(L))$$

is represented by a $q$-dimensional parameterized family of diffeomorphisms $g_t$ of $L$, where $t \in D^q$ and $g_t$ is fiber-preserving for $t \in \partial D^q$. The task is to deform the family to make all the $g_t$ fiber-preserving.
— Fix a sweepout of $L$ having Heegaard tori as the generic levels, each a union of fibers. Look at how their images under the $g_t$ meet the fixed levels. Using singularity theory, we can perturb the $g_t$ so that the tangencies are nice enough to have a version of the Rubinstein-Scharlemann graphic (this step is hard).

— From those Rubinstein-Scharlemann graphics, we can deduce that for each $t$ there is a nice image torus level—an image level that meets some fixed level so that neither torus contains a meridian disk in a complementary solid torus of the other.

— By a lot of careful isotopy of the $g_t$, we can level (or at least “straighten out”) their individual nice image levels, then all image levels, then make the $g_t$ fiber-preserving.

M-Rubinstein, Kalliongis-M, and Hong-M-Rubinstein are all written up in a preprint monograph *Diffeomorphisms of Elliptic 3-Manifolds*.

Remark: No one has been able to use Perelman’s ideas to make any progress on the Smale Conjecture for elliptic 3-manifolds.
**Heegaard splittings** (joint with Jesse Johnson)

Isotopy classes of Heegaard splittings have been extensively studied. These are actually the path components of a *space of Heegaard splittings*.

For a Heegaard splitting \((M, \Sigma)\) of a closed (orientable) 3-manifold \(M\), write \(\text{Diff}(M, \Sigma)\) for the subgroup of \(\text{Diff}(M)\) consisting of the \(f\) such that \(f(\Sigma) = \Sigma\).

Define the *space of Heegaard splittings equivalent to* \((M, \Sigma)\) to be the space of cosets

\[
\mathcal{H}(M, \Sigma) = \text{Diff}(M)/\text{Diff}(M, \Sigma).
\]

— A point in \(\mathcal{H}(M, \Sigma)\) represents a coordinate-free image of \(\Sigma\) in \(M\) under a diffeomorphism of \(M\). For two diffeomorphisms \(f, g \in \text{Diff}(M)\) satisfy \(f(\Sigma) = g(\Sigma)\) exactly when \(g^{-1}f(\Sigma) = \Sigma\), that is, when \(f\) and \(g\) represent the same coset in \(\text{Diff}(M)/\text{Diff}(M, \Sigma)\).

— A path in \(\mathcal{H}(M, \Sigma)\) is a movie of \(\Sigma\) moving around in \(M\). A loop is when it returns to its starting position, although its points may have shifted around as it moved.
A correct intuitive guess is that $\mathcal{H}(S^3, S^2) \simeq \mathbb{RP}^3$:

— $\mathcal{H}(S^3, S^2)$ is the space of positions of $S^2$ in $S^3$.

— The SC for $S^3$ says that it should be enough to consider “orthogonal” positions, that is, images of the “equatorial” $S^2$ under isometries of $S^3$. Such images correspond to their pairs of antipodal “poles,” which are arbitrary pairs of antipodal points. The space of such pairs is $\mathbb{RP}^3$.

In general, what is the homotopy type of $\mathcal{H}(M, \Sigma)$?

Since $\mathcal{H}(M, \Sigma)$ is closely related to $\text{Diff}(M)$, we expect its homotopy type to be highly affected by that of $\text{Diff}(M)$.

Notation: Write $\mathcal{H}_q(M, \Sigma)$ for $\pi_q(\mathcal{H}(M, \Sigma))$. Notice that there is a natural homomorphism

$$\pi_q(\text{Diff}(M)) \to \mathcal{H}_q(M, \Sigma).$$
Theorem 1 Suppose that $\Sigma$ has genus at least 2. Then $\pi_q(\text{Diff}(M)) \to \mathcal{H}_q(M, \Sigma)$ is an isomorphism for $q \geq 2$, and there are exact sequences

$$1 \to \pi_1(\text{Diff}(M)) \to \mathcal{H}_1(M, \Sigma) \to \mathcal{G}(M, \Sigma) \to 1,$$

$$1 \to \mathcal{G}(M, \Sigma) \to \pi_0(\text{Diff}(M, \Sigma)) \to \pi_0(\text{Diff}(M)) \to \mathcal{H}_0(M, \Sigma) \to 1.$$

Here, $\mathcal{G}(M, \Sigma)$ is the Goeritz group of the Heegaard splitting, defined to be the kernel of $\pi_0(\text{Diff}(M, \Sigma)) \to \pi_0(\text{Diff}(M))$.

Idea of the proof: Use the Cerf-Palais methodology to prove that $\text{Diff}(M) \to \text{Diff}(M)/\text{Diff}(M, \Sigma)$ is a fibration. The fiber is $\text{Diff}(M, \Sigma)$, giving a long exact sequence

$$\cdots \to \pi_q(\text{Diff}(M, \Sigma)) \to \pi_q(\text{Diff}(M)) \to \mathcal{H}_q(M, \Sigma) \to \pi_{q-1}(\text{Diff}(M, \Sigma)) \to \pi_{q-1}(\text{Diff}(M)) \to \cdots$$

Since the genus of $\Sigma$ is at least 2, $\pi_q(\text{Diff}(M, \Sigma)) = 0$ for $q \geq 2$.

For most reducible $M$, $\pi_1(\text{Diff}(M))$ is known to be non-finitely-generated (Kalliongis-M, 1996), suggesting that $\mathcal{H}(M, \Sigma)$ has a complicated homotopy type in these cases.
Theorem 1 has some nice applications:

**Corollary 2** Suppose that $M$ is irreducible and $\pi_1(M)$ is infinite, and that $M$ is not non-Haken with the Nil geometry. Then $\mathcal{H}_i(M, \Sigma) = 0$ for $i \geq 2$, and there is an exact sequence

$$1 \to \text{center}(\pi_1(M)) \to \mathcal{H}_1(M, \Sigma) \to \mathcal{G}(M, \Sigma) \to 1.$$ 

Consequently for these $(M, \Sigma)$:

(a) Each component of $\mathcal{H}(M, \Sigma)$ is aspherical.

(b) If $\pi_1(M)$ is centerless, then $\mathcal{H}(M, \Sigma)$ is a $K(\mathcal{G}(M, \Sigma), 1)$-space.

**Corollary 3** If the Hempel distance $d(M, \Sigma) > 3$, then $\mathcal{H}(M, \Sigma)$ has finitely many components, each of which is contractible. If $d(M, \Sigma) > 2\text{genus}(\Sigma)$, then $\mathcal{H}(M, \Sigma)$ is contractible.

The proof of Corollary 3 uses results of J. Hempel, J. Johnson, and A. Thompson.
For elliptic 3-manifolds, the homotopy type of $\mathcal{H}(M, \Sigma)$ is, as expected, more complicated and more difficult to calculate. But provided that the manifold satisfies the SC, we can utilize information coming from the quaternionic calculation of $\text{Isom}(M)$ to obtain a good description of $\mathcal{H}(M, \Sigma)$.

For the 3-sphere:

**Theorem 4** For $n \geq 0$ let $\Sigma_n$ be the unique Heegaard surface of genus $n$ in $S^3$.

1. $\mathcal{H}(S^3, \Sigma_0) \simeq \mathbb{RP}^3$.
2. $\mathcal{H}(S^3, \Sigma_1) \simeq \mathbb{RP}^2 \times \mathbb{RP}^2$.
3. For $n \geq 2$, $\mathcal{H}_i(S^3, \Sigma_n) \cong \pi_i(S^3 \times S^3)$ for $i \geq 2$, and there is an non-split exact sequence

$$1 \rightarrow C_2 \rightarrow \mathcal{H}_1(S^3, \Sigma_n) \rightarrow G(S^3, \Sigma_n) \rightarrow 1$$

where $C_2$ is the cyclic group of order 2.
For lens spaces:

**Theorem 5** Let $L = L(m, q)$ be a lens space with $m \geq 2$ and $1 \leq q \leq m/2$. If $L = L(2, 1)$, assume that $L$ satisfies the Smale Conjecture. For $n \geq 1$, let $\Sigma_n$ be the unique Heegaard surface of genus $n$ in $L$.

1. If $q \geq 2$, then
   
   (a) $\mathcal{H}(L, \Sigma_1)$ is contractible.

   (b) For $n \geq 2$, $\mathcal{H}_i(L, \Sigma_n) = 0$ for $i \geq 2$, and there is an exact sequence
       
       \[ 1 \to \mathbb{Z} \times \mathbb{Z} \to \mathcal{H}_1(L, \Sigma_n) \to \mathcal{G}(L, \Sigma_n) \to 1. \]

2. If $m > 2$ and $q = 1$, then
   
   (a) $\mathcal{H}(L, \Sigma_1) \simeq \mathbb{R}P^2$.

   (b) For $n \geq 2$, $\mathcal{H}_i(L, \Sigma_n) \cong \pi_i(S^3)$ for $i \geq 2$, and there are exact sequences
       
       \[ 1 \to \mathbb{Z} \to \mathcal{H}_1(L, \Sigma_n) \to \mathcal{G}(L, \Sigma_n) \to 1 \]
       
       for $m$ odd, and
       
       \[ 1 \to \mathbb{Z} \times C_2 \to \mathcal{H}_1(L, \Sigma_n) \to \mathcal{G}(L, \Sigma_n) \to 1 \]
       
       for $m$ even.
3. If \( L = L(2, 1) \), then

(a) \( \mathcal{H}(L, \Sigma_1) \cong \mathbb{RP}^2 \times \mathbb{RP}^2 \).

(b) For \( n \geq 2 \), \( \mathcal{H}_i(L, \Sigma_n) \cong \pi_i(S^3 \times S^3) \) for \( i \geq 2 \), and there is an exact sequence

\[
1 \to C_2 \times C_2 \to \mathcal{H}_1(L, \Sigma_n) \to \mathcal{G}(L, \Sigma_n) \to 1.
\]

For the other elliptic 3-manifolds:

**Theorem 6** Let \( E \) be an elliptic 3-manifold, but not \( S^3 \) or a lens space. Assume, if necessary, that \( E \) satisfies the Smale Conjecture. Let \( \Sigma \) be a Heegaard surface in \( E \).

1. If \( \pi_1(E) \cong D_{4m}^* \), or if \( E \) is one of the three manifolds with fundamental group either \( T_{24}^* \), \( O_{48}^* \), or \( I_{120}^* \), then \( \mathcal{H}_i(E, \Sigma) \cong \pi_i(S^3) \) for \( i \geq 2 \) and there is an exact sequence

\[
1 \to C_2 \to \mathcal{H}_1(E, \Sigma) \to \mathcal{G}(E, \Sigma) \to 1.
\]

2. If \( E \) is not one of the manifolds in Case (1), that is, either \( \pi_1(E) \) has a nontrivial cyclic direct factor, or \( \pi_1(E) \) is a diagonal subgroup of index 2 in \( D_{4m}^* \times C_n \) or of index 3 in \( T_{48}^* \times C_n \), then \( \mathcal{H}_i(E, \Sigma) = 0 \) for \( i \geq 2 \), and there is an exact sequence

\[
1 \to \mathbb{Z} \to \mathcal{H}_1(E, \Sigma) \to \mathcal{G}(E, \Sigma) \to 1.
\]