1. **Symplectic reduction.** Let \((M, \omega)\) be a symplectic manifold, and let \(H \in C^\infty(M)\) be a Hamiltonian generating an \(S^1\)-action, i.e., \(h_t = 1\) for some \(T > 0\), where \(h_t\) denotes the Hamiltonian flow generated by \(H\). Our goal is to prove that when the action on a regular level set is free, the quotient inherits a natural symplectic structure.

(i) Prove the law of conservation of energy: \(H \circ h_t \equiv H\).

(ii) Let \(c\) be a regular value of \(H\) and set \(\Sigma = \{H = c\}\). Prove that, for every \(x \in \Sigma\), the vector \(\text{sgrad}\ H(x)\) spans the \(\omega\)-orthogonal complement to \(T_x\Sigma\),

\[
T_x\Sigma^\omega = \{\xi \in T_xM: \omega(\xi, \eta) = 0 \forall \eta \in T_x\Sigma\}.
\]

(iii) Suppose that the \(S^1\)-action on \(\Sigma\) is free, so that the quotient \(N = \Sigma/S^1\) is a manifold. Define a symplectic form \(\sigma\) on \(N\) as follows. Let \(\pi: \Sigma \to N\) be the natural projection. Let \(x \in N\), and \(\xi, \eta \in T_xN\). Choose \(\tilde{x} \in \pi^{-1}(x)\), and \(\tilde{\xi}, \tilde{\eta} \in T_{\tilde{x}}\Sigma\) such that

\[
\pi_*\tilde{\xi} = \xi, \quad \pi_*\tilde{\eta} = \eta.
\]

Set

\[
\sigma_x(\xi, \eta) = \omega_{\tilde{x}}(\tilde{\xi}, \tilde{\eta}).
\]

Check that \(\sigma\) is a well-defined symplectic form on \(N\).

(iv) As an example, take \(M = \mathbb{R}^{2n}(p, q) \simeq \mathbb{C}^n(z = p + iq)\) with the standard symplectic form

\[
\omega_0 = \sum dp_j \wedge dq_j.
\]

Consider the Hamiltonian \(H(z) = \pi|z|^2\). Check that

\[
\text{sgrad}\ H(z) = 2\pi iz,
\]

and hence Hamilton’s equations are

\[
\dot{z}(t) = 2\pi iz(t).
\]
Therefore, the Hamiltonian flow of $H$ is $z(t) = e^{2\pi i t} z(0)$. The level set $\{ H = 1 \}$ is the $2n-1$-sphere of radius $\frac{1}{\pi}$, and the $S^1$-action is the usual $S^1$-action on $S^{2n-1}$. Therefore the reduced manifold is $S^{2n-1}/S^1 \simeq \mathbb{CP}^{n-1}$. The induced form on $\mathbb{CP}^{n-1}$ is called the Fubini-Studi form.

(v) Compute $\int_{\mathbb{CP}^1} \sigma$.

2.

(i) Let $H \in C^\infty(M)$ and let $\phi : M \to M$ be a symplectomorphism. Prove that

$$sgrad(H \circ \phi^{-1}) = \phi_* sgrad H.$$ 

Here

$$(\phi_* sgrad H)(x) = \phi_*(sgrad H(\phi^{-1}(x))).$$

(ii) Let $F_t, G_t$ be two time-dependent Hamiltonians, generating the Hamiltonian flows $f_t, g_t$, respectively. Prove that the flow of diffeomorphisms $t \mapsto f_t g_t$ is the Hamiltonian flow generated by the Hamiltonian $H_t = F_t + G_t \circ f_t^{-1}$. Deduce that the flow $t \mapsto f_t^{-1}$ is the Hamiltonian flow generated by $F_t = -F_t \circ f_t$.

3. Let $(M, \omega)$ be a symplectic manifold such that $H^1_c(M; \mathbb{R}) = 0$ (Here $H^1_c$ denotes cohomology with compact support). Prove that $\text{Symp}_0(M, \omega) = \text{Ham}(M, \omega)$. In other words, prove that every symplectomorphism of $M$ which is isotopic to the identity through symplectomorphisms is Hamiltonian.

4. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ with coordinates $(p, q)$, equipped with the symplectic form $dp \wedge dq$. Define

$$\varphi : \mathbb{T}^2 \to \mathbb{T}^2, \quad (p, q) \mapsto (p, q + v), \quad v \notin \mathbb{Z}.$$ 

Clearly, $\varphi \in \text{Symp}_0(\mathbb{T}^2)$. The goal of this exercise is to prove that $\varphi \notin \text{Ham}(\mathbb{T}^2)$.

(i) Assume on the contrary that $\varphi = \varphi_1$, where $\{\varphi_t\}_{t \in [0, 1]}$ is the Hamiltonian flow generated by the Hamiltonian $F_t$. For a loop $\gamma : S^1 \to \mathbb{T}^2$, set

$$\Sigma = \bigcup_{t \in [0, 1]} \varphi_t \gamma.$$ 

That is, $\Sigma$ is the surface swept by $\gamma$ under the isotopy $\varphi_t$. Prove that, for any loop $\gamma$,

$$(\omega, \Sigma) = 0.$$

1Recall that we denote by $\text{Symp}_0(M, \omega)$ the identity component of the group of all compactly supported symplectomorphisms of $M$, and by $\text{Ham}(M, \omega)$ its subgroup of all compactly supported Hamiltonian diffeomorphisms.
(ii) Conclude that $\varphi$ is not a Hamiltonian diffeomorphism.

**Hints**

1. (iii) If $\tilde{x}' \in \pi^{-1}(x)$ is another lift of $x$, there is $t$ such that $h_t \tilde{x} = \tilde{x}'$, and $h_t$ is a symplectomorphism. If $\tilde{\xi}' \in T_{\tilde{x}} \Sigma$ is another lift of $\xi$, then

$$\tilde{\xi}' - h_{t*} \tilde{\xi} \in \ker \pi_*.$$

Show that $\ker \pi_* = T_x \Sigma\omega$.

3. Let $f_t$ be a path in $\text{Symp}_0(M, \omega)$. Define a time-dependent vector field $\xi_t$ by

$$\frac{d}{dt} f_t x = \xi_t(f_t x).$$

Prove that the compactly supported 1-forms $i_{\xi_t} \omega$ are all closed, and hence exact. Deduce that $f_t$ is a Hamiltonian path.

4. (i) Set

$$\beta: S^1 \times [0, 1] \to \mathbb{T}^2, \quad (s, t) \mapsto \varphi_t(\gamma(s)).$$

Prove that

$$\beta^* \omega = \omega(\dot{\gamma}, \text{sgrad} \mathcal{F}_t) ds \wedge dt,$$

where $\mathcal{F}_t = F_t \circ \varphi_t$.

(ii) Consider the loops $\{q \equiv c\}$. 